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It is shown that if all solutions to $l(y) = \lambda wy$ and $l^+(y) = \lambda wy$ satisfy $\int_a^b |y|^2 w < \infty$ for some complex number λ then so do all solutions for every complex number λ. The result is derived from a corresponding one for first order vector-matrix systems.

We shall be concerned with solutions to

(1) $$l(y) = 0 \quad \text{on} \quad (a, b),$$

(2) $$l^+(y) = 0 \quad \text{on} \quad (a, b)$$

(3) $$l(y) = \lambda wy \quad \text{on} \quad (a, b), \quad \text{and}$$

(4) $$l^+(y) = \lambda wy \quad \text{on} \quad (a, b)$$

which satisfy

(5) $$\int_a^b |y|^2 w < \infty.$$

In these expressions (a, b) is an interval of the real line ($a = -\infty$ and/or $b = \infty$ is allowed), w is a weight, i.e., a positive valued continuous function on (a, b), λ is a complex number, l is an mth order linear differential operator given by

(6) $$l(y) = \sum_{k=0}^{m} a_k y^{(m-k)}$$

where each a_k is an $m - k$ times continuously differentiable complex valued function defined on (a, b), $a_0(t) \neq 0$ for all $t \in (a, b)$, and l^+ is the formal adjoint of l so that

(7) $$l^+(y) = \sum_{k=1}^{m} (-1)^{m-k} (\bar{a}_k y)^{(m-k)}.$$

In an earlier paper, **[11]**, we defined w to be a compactifying weight for l provided that every function which is a solution either of (1) or of (2) satisfies (5). If follows from Theorem 2–1 of [11] that if w is a compactifying weight for l then every function which is a solution either of (3) or of (4) satisfies (5) for every complex number λ.

The deficiency index problem (see for example [2] and [8]) for formally self-adjoint equations (where $l = l^{+}$) is concerned with finding the dimension of the linear manifold of solutions to (3) which satisfy (5). One of the results of this theory ([3], [4], [5], [6], [7], [10], and [12]) is that if this dimension is m (the order of l) for some complex number λ and $m > 1$ then it is m for every complex number λ.

While much of the theory for the self-adjoint case breaks down when $l \neq l^{+}$ we wish to show that this result carries over.

THEOREM 1. *Let each of λ_1 and λ_2 be a complex number (λ_j real, even $\lambda_j = 0$ is allowed). Let $m > 1$. If every function which is a solution of either* (3) *or* (4) *satisfies* (5) *when $\lambda = \lambda_1$ then every function which is a solution of either* (3) *or* (4) *satisfies* (5) *when $\lambda = \lambda_2$.*

This follows as a corollary to an analogous theorem (Theorem 2 below) for first order vector-matrix equations.

We consider the equations,

(8) $$Jy' = [\lambda A + B]y \quad \text{a.e. on} \quad (a, b), \quad \text{and}$$

(9) $$Jy' = [\lambda A + B^{*}]y \quad \text{a.e. on} \quad (a, b)$$

where J is a skew-symmetric ($J^{*} = -J$, $*$ denoting conjugate transpose) $m \times m$ matrix, each of A and B is a complex $m \times m$ matrix valued function which is Lebesque integrable over each compact subinterval of (a, b), λ is a complex number, and $A(t)$ is nonnegative definate a.e. on (a, b).

It was shown in [13] that, given l; J, A, and B may be chosen so that every solution of (3) satisfies (5) if and only if every solution of (8) satisfies

(10) $$\int_a^b y^{*}Ay < \infty,$$

and every solution of (4) satisfies (5) if and only if every solution of (9) satisfies (10). For the choice of J and A used in [13] it is also the case that trace $J^{-1}A \equiv 0$ when $m > 1$.

Thus Theorem 1 above follows from Theorem 2 below.

THEOREM 2. *Let each of J, A, and B satisfy the conditions imposed above. Let $m > 1$. Let each of λ_1 and λ_2 be a complex number (λ_j real, even $\lambda_j = 0$ is allowed). Let* $\int_a^b |\mathrm{tr}\, J^{-1}A| < \infty$.

If every vector function which is a solution of either (8) *or* (9) *satisfies* (10) *when* $\lambda = \lambda_1$ *then every vector function which is a solution of either* (8) *or* (9) *satisfies* (10) *when* $\lambda = \lambda_2$.

Proof. Let $Y(\lambda)$ and $Z(\lambda)$ be fundamental matrices for (8) and (9) respectively. (We will write $Y(t, \lambda)$ and $Z(t, \lambda)$ to denote the value of these functions at $t \in (a, b)$.) Let U be defined by

$$Y(\lambda_2) = Y(\lambda_1)U \quad \text{on} \quad (a, b). \tag{11}$$

Multiplying on the left by J, differentiating, and using (8) we have,

$$(\lambda_2 A + B)Y(\lambda_2) = (\lambda_1 A + B)Y(\lambda_1)U + JY(\lambda_1)U' \quad \text{a.e.} \quad \text{on} \quad (a, b).$$

From (11) we have,

$$JY(\lambda_1)U' = (\lambda_2 - \lambda_1)AY(\lambda_1)U \quad \text{a.e.} \quad \text{on} \quad (a, b).$$

Multiplying on the left by $Z^*(\lambda_1)$ we have

$$Z^*(\lambda_1)JY(\lambda_1)U' = (\lambda_2 - \lambda_1)Z^*(\lambda_1)AY(\lambda_1)U \quad \text{a.e.} \quad \text{on} \quad (a, b). \tag{12}$$

We first note that

$$\int_a^b \|Z^*(t, \lambda_1)Y(t, \lambda_1)\| dt < \infty \tag{13}$$

where $\|\cdot\|$ is any matrix norm. In order that (13) hold it is sufficient that

$$\int_a^b |z_i^*(t, \lambda_1)A(t)y_j(t, \lambda_1)| dt < \infty \tag{14}$$

whenever z_i a column of Z and y_j is a column of Y. By the Cauchy-Schwartz inequality we have a.e. on (a, b) (writing z for $z_i(t, \lambda_1)$ and y for $y_j(t, \lambda_1)$) that

$$|z^*Ay| \leqq (z^*Az)^{1/2}(yAy)^{1/2}. \tag{15}$$

From

$$0 \leqq ((z^*Az)^{1/2} - (y^*Ay)^{1/2})^2$$

we have that

(16) $$(z^*Az)^{1/2}\cdot(y^*Ay)^{1/2}\leqq\tfrac{1}{2}(z^*Az+y^*Ay).$$

From (15), (16) and the hypothesis that every solution of (8) or (9) satisfies (10) when $\lambda=\lambda_1$ we see that 14 holds.

Next we establish that

(17) $$(Z^*(\lambda_1)JY(\lambda_1))^{-1}$$

is bounded on (a,b). Let $\alpha\in(a,b)$ then by Theorem 4 of **[13]** it follows that

$$Z^*(t,\lambda_1)JY(t,\lambda_1)$$
$$=Z^*(\alpha,\lambda_1)JY(\alpha,\lambda_1)+(\lambda_1-\bar{\lambda}_1)\int_\alpha^t Z^*(s,\lambda_1)A(s)Y(s,\lambda_1)ds$$

for all $t\in(a,b)$. Thus from (13) we see that

(18) $$Z^*(t,\lambda_1)JY(t,\lambda_1)$$

has a limit as $t\to a$ and as $t\to b$. In order to show that (17) (which is continuous) is bounded it is then sufficient to show that the limits of (18) at a and at b are nonsingular. From Abel's formula for (8) and (9) (recall that $J^*=-J$, $A^*=A$, and $\operatorname{tr}PQ=\operatorname{tr}QP$ for matrices P and Q) we have that

$$\det(Z^*(t,\lambda_1)JY(t,\lambda_1))$$
$$=\det(Z^*(\alpha,\lambda_1)JY(\alpha,\lambda_1))$$
$$\cdot\exp\int_\alpha^t\operatorname{tr}((J^{-1}\lambda_1A+J^{-1}B^*)^*+J^{-1}\lambda_1A+J^{-1}B)$$
$$=\det((Z^*(\alpha,\lambda_1)jY(\alpha,\lambda_1))\exp\int_\alpha^t(\lambda_1-\bar{\lambda}_1)\operatorname{tr}J^{-1}A.$$

Since by hypothesis $\int_a^b|\operatorname{tr}J^{-1}A|<\infty$ the limits of (18) must be nonsingular.

It now follows that (12) is equivalent to an equation of the form

(19) $$U'=MU\quad\text{a.e. on}\quad(a,b)$$

where $\int_a^b\|M(t)\|\,dt<\infty$. It is well known (see, e.g. Theorem 5.4.2 of **[9]**) that all solutions of (19) are bounded.

Returning to (11) we see that every solution of (8) when $\lambda = \lambda_2$ is a bounded multiple of a solution of (8) when $\lambda = \lambda_1$.

The argument to show that every solution of (9) satisfies (10) when $\lambda = \lambda_2$ is similar.

Theorem 2 is a generalization of a result of Atkinson (Theorem 9.11.2 of **[1]**) for the case where $B^* = B$.

Theorem 1 is also valid for the quasidifferential expressions considered in **[13]** where no smoothness conditions on the coefficients of l are required.

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