

# Pacific Journal of Mathematics

## **COEFFICIENT BOUNDS FOR SOME CLASSES OF STARLIKE FUNCTIONS**

ROGER W. BARNARD AND JOHN LAWSON LEWIS

## COEFFICIENT BOUNDS FOR SOME CLASSES OF STARLIKE FUNCTIONS

ROGER BARNARD AND JOHN L. LEWIS

Let  $t$  be given,  $1/4 \leq t \leq \infty$ , and let  $S(t)$  denote the class of normalized starlike univalent functions  $f$  in  $|z| < 1$  satisfying (i)  $|f(z)/z| \geq t$ ,  $|z| < 1$ , if  $1/4 \leq t \leq 1$ , (ii)  $|f(z)/z| \leq t$ ,  $|z| < 1$ , if  $1 < t \leq \infty$ . If  $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in S(t)$  and  $n$  is a fixed positive integer, then the authors obtain sharp coefficient bounds for  $|a_n|$  when  $t$  is sufficiently large or sufficiently near  $1/4$ . In particular a sharp bound is found for  $|a_3|$  when  $1/4 \leq t \leq 1$  and  $5 \leq t \leq \infty$ . Also a sharp bound for  $|a_4|$  is found when  $1/4 \leq t \leq 1$  or  $12.259 \leq t \leq \infty$ .

**1. Introduction.** Let  $S$  denote the class of starlike univalent functions  $f$  in  $K = \{z : |z| < 1\}$  with the normalization,  $f(0) = 0$ ,  $f'(0) = 1$ . Given  $t$ ,  $1/4 \leq t \leq \infty$ , let  $S(t)$  denote the subclass of functions  $f \in S$  satisfying

$$(1.1) \quad |f(z)/z| \geq t, z \in K, \text{ if } 1/4 \leq t \leq 1,$$

$$(1.2) \quad |f(z)/z| \leq t, z \in K, \text{ if } 1 < t \leq \infty.$$

If  $1/4 < t \leq 1$ , we let  $F = F(\cdot, t)$  be defined by

$$(1.3) \quad zF'(z)/F(z) = [1 + 2(2b^2 - 1)z + z^2]^{1/2}/(1 - z), z \in K,$$

where  $0 \leq b < 1$  and  $t = [(1 + b)^{1+b} (1 - b)^{1-b}]^{-1}$ . The function  $F = F(\cdot, t)$  defined by (1.3) is in  $S(t)$  for  $1/4 < t \leq 1$ , as can be shown by a long but straightforward calculation (see Suffridge [9]). For fixed  $t$ ,  $1/4 < t \leq 1$ , this function maps  $K$  onto the complex plane minus a set

$$\{w : |w| \geq t, \quad \pi b \leq \arg w \leq 2\pi - \pi b\}.$$

If  $1 < t < \infty$ , we let  $F = F(\cdot, t) \in S(t)$  be defined by

$$(1.4) \quad \frac{F(z)}{[1 - t^{-1}F(z)]^2} = \frac{z}{(1 - z)^2}, z \in K.$$

It is well known (see Nehari [4, p. 224, ex. 4]) that the function  $F$  maps  $K$  onto a domain whose boundary consists of  $\{w : |w| = t\}$ , and a slit along the negative real axis from  $-t$  to  $-\lambda$  where  $4\lambda t^2 = (t + \lambda)^2$ . If  $t = 1/4$  or  $t = \infty$ , we let

$$F(z, 1/4) = F(z, \infty) = z/(1 - z)^2, z \in K.$$

In [2] the authors proved a subordination theorem for some classes of univalent functions. For  $S(t)$  this theorem may be stated as follows:

**THEOREM A.** *Let  $t$  be given,  $1/4 \leq t \leq \infty$ . Let  $F = F(\cdot, t)$  be as in (1.3) and (1.4). If  $f \in S(t)$ , then  $\log f(z)/z, z \in K$ , is subordinate to  $\log F(z)/z, z \in K$ .*

Theorem A implies for a given  $t, 1/4 \leq t \leq \infty$ , that  $F = F(\cdot, t)$  solves a number of extremal problems in  $S(t)$ . Some of these problems were pointed out in [2]. There, however, only general properties of subordination were used. In this note, for certain values of  $t$ , we use our specific knowledge of  $F$ , together with Theorem A, to obtain coefficient bounds for functions  $f \in S(t)$ . More specifically, we prove

**THEOREM 1.** *Let  $t$  be given,  $1/4 \leq t \leq \infty$ . Let  $F(z) = F(z, t) = z + \sum_{k=2}^{\infty} A_k(t)z^k, z \in K$ , be as in (1.3) and (1.4). Let  $f(z) = z + \sum_{k=2}^{\infty} a_k z^k, z \in K$ , be in  $S(t)$ . If  $n$  a positive integer is given ( $n > 2$ ), then there exist  $\alpha_n, \beta_n$  satisfying  $1/4 < \alpha_n \leq 1, 1 \leq \beta_n < \infty$ , with the property that*

$$(1.5) \quad |a_n| \leq A_n(t),$$

whenever  $1/4 \leq t < \alpha_n$  or  $\beta_n < t \leq \infty$ .  $\alpha_n$  and  $\beta_n$  may be chosen in such a way that equality holds in (1.5) only if  $f(z) = \eta^{-1}F(\eta z), z \in K$ , for some  $\eta, |\eta| = 1$ . In particular

$$(1.6) \quad |a_3| \leq A_3(t) \text{ if } 1/4 \leq t \leq 1 \text{ or } 5 < t \leq \infty,$$

$$(1.7) \quad |a_4| \leq A_4(t) \text{ if } 1/4 \leq t \leq 1 \text{ or } 12.259 \leq t \leq \infty.$$

Equality holds in (1.6) and (1.7) only if  $f(z) = \eta^{-1}F(\eta z), z \in K$ , for some  $\eta, |\eta| = 1$ .

Let  $f$  and  $t$  be as in Theorem 1. We note that the inequality  $|a_2| \leq A_2(t), 1/4 \leq t \leq \infty$ , is an easy consequence of Theorem A (see [2]). We also note for  $1 \leq t \leq e$  that  $|a_3| \leq 1 - t^{-2}$ , where equality holds for the function  $f \in S(t)$  defined by  $f(z) = F(z^2, t^2)^{1/2}, z \in K$ . This inequality is due to Tammi [10]. The problem of finding a sharp upper bound for  $|a_3|$  when  $f \in S(t), e < t < 5$ , is still open. However, Barnard [1] has shown that the function which maximizes  $|a_3|$  in  $S(t)$  is either  $F$  or a function which maps  $K$  onto a domain whose boundary consists  $\{w : |w| = t\}$  and two radial slits of equal length.

We remark that several authors have considered similar problems in the class  $U(t)$  of normalized univalent functions  $f$  (i.e.,  $f(0) = 0$ ,

$f'(0) = 1$ ) bounded above by  $t$ ,  $1 < t < \infty$ . If  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ ,  $z \in K$ , is in  $U(t)$ , then Schiffer and Tammi [6] showed that  $|a_4| \leq A_4(t)$ , for  $t \geq 33\frac{1}{3}$ . If in addition  $f$  has real coefficients, then Singh [8] proved that  $|a_4| \leq A_4(t)$  for  $t \geq 11$ . Moreover, Schiffer and Tammi [7] have proved for each positive integer  $n \geq 2$ , that there exists  $\delta_n$ ,  $1 < \delta_n < \infty$ , with the following property: If  $f \in U(t)$  and  $1 < t \leq \delta_n$ , then

$$|a_n| \leq \frac{2}{n-1} (1 - t^{1-n}).$$

Here equality holds for  $f(z) = F(z^{n-1}, t^{n-1})^{1/(n-1)}$ ,  $z \in K$ , which in fact is in  $S(t)$ . Hence the above inequality is also sharp for functions in  $S(t)$  when  $1 < t \leq \delta_n$ . Finally we remark that Schiffer and Tammi [6] have shown that it suffices to take  $\delta_4 \leq 34/19$ .

2. *Proof of Theorem 1.* Let  $G, \omega$ , be analytic in  $K$  and suppose that

$$(2.1) \quad \omega(0) = 0,$$

$$(2.2) \quad |\omega(z)| \leq 1, z \in K.$$

Put  $g(z) = G[\omega(z)]$ ,  $z \in K$ . Suppose that  $G(z) = \sum_{k=1}^{\infty} c_k z^k$ , and  $g(z) = \sum_{k=1}^{\infty} b_k z^k$ . Then Rogosinski [5, Thm. VI] proved

**THEOREM B.** *Let  $n$  be a fixed positive integer. If  $c_n > 0$  and if there exists an analytic function  $P$  in  $K$  with positive real part satisfying*

$$P(z) = \frac{c_n}{2} + c_{n-1}z + c_{n-2}z^2 + \dots + c_1z^{n-1} + \sum_{k=n}^{\infty} d_k z^k$$

for  $z \in K$ , then  $|b_n| \leq |c_n|$ . Equality can occur only if  $g(z) = G(\eta z)$  for some  $\eta$ ,  $|\eta| = 1$ , or if  $n > 1$  and  $P$  has the form,

$$(2.3) \quad P(z) = \sum_{i=1}^J \lambda_i \left( \frac{1 + \varepsilon_i z}{1 - \varepsilon_i z} \right), z \in K,$$

where  $\lambda_i > 0$ ,  $|\varepsilon_i| = 1$ ,  $1 \leq i \leq J$ , and  $J \leq n - 1$ .

Furthermore, Carathéodory (see Tsuji [11, Ch. 4 §7]) proved

**THEOREM C.** *The function  $P$  in Theorem B exists if and only if the  $n$  by  $n$  matrix*

$$\begin{pmatrix} c_n & c_{n-1} & c_{n-2} & \cdots & c_1 \\ c_{n-1} & c_n & c_{n-1} & \cdots & c_2 \\ c_{n-2} & c_{n-1} & c_n & \cdots & c_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_1 & c_2 & c_3 & \cdots & c_n \end{pmatrix}$$

is positive semi definite. If  $P$  exists, then  $P$  has the form (2.3) only if the above matrix has determinant zero.

We now use Theorems A, B, and C to prove Theorem 1. Let  $t$  be fixed,  $1/4 \leq t \leq \infty$ , and  $f \in S(t)$ . Then Theorem A implies there exists a function  $\omega$  satisfying (2.1) and (2.2) for which  $f(z)/z = F[\omega(z)]/\omega(z)$ ,  $z \in K$ . Hence we may use Theorems B and C with  $g(z) = f(z)/z - 1$ ,  $G(z) = (F(z)/z) - 1$ ,  $z \in K$ , and  $c_i = A_{i+1}(t)$ ,  $1 \leq i \leq n-1$ , to prove Theorem 1. To do so we shall want some notation.

Let  $n$  and  $k$  be fixed positive integers satisfying  $2 \leq k \leq n$ . Let  $\delta(k, n, t)$  be the  $k-1$  by  $k-1$  matrix

$$(2.4) \quad \delta(k, n, t) = \begin{pmatrix} A_n(t) & A_{n-1}(t) & \cdots & A_{n-k+2}(t) \\ A_{n-1}(t) & A_n(t) & \cdots & A_{n-k+3}(t) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ A_{n-k+2}(t) & A_{n-k+3}(t) & \cdots & A_n(t) \end{pmatrix}$$

Let  $|\delta(k, n, t)|$  denote the determinant of  $\delta(k, n, t)$ . Then it is well known (see Hohn [3, Thm. 9, 17.3]) that  $\delta(n, n, t)$  is positive definite if and only if  $|\delta(k, n, t)| > 0$  for  $2 \leq k \leq n$ .

We note that  $A_n(\infty) = A_n(1/4) = n$  for  $n \geq 2$ . Using this fact we obtain that  $|\delta(k, n, \infty)| = |\delta(k, n, 1/4)| = (2n+2-k) 2^{k-3}$  for  $2 \leq k \leq n$  and  $n > 2$ . Since (1.3) and (1.4) imply  $A_n$  is continuous as a function of  $t$ ,  $1/4 \leq t \leq \infty$ , it follows that

$$\lim_{t \rightarrow \infty} |\delta(k, n, t)| = |\delta(k, n, \infty)| = \lim_{t \rightarrow 1/4} |\delta(k, n, t)| > 0$$

for each positive integer  $n > 2$  and  $2 \leq k \leq n$ . From this inequality and our previous remark we see that  $\delta(n, n, t)$  is positive definite for

sufficiently large  $t$  and  $t$  near  $1/4$ , say  $1/4 \leq t < \alpha_n$ ,  $\beta_n < t \leq \infty$ . Using Theorems A, B, and C, it follows that (1.5) is true.

To prove (1.6) and (1.7) we make some explicit calculations. The case  $t = 1$  is trivial since then  $S(t)$  consists only of the identity function. First from (1.4) we find for  $x = t^{-1}$ , and  $1 < t \leq \infty$ , that

$$\begin{aligned}
 (2.5) \quad A_2(t) &= 2(1 - x), \\
 A_3(t) &= (3 - 5x)(1 - x), \\
 A_4(t) &= (4 + 14x^2 - 16x)(1 - x).
 \end{aligned}$$

Second if  $1/4 \leq t < 1$  and  $a = 2b^2 - 1$  [ $b$  as in (1.3)], then from (1.3) we get

$$\begin{aligned}
 (2.6) \quad A_2(t) &= 1 + a, \\
 A_3(t) &= (1 + a)(5 + a)/4, \\
 A_4(t) &= (1 + a)(17 + 6a + a^2)/12.
 \end{aligned}$$

Here  $-1 < a \leq 1$ .

To prove (1.6) it suffices, by the previous argument, to show that  $A_2(t) > 0$  and

$$|\delta(3, 3, t)| = A_3(t)^2 - A_2(t)^2 > 0$$

for  $5 < t < \infty$  or  $1/4 \leq t < 1$ . From (2.5) and (2.6) we see that these inequalities are valid for the above values of  $t$ . To prove (1.7), we need to show that  $\delta(3, 4, t) > 0$ ,  $\delta(4, 4, t) > 0$ , for the stipulated values of  $t$  in Theorem 1. To do this we consider two cases. If  $1 < t \leq \infty$ , and  $x = 1/t$ , then from (2.4) and (2.5) we have

$$|\delta(4, 4, t)| = (1 - x)^3 \begin{vmatrix} 4 + 14x^2 - 16x & 3 - 5x & 2 \\ 3 - 5x & 4 + 14x^2 - 16x & 3 - 5x \\ 2 & 3 - 5x & 4 + 14x^2 - 16x \end{vmatrix}$$

Adding the second row to the first and third rows we get

$$|\delta(4, 4, t)| = (1 - x)^5 \begin{vmatrix} 7(1 - 2x) & 1(1 \times 2x) & 5 \\ 3 - 5x & 4 + 14x^2 - 16x & 3 - 5x \\ 5 & 7(1 - 2x) & 7(1 - 2x) \end{vmatrix}$$

Evaluating this determinant we obtain

$$|\delta(4,4,t)| = 4(1-x)^5(1-7x)[3-47x+126x^2-98x^3] > 0$$

for  $12.259 \leq t \leq \infty$ . It is easily checked that  $|\delta(3,4,t)| = A_4^2(t) - A_3^3(t) > 0$  for  $12.259 \leq t \leq \infty$ . Hence (1.7) is true for  $12.259 \leq t \leq \infty$ .

If  $1/4 \leq t < 1$ , then from (2.4), (2.6), we obtain

$$(12)^3 |\delta(4,4,t)| = (1+a)^3 \begin{vmatrix} 17+6a+a^2 & 3(5+a) & 12 \\ 3(5+a) & 17+6a+a^2 & 3(5+a) \\ 12 & 3(5+a) & 17+6a+a^2 \end{vmatrix}$$

Subtracting the second row from the first and third rows, we get

$$(12)^3 |\delta(4,4,t)| = (1+a)^3 \begin{vmatrix} a+2 & -a-2 & -3 \\ 3(5+a) & 17+6a+a^2 & 3(5+a) \\ -3 & -a-2 & a+2 \end{vmatrix}$$

Adding six times the first and third rows to the second of this determinant, we find that

$$(12)^3 |\delta(4,4,t)| = (1+a)^6 \begin{vmatrix} a+2 & -a-2 & -3 \\ 9 & a-7 & 9 \\ -3 & -a-2 & a+2 \end{vmatrix}$$

Evaluating this determinant we obtain

$$(12)^3 |\delta(4,4,t)| = (1+a)^6(a^3+15a^2+93a+215) > 0$$

for  $-1 < a \leq 1$ . Hence  $|\delta(4,4,t)| > 0$  for  $1/4 \leq t < 1$ . It is easily checked that  $|\delta(3,4,t)| > 0$  for  $1/4 \leq t < 1$ . We conclude that (1.7) is true for  $1/4 \leq t < 1$ . The proof of Theorem 1 is now complete.

Finally we remark for  $1/4 \leq t < 1$  that

$$48A_5(t) = (1+a)(74+38a+10a^2-2a^3) < 48A_4(t)$$

for  $t$  near 1,  $t < 1$ . It follows that  $|\delta(3,5,t)| < 0$  for  $t$  near 1,  $t < 1$ . Hence our method does not imply for all  $t$ ,  $1/4 \leq t \leq 1$ , that

$|a_5| \leq A_5(t)$ . However, it is still possible our method implies that  $\alpha_n$  in Theorem 1 can be chosen independent of  $n$ .

## REFERENCES

1. R. Barnard, *A variational technique for bounded starlike functions*, to appear.
2. R. Barnard and J. Lewis, *Subordination theorems for some classes of starlike functions*, to appear.
3. F. Hohn, *Elementary Matrix Algebra*, the Macmillan Company, 1966.
4. Z. Nehari, *Conformal Mapping*, McGraw-Hill, 1952.
5. W. Rogosinski, *On the coefficients of subordinate functions*, Proc. London Math. Soc., **48** (1943), 48–82.
6. M. Schiffer and O. Tammi, *On the fourth coefficient of bounded univalent functions*, Trans. Amer. Math. Soc., **119**, 67–78.
7. M. Schiffer and O. Tammi, *On bounded univalent functions which are close to identity*, Ann. Acad. Sci. Fenn. Ser. AI, no. 435, 1969.
8. V. Singh, *Grunsky inequalities and coefficients of bounded Schlicht functions*, Ann. Acad. Sci. Fenn. Ser. AI, no. 310, 1962.
9. T. Suffridge, *A coefficient problem for a class of univalent functions*, Mich. Math. J., **16** (1969), 33–42.
10. O. Tammi, *On the maximalization of the coefficient  $a_3$  of bounded Schlicht functions*, Ann. Acad. Sci. Fenn. Ser. AI, no. 149, 1954.
11. M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen Company, 1959.

Received December 4, 1973.

UNIVERSITY OF KENTUCKY





# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

RICHARD ARENS (Managing Editor)

University of California  
Los Angeles, California 90024

J. DUGUNDJI

Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

R. A. BEAUMONT

University of Washington  
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM

Stanford University  
Stanford, California 94305

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA  
STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON

\* \* \*

AMERICAN MATHEMATICAL SOCIETY

---

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate, may be sent to any one of the four editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$ 72.00 a year (6 Vols., 12 issues). Special rate: \$ 36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION  
Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

Copyright © 1975 Pacific Journal of Mathematics  
All Rights Reserved

Ralph Alexander, <i>Generalized sums of distances</i> .....	297
Zvi Arad and George Isaac Glauberan, <i>A characteristic subgroup of a group of odd order</i> .....	305
B. Aupetit, <i>Continuité du spectre dans les algèbres de Banach avec involution</i> .....	321
Roger W. Barnard and John Lawson Lewis, <i>Coefficient bounds for some classes of starlike functions</i> .....	325
Roger W. Barnard and John Lawson Lewis, <i>Subordination theorems for some classes of starlike functions</i> .....	333
Ladislav Bican, <i>Preradicals and injectivity</i> .....	367
James Donnell Buckholtz and Ken Shaw, <i>Series expansions of analytic functions. II</i> .....	373
Richard D. Carmichael and E. O. Milton, <i>Distributional boundary values in the dual spaces of spaces of type <math>\mathcal{S}</math></i> .....	385
Edwin Duda, <i>Weak-unicoherence</i> .....	423
Albert Edrei, <i>The Padé table of functions having a finite number of essential singularities</i> .....	429
Joel N. Franklin and Solomon Wolf Golomb, <i>A function-theoretic approach to the study of nonlinear recurring sequences</i> .....	455
George Isaac Glauberan, <i>On Burnside's other <math>p^a q^b</math> theorem</i> .....	469
Arthur D. Grainger, <i>Invariant subspaces of compact operators on topological vector spaces</i> .....	477
Jon Craig Helton, <i>Mutual existence of sum and product integrals</i> .....	495
Franklin Takashi Iha, <i>On boundary functionals and operators with finite-dimensional null spaces</i> .....	517
Gerald J. Janusz, <i>Generators for the Schur group of local and global number fields</i> .....	525
A. Katsaras and Dar-Biau Liu, <i>Integral representations of weakly compact operators</i> .....	547
W. J. Kim, <i>On the first and the second conjugate points</i> .....	557
Charles Philip Lanski, <i>Regularity and quotients in rings with involution</i> ....	565
Ewing L. Lusk, <i>An obstruction to extending isotopies of piecewise linear manifolds</i> .....	575
Saburou Saitoh, <i>On some completenesses of the Bergman kernel and the Rudin kernel</i> .....	581
Stephen Jeffrey Willson, <i>The converse to the Smith theorem for <math>Z_p</math>-homology spheres</i> .....	597