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**A NOTE ON DIFFERENTIAL EQUATIONS WITH ALL
SOLUTIONS OF INTEGRABLE-SQUARE**

PHILIP WILLIAM WALKER

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It is shown that if all solutions to $l(y) = \lambda wy$ and $l^+(y) = \lambda wy$ satisfy $\int_a^b |y|^2 w < \infty$ for some complex number λ then so do all solutions for every complex number λ . The result is derived from a corresponding one for first order vector-matrix systems.

We shall be concerned with solutions to

- (1) $l(y) = 0$ on (a, b) ,
- (2) $l^+(y) = 0$ on (a, b)
- (3) $l(y) = \lambda wy$ on (a, b) , and
- (4) $l^+(y) = \lambda wy$ on (a, b)

which satisfy

$$(5) \quad \int_a^b |y|^2 w < \infty.$$

In these expressions (a, b) is an interval of the real line ($a = -\infty$ and/or $b = \infty$ is allowed), w is a weight, i.e., a positive valued continuous function on (a, b) , λ is a complex number, l is an m th order linear differential operator given by

$$(6) \quad l(y) = \sum_{k=0}^m a_k y^{(m-k)}$$

where each a_k is an $m - k$ times continuously differentiable complex valued function defined on (a, b) , $a_0(t) \neq 0$ for all $t \in (a, b)$, and l^+ is the formal adjoint of l so that

$$(7) \quad l^+(y) = \sum_{k=1}^m (-1)^{m-k} (\bar{a}_k y)^{(m-k)}.$$

In an earlier paper, [11], we defined w to be a compactifying weight

for l provided that every function which is a solution either of (1) or of (2) satisfies (5). It follows from Theorem 2-1 of [11] that if w is a compactifying weight for l then every function which is a solution either of (3) or of (4) satisfies (5) for every complex number λ .

The deficiency index problem (see for example [2] and [8]) for formally self-adjoint equations (where $l = l^+$) is concerned with finding the dimension of the linear manifold of solutions to (3) which satisfy (5). One of the results of this theory ([3], [4], [5], [6], [7], [10], and [12]) is that if this dimension is m (the order of l) for some complex number λ and $m > 1$ then it is m for every complex number λ .

While much of the theory for the self-adjoint case breaks down when $l \neq l^+$ we wish to show that this result carries over.

THEOREM 1. *Let each of λ_1 and λ_2 be a complex number (λ_j real, even $\lambda_j = 0$ is allowed). Let $m > 1$. If every function which is a solution of either (3) or (4) satisfies (5) when $\lambda = \lambda_1$, then every function which is a solution of either (3) or (4) satisfies (5) when $\lambda = \lambda_2$.*

This follows as a corollary to an analogous theorem (Theorem 2 below) for first order vector-matrix equations.

We consider the equations,

$$(8) \quad Jy' = [\lambda A + B]y \quad \text{a.e. on } (a, b), \quad \text{and}$$

$$(9) \quad Jy' = [\lambda A + B^*]y \quad \text{a.e. on } (a, b)$$

where J is a skew-symmetric ($J^* = -J$, * denoting conjugate transpose) $m \times m$ matrix, each of A and B is a complex $m \times m$ matrix valued function which is Lebesgue integrable over each compact subinterval of (a, b) , λ is a complex number, and $A(t)$ is nonnegative definite a.e. on (a, b) .

It was shown in [13] that, given l ; J , A , and B may be chosen so that every solution of (3) satisfies (5) if and only if every solution of (8) satisfies

$$(10) \quad \int_a^b y^* A y < \infty,$$

and every solution of (4) satisfies (5) if and only if every solution of (9) satisfies (10). For the choice of J and A used in [13] it is also the case that $\text{trace } J^{-1}A \equiv 0$ when $m > 1$.

Thus Theorem 1 above follows from Theorem 2 below.

THEOREM 2. *Let each of J , A , and B satisfy the conditions imposed above. Let $m > 1$. Let each of λ_1 and λ_2 be a complex number (λ_j real, even $\lambda_j = 0$ is allowed). Let $\int_a^b |\text{tr } J^{-1}A| < \infty$.*

If every vector function which is a solution of either (8) or (9) satisfies (10) when $\lambda = \lambda_1$ then every vector function which is a solution of either (8) or (9) satisfies (10) when $\lambda = \lambda_2$.

Proof. Let $Y(\lambda)$ and $Z(\lambda)$ be fundamental matrices for (8) and (9) respectively. (We will write $Y(t, \lambda)$ and $Z(t, \lambda)$ to denote the value of these functions at $t \in (a, b)$.) Let U be defined by

$$(11) \quad Y(\lambda_2) = Y(\lambda_1)U \quad \text{on } (a, b).$$

Multiplying on the left by J , differentiating, and using (8) we have,

$$\begin{aligned} (\lambda_2 A + B)Y(\lambda_2) &= (\lambda_1 A + B)Y(\lambda_1)U \\ &+ JY(\lambda_1)U' \quad \text{a.e. on } (a, b). \end{aligned}$$

From (11) we have,

$$JY(\lambda_1)U' = (\lambda_2 - \lambda_1)AY(\lambda_1)U \quad \text{a.e. on } (a, b).$$

Multiplying on the left by $Z^*(\lambda_1)$ we have

$$(12) \quad Z^*(\lambda_1)JY(\lambda_1)U' = (\lambda_2 - \lambda_1)Z^*(\lambda_1)AY(\lambda_1)U \quad \text{a.e. on } (a, b).$$

We first note that

$$(13) \quad \int_a^b \|Z^*(t, \lambda_1)Y(t, \lambda_1)\| dt < \infty$$

where $\|\cdot\|$ is any matrix norm. In order that (13) hold it is sufficient that

$$(14) \quad \int_a^b |z_i^*(t, \lambda_1)A(t)y_j(t, \lambda_1)| dt < \infty$$

whenever z_i a column of Z and y_j is a column of Y . By the Cauchy-Schwartz inequality we have a.e. on (a, b) (writing z for $z_i(t, \lambda_1)$ and y for $y_j(t, \lambda_1)$) that

$$(15) \quad |z^*Ay| \leq (z^*Az)^{1/2}(yAy)^{1/2}.$$

From

$$0 \leq ((z^*Az)^{1/2} - (y^*Ay)^{1/2})^2$$

we have that

$$(16) \quad (z^*Az)^{1/2} \cdot (y^*Ay)^{1/2} \leq \frac{1}{2}(z^*Az + y^*Ay).$$

From (15), (16) and the hypothesis that every solution of (8) or (9) satisfies (10) when $\lambda = \lambda_1$ we see that (14) holds.

Next we establish that

$$(17) \quad (Z^*(\lambda_1)JY(\lambda_1))^{-1}$$

is bounded on (a, b) . Let $\alpha \in (a, b)$ then by Theorem 4 of [13] it follows that

$$\begin{aligned} & Z^*(t, \lambda_1)JY(t, \lambda_1) \\ &= Z^*(\alpha, \lambda_1)JY(\alpha, \lambda_1) + (\lambda_1 - \bar{\lambda}_1) \int_{\alpha}^t Z^*(s, \lambda_1)A(s)Y(s, \lambda_1)ds \end{aligned}$$

for all $t \in (a, b)$. Thus from (13) we see that

$$(18) \quad Z^*(t, \lambda_1)JY(t, \lambda_1)$$

has a limit as $t \rightarrow a$ and as $t \rightarrow b$. In order to show that (17) (which is continuous) is bounded it is then sufficient to show that the limits of (18) at a and at b are nonsingular. From Abel's formula for (8) and (9) (recall that $J^* = -J$, $A^* = A$, and $\text{tr } PQ = \text{tr } QP$ for matrices P and Q) we have that

$$\begin{aligned} & \det(Z^*(t, \lambda_1)JY(t, \lambda_1)) \\ &= \det(Z^*(\alpha, \lambda_1)JY(\alpha, \lambda_1)) \\ & \cdot \exp \int_{\alpha}^t \text{tr}((J^{-1}\lambda_1 A + J^{-1}B^*)^* + J^{-1}\lambda_1 A + J^{-1}B) \\ &= \det((Z^*(\alpha, \lambda_1)JY(\alpha, \lambda_1)) \exp \int_{\alpha}^t (\lambda_1 - \bar{\lambda}_1) \text{tr } J^{-1}A). \end{aligned}$$

Since by hypothesis $\int_a^b |\text{tr } J^{-1}A| < \infty$ the limits of (18) must be nonsingular.

It now follows that (12) is equivalent to an equation of the form

$$(19) \quad U' = MU \quad \text{a.e. on } (a, b)$$

where $\int_a^b \|M(t)\| dt < \infty$. It is well known (see, e.g. Theorem 5.4.2 of [9]) that all solutions of (19) are bounded.

Returning to (11) we see that every solution of (8) when $\lambda = \lambda_2$ is a bounded multiple of a solution of (8) when $\lambda = \lambda_1$.

The argument to show that every solution of (9) satisfies (10) when $\lambda = \lambda_2$ is similar.

Theorem 2 is a generalization of a result of Atkinson (Theorem 9.11.2 of [1]) for the case where $B^* = B$.

Theorem 1 is also valid for the quasidifferential expressions considered in [13] where no smoothness conditions on the coefficients of l are required.

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