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COEFFICIENT BOUNDS FOR SOME CLASSES OF STARLIKE FUNCTIONS

ROGER W. BARNARD AND JOHN LAWSON LEWIS

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Let t be given, $1/4 \leq t \leq \infty$, and let $S(t)$ denote the class of normalized starlike univalent functions f in $|z| < 1$ satisfying (i) $|f(z)/z| \geq t$, $|z| < 1$, if $1/4 \leq t \leq 1$, (ii) $|f(z)/z| \leq t$, $|z| < 1$, if $1 < t \leq \infty$. If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in S(t)$ and n is a fixed positive integer, then the authors obtain sharp coefficient bounds for $|a_n|$ when t is sufficiently large or sufficiently near $1/4$. In particular a sharp bound is found for $|a_3|$ when $1/4 \leq t \leq 1$ and $5 \leq t \leq \infty$. Also a sharp bound for $|a_4|$ is found when $1/4 \leq t \leq 1$ or $12.259 \leq t \leq \infty$.

1. Introduction. Let S denote the class of starlike univalent functions f in $K = \{z : |z| < 1\}$ with the normalization, $f(0) = 0$, $f'(0) = 1$. Given t , $1/4 \leq t \leq \infty$, let $S(t)$ denote the subclass of functions $f \in S$ satisfying

$$(1.1) \quad |f(z)/z| \geq t, z \in K, \text{ if } 1/4 \leq t \leq 1,$$

$$(1.2) \quad |f(z)/z| \leq t, z \in K, \text{ if } 1 < t \leq \infty.$$

If $1/4 < t \leq 1$, we let $F = F(\cdot, t)$ be defined by

$$(1.3) \quad zF'(z)/F(z) = [1 + 2(2b^2 - 1)z + z^2]^{1/2}/(1 - z), z \in K,$$

where $0 \leq b < 1$ and $t = [(1 + b)^{1+b} (1 - b)^{1-b}]^{-1}$. The function $F = F(\cdot, t)$ defined by (1.3) is in $S(t)$ for $1/4 < t \leq 1$, as can be shown by a long but straightforward calculation (see Suffridge [9]). For fixed t , $1/4 < t \leq 1$, this function maps K onto the complex plane minus a set

$$\{w : |w| \geq t, \quad \pi b \leq \arg w \leq 2\pi - \pi b\}.$$

If $1 < t < \infty$, we let $F = F(\cdot, t) \in S(t)$ be defined by

$$(1.4) \quad \frac{F(z)}{[1 - t^{-1}F(z)]^2} = \frac{z}{(1 - z)^2}, z \in K.$$

It is well known (see Nehari [4, p. 224, ex. 4]) that the function F maps K onto a domain whose boundary consists of $\{w : |w| = t\}$, and a slit along the negative real axis from $-t$ to $-\lambda$ where $4\lambda t^2 = (t + \lambda)^2$. If $t = 1/4$ or $t = \infty$, we let

$$F(z, 1/4) = F(z, \infty) = z/(1-z)^2, z \in K.$$

In [2] the authors proved a subordination theorem for some classes of univalent functions. For $S(t)$ this theorem may be stated as follows:

THEOREM A. *Let t be given, $1/4 \leq t \leq \infty$. Let $F = F(\cdot, t)$ be as in (1.3) and (1.4). If $f \in S(t)$, then $\log f(z)/z, z \in K$, is subordinate to $\log F(z)/z, z \in K$.*

Theorem A implies for a given $t, 1/4 \leq t \leq \infty$, that $F = F(\cdot, t)$ solves a number of extremal problems in $S(t)$. Some of these problems were pointed out in [2]. There, however, only general properties of subordination were used. In this note, for certain values of t , we use our specific knowledge of F , together with Theorem A, to obtain coefficient bounds for functions $f \in S(t)$. More specifically, we prove

THEOREM 1. *Let t be given, $1/4 \leq t \leq \infty$. Let $F(z) = F(z, t) = z + \sum_{k=2}^{\infty} A_k(t)z^k, z \in K$, be as in (1.3) and (1.4). Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k, z \in K$, be in $S(t)$. If n a positive integer is given ($n > 2$), then there exist α_n, β_n satisfying $1/4 < \alpha_n \leq 1, 1 \leq \beta_n < \infty$, with the property that*

$$(1.5) \quad |a_n| \leq A_n(t),$$

whenever $1/4 \leq t < \alpha_n$ or $\beta_n < t \leq \infty$. α_n and β_n may be chosen in such a way that equality holds in (1.5) only if $f(z) = \eta^{-1}F(\eta z), z \in K$, for some $\eta, |\eta| = 1$. In particular

$$(1.6) \quad |a_3| \leq A_3(t) \text{ if } 1/4 \leq t \leq 1 \text{ or } 5 < t \leq \infty,$$

$$(1.7) \quad |a_4| \leq A_4(t) \text{ if } 1/4 \leq t \leq 1 \text{ or } 12.259 \leq t \leq \infty.$$

Equality holds in (1.6) and (1.7) only if $f(z) = \eta^{-1}F(\eta z), z \in K$, for some $\eta, |\eta| = 1$.

Let f and t be as in Theorem 1. We note that the inequality $|a_2| \leq A_2(t), 1/4 \leq t \leq \infty$, is an easy consequence of Theorem A (see [2]). We also note for $1 \leq t \leq e$ that $|a_3| \leq 1 - t^{-2}$, where equality holds for the function $f \in S(t)$ defined by $f(z) = F(z^2, t^2)^{1/2}, z \in K$. This inequality is due to Tammi [10]. The problem of finding a sharp upper bound for $|a_3|$ when $f \in S(t), e < t < 5$, is still open. However, Barnard [1] has shown that the function which maximizes $|a_3|$ in $S(t)$ is either F or a function which maps K onto a domain whose boundary consists $\{w : |w| = t\}$ and two radial slits of equal length.

We remark that several authors have considered similar problems in the class $U(t)$ of normalized univalent functions f (i.e., $f(0) = 0$,

$f'(0) = 1$) bounded above by t , $1 < t < \infty$. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $z \in K$, is in $U(t)$, then Schiffer and Tammi [6] showed that $|a_4| \leq A_4(t)$, for $t \geq 33\frac{1}{3}$. If in addition f has real coefficients, then Singh [8] proved that $|a_4| \leq A_4(t)$ for $t \geq 11$. Moreover, Schiffer and Tammi [7] have proved for each positive integer $n \geq 2$, that there exists δ_n , $1 < \delta_n < \infty$, with the following property: If $f \in U(t)$ and $1 < t \leq \delta_n$, then

$$|a_n| \leq \frac{2}{n-1} (1 - t^{1-n}).$$

Here equality holds for $f(z) = F(z^{n-1}, t^{n-1})^{1/(n-1)}$, $z \in K$, which in fact is in $S(t)$. Hence the above inequality is also sharp for functions in $S(t)$ when $1 < t \leq \delta_n$. Finally we remark that Schiffer and Tammi [6] have shown that it suffices to take $\delta_4 \leq 34/19$.

2. Proof of Theorem 1. Let G, ω , be analytic in K and suppose that

$$(2.1) \quad \omega(0) = 0,$$

$$(2.2) \quad |\omega(z)| \leq 1, z \in K.$$

Put $g(z) = G[\omega(z)]$, $z \in K$. Suppose that $G(z) = \sum_{k=1}^{\infty} c_k z^k$, and $g(z) = \sum_{k=1}^{\infty} b_k z^k$. Then Rogosinski [5, Thm. VI] proved

THEOREM B. *Let n be a fixed positive integer. If $c_n > 0$ and if there exists an analytic function P in K with positive real part satisfying*

$$P(z) = \frac{c_n}{2} + c_{n-1}z + c_{n-2}z^2 + \dots + c_1z^{n-1} + \sum_{k=n}^{\infty} d_k z^k$$

for $z \in K$, then $|b_n| \leq |c_n|$. Equality can occur only if $g(z) = G(\eta z)$ for some η , $|\eta| = 1$, or if $n > 1$ and P has the form,

$$(2.3) \quad P(z) = \sum_{i=1}^J \lambda_i \left(\frac{1 + \varepsilon_i z}{1 - \varepsilon_i z} \right), z \in K,$$

where $\lambda_i > 0$, $|\varepsilon_i| = 1$, $1 \leq i \leq J$, and $J \leq n - 1$.

Furthermore, Carathéodory (see Tsuji [11, Ch. 4 §7]) proved

THEOREM C. *The function P in Theorem B exists if and only if the n by n matrix*

$$\begin{pmatrix} c_n & c_{n-1} & c_{n-2} & \cdots & c_1 \\ c_{n-1} & c_n & c_{n-1} & \cdots & c_2 \\ c_{n-2} & c_{n-1} & c_n & \cdots & c_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_1 & c_2 & c_3 & \cdots & c_n \end{pmatrix}$$

is positive semi definite. If P exists, then P has the form (2.3) only if the above matrix has determinant zero.

We now use Theorems A, B, and C to prove Theorem 1. Let t be fixed, $1/4 \leq t \leq \infty$, and $f \in S(t)$. Then Theorem A implies there exists a function ω satisfying (2.1) and (2.2) for which $f(z)/z = F[\omega(z)]/\omega(z)$, $z \in K$. Hence we may use Theorems B and C with $g(z) = f(z)/z - 1$, $G(z) = (F(z)/z) - 1$, $z \in K$, and $c_i = A_{i+1}(t)$, $1 \leq i \leq n - 1$, to prove Theorem 1. To do so we shall want some notation.

Let n and k be fixed positive integers satisfying $2 \leq k \leq n$. Let $\delta(k, n, t)$ be the $k - 1$ by $k - 1$ matrix

$$(2.4) \quad \delta(k, n, t) = \begin{pmatrix} A_n(t) & A_{n-1}(t) & \cdots & A_{n-k+2}(t) \\ A_{n-1}(t) & A_n(t) & \cdots & A_{n-k+3}(t) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ A_{n-k+2}(t) & A_{n-k+3}(t) & \cdots & A_n(t) \end{pmatrix}$$

Let $|\delta(k, n, t)|$ denote the determinant of $\delta(k, n, t)$. Then it is well known (see Hohn [3, Thm. 9, 17.3]) that $\delta(n, n, t)$ is positive definite if and only if $|\delta(k, n, t)| > 0$ for $2 \leq k \leq n$.

We note that $A_n(\infty) = A_n(1/4) = n$ for $n \geq 2$. Using this fact we obtain that $|\delta(k, n, \infty)| = |\delta(k, n, 1/4)| = (2n + 2 - k) 2^{k-3}$ for $2 \leq k \leq n$ and $n > 2$. Since (1.3) and (1.4) imply A_n is continuous as a function of t , $1/4 \leq t \leq \infty$, it follows that

$$\lim_{t \rightarrow \infty} |\delta(k, n, t)| = |\delta(k, n, \infty)| = \lim_{t \rightarrow 1/4} |\delta(k, n, t)| > 0$$

for each positive integer $n > 2$ and $2 \leq k \leq n$. From this inequality and our previous remark we see that $\delta(n, n, t)$ is positive definite for

sufficiently large t and t near $1/4$, say $1/4 \leq t < \alpha_n, \beta_n < t \leq \infty$. Using Theorems A, B, and C, it follows that (1.5) is true.

To prove (1.6) and (1.7) we make some explicit calculations. The case $t = 1$ is trivial since then $S(t)$ consists only of the identity function. First from (1.4) we find for $x = t^{-1}$, and $1 < t \leq \infty$, that

$$(2.5) \quad \begin{aligned} A_2(t) &= 2(1 - x), \\ A_3(t) &= (3 - 5x)(1 - x), \\ A_4(t) &= (4 + 14x^2 - 16x)(1 - x). \end{aligned}$$

Second if $1/4 \leq t < 1$ and $a = 2b^2 - 1$ [b as in (1.3)], then from (1.3) we get

$$(2.6) \quad \begin{aligned} A_2(t) &= 1 + a, \\ A_3(t) &= (1 + a)(5 + a)/4, \\ A_4(t) &= (1 + a)(17 + 6a + a^2)/12. \end{aligned}$$

Here $-1 < a \leq 1$.

To prove (1.6) it suffices, by the previous argument, to show that $A_2(t) > 0$ and

$$|\delta(3, 3, t)| = A_3(t)^2 - A_2(t)^2 > 0$$

for $5 < t < \infty$ or $1/4 \leq t < 1$. From (2.5) and (2.6) we see that these inequalities are valid for the above values of t . To prove (1.7), we need to show that $\delta(3, 4, t) > 0, \delta(4, 4, t) > 0$, for the stipulated values of t in Theorem 1. To do this we consider two cases. If $1 < t \leq \infty$, and $x = 1/t$, then from (2.4) and (2.5) we have

$$|\delta(4, 4, t)| = (1 - x)^3 \begin{vmatrix} 4 + 14x^2 - 16x & 3 - 5x & 2 \\ 3 - 5x & 4 + 14x^2 - 16x & 3 - 5x \\ 2 & 3 - 5x & 4 + 14x^2 - 16x \end{vmatrix}$$

Adding the second row to the first and third rows we get

$$|\delta(4, 4, t)| = (1 - x)^5 \begin{vmatrix} 7(1 - 2x) & 1(1 \times 2x) & 5 \\ 3 - 5x & 4 + 14x^2 - 16x & 3 - 5x \\ 5 & 7(1 - 2x) & 7(1 - 2x) \end{vmatrix}$$

Evaluating this determinant we obtain

$$|\delta(4,4,t)| = 4(1-x)^5(1-7x)[3-47x+126x^2-98x^3] > 0$$

for $12.259 \leq t \leq \infty$. It is easily checked that $|\delta(3,4,t)| = A_4^2(t) - A_3^2(t) > 0$ for $12.259 \leq t \leq \infty$. Hence (1.7) is true for $12.259 \leq t \leq \infty$.

If $1/4 \leq t < 1$, then from (2.4), (2.6), we obtain

$$(12)^3 |\delta(4,4,t)| = (1+a)^3 \begin{vmatrix} 17+6a+a^2 & 3(5+a) & 12 \\ 3(5+a) & 17+6a+a^2 & 3(5+a) \\ 12 & 3(5+a) & 17+6a+a^2 \end{vmatrix}$$

Subtracting the second row from the first and third rows, we get

$$(12)^3 |\delta(4,4,t)| = (1+a)^5 \begin{vmatrix} a+2 & -a-2 & -3 \\ 3(5+a) & 17+6a+a^2 & 3(5+a) \\ -3 & -a-2 & a+2 \end{vmatrix}$$

Adding six times the first and third rows to the second of this determinant, we find that

$$(12)^3 |\delta(4,4,t)| = (1+a)^6 \begin{vmatrix} a+2 & -a-2 & -3 \\ 9 & a-7 & 9 \\ -3 & -a-2 & a+2 \end{vmatrix}$$

Evaluating this determinant we obtain

$$(12)^3 |\delta(4,4,t)| = (1+a)^6 (a^3 + 15a^2 + 93a + 215) > 0$$

for $-1 < a \leq 1$. Hence $|\delta(4,4,t)| > 0$ for $1/4 \leq t < 1$. It is easily checked that $|\delta(3,4,t)| > 0$ for $1/4 \leq t < 1$. We conclude that (1.7) is true for $1/4 \leq t < 1$. The proof of Theorem 1 is now complete.

Finally we remark for $1/4 \leq t < 1$ that

$$48A_5(t) = (1+a)(74+38a+10a^2-2a^3) < 48A_4(t)$$

for t near 1, $t < 1$. It follows that $|\delta(3,5,t)| < 0$ for t near 1, $t < 1$. Hence our method does not imply for all t , $1/4 \leq t \leq 1$, that

$|a_5| \leq A_5(t)$. However, it is still possible our method implies that α_n in Theorem 1 can be chosen independent of n .

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