COEFFICIENT BOUNDS FOR SOME CLASSES OF STARLIKE FUNCTIONS

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Let $t$ be given, $1/4 \leq t \leq \infty$, and let $S(t)$ denote the class of normalized starlike univalent functions $f$ in $|z| < 1$ satisfying

(i) $|f(z)/z| \geq t$, $|z| < 1$, if $1/4 \leq t \leq 1$, 
(ii) $|f(z)/z| \leq t$, $|z| < 1$, if $1 < t \leq \infty$.

If $f(z) = z + \sum_{k=2}^{\infty} a_n z^k \in S(t)$ and $n$ is a fixed positive integer, then the authors obtain sharp coefficient bounds for $|a_n|$ when $t$ is sufficiently large or sufficiently near $1/4$. In particular a sharp bound is found for $|a_4|$ when $1/4 \leq t \leq 1$ and $5 \leq t \leq \infty$. Also a sharp bound for $|a_4|$ is found when $1/4 \leq t \leq 1$ or $12.259 \leq t \leq \infty$.

1. Introduction. Let $S$ denote the class of starlike univalent functions $f$ in $K = \{z : |z| < 1\}$ with the normalization, $f(0) = 0, f'(0) = 1$. Given $t$, $1/4 \leq t \leq \infty$, let $S(t)$ denote the subclass of functions $f \in S$ satisfying

\begin{align*}
(1.1) & \quad |f(z)/z| \geq t, \ z \in K, \text{ if } 1/4 \leq t \leq 1, \\
(1.2) & \quad |f(z)/z| \leq t, \ z \in K, \text{ if } 1 < t \leq \infty.
\end{align*}

If $1/4 < t \leq 1$, we let $F = F(\cdot, t)$ be defined by

\begin{equation}
\frac{zF'(z)}{F(z)} = \frac{[1 + 2(2b^2 - 1)z + z^2]^{1/2}}{(1 - z)}, \ z \in K,
\end{equation}

where $0 \leq b < 1$ and $t = [(1 + b)^{1-b} (1 - b)^{1-b}]^{-1}$. The function $F = F(\cdot, t)$ defined by (1.3) is in $S(t)$ for $1/4 < t \leq 1$, as can be shown by a long but straightforward calculation (see Suffridge [9]). For fixed $t$, $1/4 < t \leq 1$, this function maps $K$ onto the complex plane minus a set

$$\{w : |w| \geq t, \ \pi b \leq \arg w \leq 2\pi - \pi b\}.$$

If $1 < t < \infty$, we let $F = F(\cdot, t) \in S(t)$ be defined by

\begin{equation}
\frac{F(z)}{[1 - t^{-1}F(z)]^2} = \frac{z}{(1 - z)^2}, \ z \in K.
\end{equation}

It is well known (see Nehari [4, p. 224, ex. 4]) that the function $F$ maps $K$ onto a domain whose boundary consists of $\{w : |w| = t\}$, and a slit along the negative real axis from $-t$ to $-\lambda$ where $4\lambda^2 = (t + \lambda)^2$. If $t = 1/4$ or $t = \infty$, we let
\[ F(z, 1/4) = F(z) = z/(1 - z), z \in K. \]

In [2] the authors proved a subordination theorem for some classes of univalent functions. For \(S(t)\) this theorem may be stated as follows:

**Theorem A.** Let \( t \) be given, \( 1/4 \leq t \leq \infty \). Let \( F = F(\cdot, t) \) be as in (1.3) and (1.4). If \( f \in S(t) \), then \( \log f(z)/z, z \in K \), is subordinate to \( \log F(z)/z, z \in K \).

Theorem A implies for a given \( t, 1/4 \leq t \leq \infty \), that \( F = F(\cdot, t) \) solves a number of extremal problems in \(S(t)\). Some of these problems were pointed out in [2]. There, however, only general properties of subordination were used. In this note, for certain values of \( t \), we use our specific knowledge of \( F \), together with Theorem A, to obtain coefficient bounds for functions \( f \in S(t) \). More specifically, we prove

**Theorem 1.** Let \( t \) be given, \( 1/4 \leq t \leq \infty \). Let \( F(z) = F(z, t) = z + \sum_{k=2}^{\infty} A_k(t)z^k, z \in K \), be as in (1.3) and (1.4). Let \( f(z) = z + \sum_{k=2}^{\infty} a_kz^k, z \in K \), be in \(S(t)\). If \( n \) a positive integer is given (\( n > 2 \)), then there exist \( \alpha_n, \beta_n \) satisfying \( 1/4 < \alpha_n \leq 1, 1 \leq \beta_n < \infty \), with the property that

\[(1.5) \quad |a_n| \leq A_n(t), \]

whenever \( 1/4 \leq t < \alpha_n \) or \( \beta_n < t \leq \infty \). \( \alpha_n \) and \( \beta_n \) may be chosen in such a way that equality holds in (1.5) only if \( f(z) = \eta^{-1}F(\eta z), z \in K \), for some \( \eta, |\eta| = 1 \). In particular

\[(1.6) \quad |a_3| \leq A_3(t) \text{ if } 1/4 \leq t \leq 1 \text{ or } 5 < t \leq \infty, \]
\[(1.7) \quad |a_4| \leq A_4(t) \text{ if } 1/4 \leq t \leq 1 \text{ or } 12.259 \leq t \leq \infty. \]

Equality holds in (1.6) and (1.7) only if \( f(z) = \eta^{-1}F(\eta z), z \in K \), for some \( \eta, |\eta| = 1 \).

Let \( f \) and \( t \) be as in Theorem 1. We note that the inequality \( |a_2| \leq A_2(t), 1/4 \leq t \leq \infty \), is an easy consequence of Theorem A (see [2]). We also note for \( 1 \leq t \leq e \) that \( |a_3| \leq 1 - t^2 \), where equality holds for the function \( f \in S(t) \) defined by \( f(z) = F(z^2, t^2) \), \( z \in K \). This inequality is due to Tammi [10]. The problem of finding a sharp upper bound for \( |a_3| \) when \( f \in S(t), e < t < 5 \), is still open. However, Barnard [1] has shown that the function which maximizes \( |a_3| \) in \(S(t)\) is either \( F \) or a function which maps \( K \) onto a domain whose boundary consists \( \{w : |w| = t\} \) and two radial slits of equal length.

We remark that several authors have considered similar problems in the class \( U(t) \) of normalized univalent functions \( f \) (i.e., \( f(0) = 0, \)

\( f'(0) = 1 \) bounded above by \( t \), \( 1 < t < \infty \). If \( f(z) = z + \sum_{n=2}^\infty a_n z^n \), \( z \in K \), is in \( U(t) \), then Schiffer and Tammi [6] showed that \( |a_4| \leq A_4(t) \), for \( t \geq 33 \frac{1}{3} \). If in addition \( f \) has real coefficients, then Singh [8] proved that \( |a_4| \leq A_4(t) \) for \( t \geq 11 \). Moreover, Schiffer and Tammi [7] have proved for each positive integer \( n \geq 2 \), that there exists \( \delta_n, 1 < \delta_n < \infty \), with the following property: If \( f \in U(t) \) and \( 1 < t \leq \delta_n \), then

\[
|a_n| \leq \frac{2}{n-1} (1 - t^{1-n}).
\]

Here equality holds for \( f(z) = F(z^{n-1}, t^{n-1})^{1/(n-1)}, z \in K \), which in fact is in \( S(t) \). Hence the above inequality is also sharp for functions in \( S(t) \) when \( 1 < t \leq \delta_n \). Finally we remark that Schiffer and Tammi [6] have shown that if suffices to take \( \delta_4 \leq 34/19 \).

2. Proof of Theorem 1. Let \( G, \omega \), be analytic in \( K \) and suppose that

\[
\omega(0) = 0,
\]

\[
|\omega(z)| \leq 1, z \in K.
\]

Put \( g(z) = G[\omega(z)], z \in K \). Suppose that \( G(z) = \sum_{k=1}^\infty c_k z^k \), and \( g(z) = \sum_{k=1}^\infty b_k z^k \). Then Rogosinski [5, Thm. VI] proved

**Theorem B.** Let \( n \) be a fixed positive integer. If \( c_n > 0 \) and if there exists an analytic function \( P \) in \( K \) with positive real part satisfying

\[
P(z) = \frac{c_n}{2} + c_{n-1} z + c_{n-2} z^2 + \cdots + c_1 z^{n-1} + \sum_{k=n}^\infty d_k z^k
\]

for \( z \in K \), then \( |b_n| \leq |c_n| \). Equality can occur only if \( g(z) = G(\eta z) \) for some \( \eta, |\eta| = 1 \), or if \( n > 1 \) and \( P \) has the form,

\[
P(z) = \sum_{i=1}^J \lambda_i \left( \frac{1 + \varepsilon_i z}{1 - \varepsilon_i z} \right), \quad z \in K,
\]

where \( \lambda_i > 0, |\varepsilon_i| = 1, 1 \leq i \leq J \), and \( J \leq n - 1 \).

Furthermore, Carathéodory (see Tsuji [11, Ch. 4 §7]) proved

**Theorem C.** The function \( P \) in Theorem B exists if and only if the \( n \) by \( n \) matrix
is positive semi definite. If \( P \) exists, then \( P \) has the form (2.3) only if the above matrix has determinant zero.

We now use Theorems A, B, and C to prove Theorem 1. Let \( t \) be fixed, \( 1/4 \leq t \leq \infty \), and \( f \in S(t) \). Then Theorem A implies there exists a function \( \omega \) satisfying (2.1) and (2.2) for which

\[
\frac{f(z)}{z} = \frac{F[\omega(z)]}{\omega(z)}, \quad z \in K.
\]

Hence we may use Theorems B and C with \( g(z) = \frac{f(z)}{z} - 1 \), \( G(z) = (F(z)/z) - 1 \), \( z \in K \), and \( c_i = A_{i-1}(t) \), \( 1 \leq i \leq n - 1 \), to prove Theorem 1. To do so we shall want some notation.

Let \( n \) and \( k \) be fixed positive integers satisfying \( 2 \leq k \leq n \). Let \( \delta(k, n, t) \) be the \( k \times k - 1 \) by \( k - 1 \) matrix

\[
\delta(k, n, t) = \begin{pmatrix}
A_n(t) & A_{n-1}(t) & \cdots & A_{n-k+3}(t) \\
A_{n-1}(t) & A_n(t) & \cdots & A_{n-k+2}(t) \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots \\
& & & A_{n-k+2}(t) & A_{n-k+3}(t) & \cdots & A_n(t)
\end{pmatrix}
\]

Let \( |\delta(k, n, t)| \) denote the determinant of \( \delta(k, n, t) \). Then it is well known (see Hohn [3, Thm. 9.17.3]) that \( \delta(n, n, t) \) is positive definite if and only if \( |\delta(k, n, t)| > 0 \) for \( 2 \leq k \leq n \).

We note that \( A_n(\infty) = A_n(1/4) = n \) for \( n \geq 2 \). Using this fact we obtain that \( |\delta(k, n, \infty)| = |\delta(k, n, 1/4)| = (2n + 2 - k) 2^{k-3} \) for \( 2 \leq k \leq n \) and \( n > 2 \). Since (1.3) and (1.4) imply \( A_n \) is continuous as a function of \( t \), \( 1/4 \leq t \leq \infty \), it follows that

\[
\lim_{t \to \infty} |\delta(k, n, t)| = |\delta(k, n, \infty)| = \lim_{t \to 1/4} |\delta(k, n, t)| > 0
\]

for each positive integer \( n > 2 \) and \( 2 \leq k \leq n \). From this inequality and our previous remark we see that \( \delta(n, n, t) \) is positive definite for
sufficiently large $t$ and $t$ near $1/4$, say $1/4 \leq t < \alpha_n, \beta_n < t \leq \infty$. Using Theorems A, B, and C, it follows that (1.5) is true.

To prove (1.6) and (1.7) we make some explicit calculations. The case $t = 1$ is trivial since then $S(t)$ consists only of the identity function. First from (1.4) we find for $x = t^{-1}$, and $1 < t \leq \infty$, that

\begin{align*}
A_2(t) &= 2(1 - x), \\
A_3(t) &= (3 - 5x)(1 - x), \\
A_4(t) &= (4 + 14x^2 - 16x)(1 - x).
\end{align*}

Second if $1/4 \leq t < 1$ and $a = 2b^2 - 1$ [as in (1.3)], then from (1.3) we get

\begin{align*}
A_2(t) &= 1 + a, \\
A_3(t) &= (1 + a)(5 + a)/4, \\
A_4(t) &= (1 + a)(17 + 6a + a^2)/12.
\end{align*}

Here $-1 < a \leq 1$.

To prove (1.6) it suffices, by the previous argument, to show that $A_2(t) > 0$ and

\[ |\delta(3,3,t)| = A_3(t)^2 - A_2(t)^2 > 0 \]

for $5 < t < \infty$ or $1/4 \leq t < 1$. From (2.5) and (2.6) we see that these inequalities are valid for the above values of $t$. To prove (1.7), we need to show that $\delta(3,4,t) > 0, \delta(4,4,t) > 0$, for the stipulated values of $t$ in Theorem 1. To do this we consider two cases. If $1 < t \leq \infty$, and $x = 1/t$, then from (2.4) and (2.5) we have

\[ |\delta(4,4,t)| = (1 - x)^3 \begin{vmatrix}
4 + 14x^2 - 16x & 3 - 5x & 2 \\
3 - 5x & 4 + 14x^2 - 16x & 3 - 5x \\
2 & 3 - 5x & 4 + 14x^2 - 16x
\end{vmatrix} \]

Adding the second row to the first and third rows we get

\[ |\delta(4,4,t)| = (1 - x)^3 \begin{vmatrix}
7(1 - 2x) & 1(1 \times 2x) & 5 \\
3 - 5x & 4 + 14x^2 - 16x & 3 - 5x \\
5 & 7(1 - 2x) & 7(1 - 2x)
\end{vmatrix} \]
Evaluating this determinant we obtain

$$|\delta(4,4,t)| = 4(1-x)^5 (1-7x) [3 - 47x + 126x^2 - 98x^3] > 0$$

for $12.259 \leq t \leq \infty$. It is easily checked that $|\delta(3,4,t)| = A_3^2(t) - A_3^2(t) > 0$ for $12.259 \leq t \leq \infty$. Hence (1.7) is true for $12.259 \leq t \leq \infty$.

If $1/4 \leq t < 1$, then from (2.4), (2.6), we obtain

$$\begin{vmatrix}
17 + 6a + a^2 & 3(5 + a) & 12 \\
3(5 + a) & 17 + 6a + a^2 & 3(5 + a) \\
12 & 3(5 + a) & 17 + 6a + a^2
\end{vmatrix}$$

Subtracting the second row from the first and third rows, we get

$$\begin{vmatrix}
a + 2 & -a - 2 & -3 \\
3(5 + a) & 17 + 6a + a^2 & 3(5 + a) \\
-3 & -a - 2 & a + 2
\end{vmatrix}$$

Adding six times the first and third rows to the second of this determinant, we find that

$$\begin{vmatrix}
a + 2 & -a - 2 & -3 \\
9 & a - 7 & 9 \\
-3 & -a - 2 & a + 2
\end{vmatrix}$$

Evaluating this determinant we obtain

$$(12)^3|\delta(4,4,t)| = (1 + a)^3 (15a^2 + 93a + 215) > 0$$

for $-1 < a \leq 1$. Hence $|\delta(4,4,t)| > 0$ for $1/4 \leq t < 1$. It is easily checked that $|\delta(3,4,t)| > 0$ for $1/4 \leq t < 1$. We conclude that (1.7) is true for $1/4 \leq t < 1$. The proof of Theorem 1 is now complete.

Finally we remark for $1/4 \leq t < 1$ that

$$48A_3(t) = (1 + a) (74 + 38a + 10a^2 - 2a^3) < 48A_3(t)$$

for $t$ near 1, $t < 1$. It follows that $|\delta(3,5,t)| < 0$ for $t$ near 1, $t < 1$. Hence our method does not imply for all $t$, $1/4 \leq t \leq 1$, that
\[ |a_5| \leq A_3(t). \] However, it is still possible our method implies that \( \alpha_n \) in Theorem 1 can be chosen independent of \( n \).

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Received December 4, 1973.

UNIVERSITY OF KENTUCKY
Pacific Journal of Mathematics
Vol. 56, No. 2 December, 1975

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