A NOTE ON DIFFERENTIAL EQUATIONS WITH ALL SOLUTIONS OF INTEGRABLE-SQUARE

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It is shown that if all solutions to \( l(y) = \lambda wy \) and \( l^*(y) = \lambda wy \) satisfy \( \int_{a}^{b} |y|^2w < \infty \) for some complex number \( \lambda \) then so do all solutions for every complex number \( \lambda \). The result is derived from a corresponding one for first order vector-matrix systems.

We shall be concerned with solutions to

\[
\begin{align*}
(1) & \quad l(y) = 0 \quad \text{on} \quad (a, b), \\
(2) & \quad l^*(y) = 0 \quad \text{on} \quad (a, b) \\
(3) & \quad l(y) = \lambda wy \quad \text{on} \quad (a, b), \quad \text{and} \\
(4) & \quad l^*(y) = \lambda wy \quad \text{on} \quad (a, b)
\end{align*}
\]

which satisfy

\[
\int_{a}^{b} |y|^2w < \infty.
\]

In these expressions \( (a, b) \) is an interval of the real line (\( a = -\infty \) and/or \( b = \infty \) is allowed), \( w \) is a weight, i.e., a positive valued continuous function on \( (a, b) \), \( \lambda \) is a complex number, \( l \) is an \( m \)th order linear differential operator given by

\[
l(y) = \sum_{k=0}^{m} a_k y^{(m-k)}
\]

where each \( a_k \) is an \( m - k \) times continuously differentiable complex valued function defined on \( (a, b) \), \( a_0(t) \neq 0 \) for all \( t \in (a, b) \), and \( l^* \) is the formal adjoint of \( l \) so that

\[
l^*(y) = \sum_{k=1}^{m} (-1)^{m-k} (\bar{a}_k y)^{(m-k)}.
\]

In an earlier paper, [11], we defined \( w \) to be a compactifying weight.
for \( l \) provided that every function which is a solution either of (1) or of (2) satisfies (5). If follows from Theorem 2–1 of [11] that if \( w \) is a compactifying weight for \( l \) then every function which is a solution either of (3) or of (4) satisfies (5) for every complex number \( \lambda \).

The deficiency index problem (see for example [2] and [8]) for formally self-adjoint equations (where \( l = l' \)) is concerned with finding the dimension of the linear manifold of solutions to (3) which satisfy (5). One of the results of this theory ([3], [4], [5], [6], [7], [10], and [12]) is that if this dimension is \( m \) (the order of \( l \)) for some complex number \( \lambda \) and \( m > 1 \) then it is \( m \) for every complex number \( \lambda \).

While much of the theory for the self-adjoint case breaks down when \( l \neq l' \) we wish to show that this result carries over.

**Theorem 1.** Let each of \( \lambda_1 \) and \( \lambda_2 \) be a complex number (\( \lambda_1 \) real, even \( \lambda_1 = 0 \) is allowed). Let \( m > 1 \). If every function which is a solution of either (3) or (4) satisfies (5) when \( \lambda = \lambda_1 \) then every function which is a solution of either (3) or (4) satisfies (5) when \( \lambda = \lambda_2 \).

This follows as a corollary to an analogous theorem (Theorem 2 below) for first order vector-matrix equations.

We consider the equations,

\[
Jy' = [\lambda A + B]y \quad \text{a.e. on } (a, b), \quad \text{and} \\
Jy' = [\lambda A + B^*]y \quad \text{a.e. on } (a, b)
\]

where \( J \) is a skew-symmetric \((J^* = -J, * \text{ denoting conjugate transpose})\) \( m \times m \) matrix, each of \( A \) and \( B \) is a complex \( m \times m \) matrix valued function which is Lebesque integrable over each compact subinterval of \((a, b)\), \( \lambda \) is a complex number, and \( A(t) \) is nonnegative definite a.e. on \((a, b)\).

It was shown in [13] that, given \( l; J, A, \) and \( B \) may be chosen so that every solution of (3) satisfies (5) if and only if every solution of (8) satisfies

\[
\int_a^b y^* A y < \infty,
\]

and every solution of (4) satisfies (5) if and only if every solution of (9) satisfies (10). For the choice of \( J \) and \( A \) used in [13] it is also the case that trace \( J^{-1} A = 0 \) when \( m > 1 \).

Thus Theorem 1 above follows from Theorem 2 below.

**Theorem 2.** Let each of \( J, A, \) and \( B \) satisfy the conditions imposed above. Let \( m > 1 \). Let each of \( \lambda_1 \) and \( \lambda_2 \) be a complex number (\( \lambda_1 \) real, even \( \lambda_1 = 0 \) is allowed). Let \( \int_a^b |\text{tr} J^{-1} A| < \infty \).
If every vector function which is a solution of either (8) or (9) satisfies (10) when \( \lambda = \lambda_1 \) then every vector function which is a solution of either (8) or (9) satisfies (10) when \( \lambda = \lambda_2 \).

**Proof.** Let \( Y(\lambda) \) and \( Z(\lambda) \) be fundamental matrices for (8) and (9) respectively. (We will write \( Y(t, \lambda) \) and \( Z(t, \lambda) \) to denote the value of these functions at \( t \in (a, b) \).) Let \( U \) be defined by

\[
Y(\lambda_2) = Y(\lambda_1)U \quad \text{on} \quad (a, b).
\]

Multiplying on the left by \( I \), differentiating, and using (8) we have,

\[
(\lambda_2A + B)Y(\lambda_2) = (\lambda_1A + B)Y(\lambda_1)U + JY(\lambda_1)U' \quad \text{a.e. on} \quad (a, b).
\]

From (11) we have,

\[
JY(\lambda_1)U' = (\lambda_2 - \lambda_1)AY(\lambda_1)U \quad \text{a.e. on} \quad (a, b).
\]

Multiplying on the left by \( Z^*(\lambda_1) \) we have

\[
Z^*(\lambda_1)Y(\lambda_1)U' = (\lambda_2 - \lambda_1)Z^*(\lambda_1)AY(\lambda_1)U \quad \text{a.e. on} \quad (a, b).
\]

We first note that

\[
\int_a^b \|Z^*(t, \lambda_1)Y(t, \lambda_1)\|dt < \infty
\]

where \( \| \cdot \| \) is any matrix norm. In order that (13) hold it is sufficient that

\[
\int_a^b |z^*(t, \lambda_1)A(t)y_j(t, \lambda_1)|dt < \infty
\]

whenever \( z_i \) a column of \( Z \) and \( y_j \) is a column of \( Y \). By the Cauchy-Schwartz inequality we have a.e. on \( (a, b) \) (writing \( z \) for \( z_i(t, \lambda_1) \) and \( y \) for \( y_j(t, \lambda_1) \)) that

\[
|z^*Ay| \leq (z^*Az)^{1/2}(yAy)^{1/2}.
\]

From

\[
0 \leq ((z^*Az)^{1/2} - (y^*Ay)^{1/2})^2
\]

we have that
From (15), (16) and the hypothesis that every solution of (8) or (9) satisfies (10) when $\lambda = \lambda_1$, we see that 14 holds.

Next we establish that

$$\begin{align*}
(Z^*(\lambda_1)JY(\lambda_1))^{-1}
\end{align*}$$

is bounded on $(a,b)$. Let $\alpha \in (a,b)$ then by Theorem 4 of [13] it follows that

$$\begin{align*}
Z^*(t, \lambda_1)JY(t, \lambda_1)
&= Z^*(\alpha, \lambda_1)JY(\alpha, \lambda_1) + (\lambda_1 - \lambda_1) \int_{\alpha}^{t} Z^*(s, \lambda_1)A(s)Y(s, \lambda_1)ds
\end{align*}$$

for all $t \in (a,b)$. Thus from (13) we see that

$$(18) \quad Z^*(t, \lambda_1)JY(t, \lambda_1)$$

has a limit as $t \to a$ and as $t \to b$. In order to show that (17) (which is continuous) is bounded it is then sufficient to show that the limits of (18) at $a$ and at $b$ are nonsingular. From Abel's formula for (8) and (9) (recall that $J^* = -J$, $A^* = A$, and $\text{tr} PQ = \text{tr} QP$ for matrices $P$ and $Q$) we have that

$$\begin{align*}
\det(Z^*(t, \lambda_1)JY(t, \lambda_1))
&= \det(Z^*(\alpha, \lambda_1)JY(\alpha, \lambda_1)) \\
&\cdot \exp \int_{\alpha}^{t} \text{tr}(J^{-1}\lambda_1A + J^{-1}B^*) + J^{-1}\lambda_1A + J^{-1}B) \\
&= \det((Z^*(\alpha, \lambda_1)JY(\alpha, \lambda_1)) \exp \int_{\alpha}^{t} (\lambda_1 - \lambda_1) \text{tr} J^{-1}A.
\end{align*}$$

Since by hypothesis $\int_{a}^{b} |\text{tr} J^{-1}A| < \infty$ the limits of (18) must be nonsingular.

It now follows that (12) is equivalent to an equation of the form

$$(19) \quad U' = MU \quad \text{a.e. on} \quad (a,b)$$

where $\int_{a}^{b} \|M(t)\| dt < \infty$. It is well known (see, e.g. Theorem 5.4.2 of [9]) that all solutions of (19) are bounded.
Returning to (11) we see that every solution of (8) when \( \lambda = \lambda_2 \) is a bounded multiple of a solution of (8) when \( \lambda = \lambda_1 \).

The argument to show that every solution of (9) satisfies (10) when \( \lambda = \lambda_2 \) is similar.

Theorem 2 is a generalization of a result of Atkinson (Theorem 9.11.2 of [1]) for the case where \( B^* = B \).

Theorem 1 is also valid for the quasidifferential expressions considered in [13] where no smoothness conditions on the coefficients of \( I \) are required.

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