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For every real $r \geqq 0$, there is a sequence $\{b_n^{(r)}\}$ defined by

$$b_0^{(r)} = 1, \quad b_{n+1}^{(r)} = \prod_{i=1}^{n} b_i^{(r)} + r \text{ for } n \geqq 0. \tag{1}$$

These sequences were considered previously, in [1], for integer values of r, and it was shown that there is a constant $\theta = \theta(r)$ such that

$$b_{n+1}^{(r)} \sim \theta^{2^n}, \quad n \to \infty, \tag{2}$$

for each $r = 1, 2, 3, \cdots$. It was observed that

$$b_{n+1}^{(2)} = 2^{2^n} + 1, \quad n \geqq 0, \tag{3}$$

whereby $\theta(2) = 2$, and the problem was proposed "to determine the algebraic or transcendental character of the real numbers $\theta(r)$ for $r = 1, 3, 4, 5, 6, \cdots$."

In this paper, we observe explicitly (in §II) that

$$b_n^{(4)} = \tau^{2^n} + \tau^{-2^n} + 2, \quad n \geqq 1, \tag{4}$$

where $\tau = (\sqrt{5} + 1)/2 = 1.618 \cdots$ is the "Golden Mean", and thus $\theta(4) = \tau^2 = (\sqrt{5} + 3)/2 = 2.618 \cdots$.

Moreover, we extend the result (2) by showing, for every *real* $r > 0$, there is a real constant $\theta = \theta(r) > 1$ such that

$$b_{n+1}^{(r)} = \theta^{2^n} + \frac{r}{2} + \frac{1}{8} r(r-2)\theta^{-2^n} + O(\theta^{-2^{n+2}}), \quad n \to \infty. \tag{5}$$

Thus for $r \neq 2$ the sequence $\{\beta_n^{(r)}\} = \{b_n^{(r)} - (r/2)\}$ differs from the sequence $\{\theta^{2^{n-1}}(r)\}$ by an amount which approaches 0 exponentially as $n \to \infty$. The case $r = 4$, described in (4), is illustrative of this behavior, while the case $r = 2$, described in (3), is exceptional in that the error term is identically 0.

For $r = 0$, $b_n^{(0)} = \beta_n^{(0)} = 1$ for all $n \geqq 0$, so that $\theta(0) = 1$; and $\theta(r)$ is a continuous, monotone increasing function of $r \geqq 0$.

The basic tool used in treating the general case is a new theorem in function theory (§III), which is ideally suited to the study of sequences

generated by nonlinear (and especially polynomial) recursions. The function-theoretic approach also reveals (§V) an unexpected connection between the class of nonlinear recursions considered here, and the enumeration problem for rooted trees.

II. The case $r = 4$. We observe that $\tau^2 = (3+\sqrt{5})/2$ and $\tau^{-2} = (3-\sqrt{5})/2$. Thus $\tau^{2^1} + \tau^{-2^1} = 3 = b_1^{(4)} - 2 = \beta_1^{(4)}$. Setting $r = 4$ in (1), we see

$$b_{n+1}^{(4)} = b_n^{(4)} \cdot \prod_{i=1}^{n-1} b_i^{(4)} + 4 = b_n^{(4)}(b_n^{(4)} - 4) + 4 = (b_n^{(4)} - 2)^2.$$

The general identity

$$b_n^{(4)} = \tau^{2^n} + \tau^{-2^n} + 2, \quad n \geqq 1 \tag{6}$$

is established inductively by

$$b_{n+1}^{(4)} = (b_n^{(4)} - 2)^2 = (\tau^{2^n} + \tau^{-2^n})^2 = \tau^{2^{n+1}} + \tau^{-2^{n+1}} + 2, \tag{7}$$

since we have already verified (6) for the case $n = 1$.

III. A function-theoretic approach. The behavior of $b_n^{(r)}$ as $n \to \infty$ will be elucidated by means of the following.

THEOREM. *For $|z| \geqq R > 0$ let $p(z)$ be analytic, and let $|p(z)| \geqq A|z|$, where $A > 1$. Then there is a unique positive integer, m, a unique complex constant, $c \neq 0$, and a unique function, $f(z)$, analytic and nonzero for $|z| > R$, such that $f(z) \cong z$ as $z \to \infty$, and such that*

$$f(p(z)) \equiv c\{f(z)\}^m. \tag{8}$$

Proof. Since $p(z) \to \infty$ as $z \to \infty$, this function has a pole at infinity. If the pole is of order M:

$$p(z) \cong Cz^M \quad \text{as} \quad z \to \infty,$$

and if a functional equation of the form (8) holds, then since $f(z)$ is required to be $\cong z$ at ∞, we conclude that $c = C$ and $m = M$.

Let the unknown function $f(z)$ be represented by $f(z) = z \exp \varphi(z)$ where $\varphi(z) \to 0$ as $z \to \infty$. The functional equation (8) now takes the form

$$p(z) \exp \varphi(p(z)) = cz^m \exp[m\varphi(z)]$$

which is true if

$$\varphi(z) = q(z) + \frac{1}{m}\varphi(p(z)) \tag{9}$$

where $mq(z)$ is the logarithm of $p(z)/(cz^m)$ which tends to zero as $z \to \infty$.

A solution of the functional equation (9) can be found by the method of iteration. Let $\varphi_0(z) = q(z)$, and define $\varphi_1(z), \varphi_2(z), \cdots$ by

$$\varphi_{n+1}(z) = q(z) + \frac{1}{m}\varphi_n(p(z)) \qquad (n = 0, 1, \cdots). \tag{10}$$

Since $|p(z)| \geqq A|z| > |z|$, all the functions $\varphi_n(z)$ are analytic for $|z| \geqq R$. Since $q(z)$ is regular and equal to zero at $z = \infty$, there is a constant Q such that

$$|\varphi_0(z)| = |q(z)| \leqq Q/|z| \quad (|z| \geqq R). \tag{11}$$

We can now prove by induction that, for $n = 1, 2, \cdots$,

$$|\varphi_n(z) - \varphi_{n-1}(z)| \leqq Q/(A^n|z|). \tag{12}$$

For $n = 1$, we deduce from (10) and (11)

$$|\varphi_1(z) - \varphi_0(z)| = \frac{1}{m}|\varphi_0(p(z))| \leqq \frac{1}{m} Q/|p(z)| \leqq Q/(A|z|).$$

For $n > 1$, we have

$$\varphi_{n+1}(z) - \varphi_n(z) = \frac{1}{m}\{\varphi_n(p(z)) - \varphi_{n-1}(p(z))\}.$$

Assuming (12), we find

$$|\varphi_{n+1}(z) - \varphi_n(z)| \leqq Q/(A^n|p(z)|) \leqq Q/(A^{n+1}|z|).$$

This completes the inductive proof of (12).

We now conclude that $\varphi_n(z)$ tends to a limit as $n \to \infty$, namely

$$\lim_{n\to\infty} \varphi_n(z) = \varphi_0(z) + \sum_{n=1}^{\infty} \{\varphi_n(z) - \varphi_{n-1}(z)\} = \varphi(z)$$

since the infinite series is dominated by $(Q/|z|)\Sigma A^{-n} < \infty$. The convergence is uniform for $|z| \geqq R$. Therefore, the limit, $\varphi(z)$, is analytic for $|z| > R$.

It remains to prove that there is no other function, $\psi(z)$, analytic for $|z|>R$, with $\psi(\infty)=0$, such that

$$\psi(z)=q(z)+\frac{1}{m}\,\psi(p(z)). \tag{13}$$

Let $\varphi(z)-\psi(z)=w(z)$. Then (9) and (13) imply

$$w(z)=\frac{1}{m}\,w(p(z)). \tag{14}$$

If $w(z)\neq 0$, then $w(z)\cong Wz^{-N}$ as $z\to\infty$, where $W\neq 0$ and N is a positive integer. Then (14) implies

$$Wz^{-N}\cong\frac{1}{m}\,W(p(z))^{-N} \quad \text{as} \quad z\to\infty$$

which is absurd, since $m\geqq 1$ and $|p(z)/z|\geqq A>1$. This completes the proof of the theorem.

IV. Application to the quadratic recursion. Keeping $r>0$ fixed, we define $\beta_n=\beta_n^{(r)}=b_n^{(r)}-(r/2)$. Then (1) implies

$$\beta_{n+1}=\beta_n^2-\rho, \quad \text{where} \quad \rho=r(r-2)/4, \qquad n=1,2,3,\cdots \tag{15}$$

Then $\beta_n\to\infty$ because $\beta_n\geqq 1+\frac{1}{2}r$ if $n\geqq 1$, and hence, $\beta_{n+1}\geqq \beta_n+r$. Define

$$p(z)=z^2-\rho. \tag{16}$$

Then (15) takes the form

$$\beta_{n+1}=p(\beta_n) \qquad (n=1,2,\cdots). \tag{17}$$

If $A>1$, then

$$|p(z)|=|z^2-\rho|\geqq|z^2|-|\rho|\geqq A|z| \quad \text{if} \quad |z|\geqq R \tag{18}$$

provided that we define

$$R=\tfrac{1}{2}(A+(A^2+4|\rho|)^{\frac{1}{2}}) \tag{19}$$

since this is the positive root of $R^2-|\rho|=AR$.

According to our theorem, there is a unique function of the form

$$f(z) = z + c_0 + \frac{c_1}{z} + \cdots (|z| > R) \tag{20}$$

such that, if $p(z) = z^2 + \cdots$, then

$$f(p(z)) \equiv \{f(z)\}^2 \quad \text{for} \quad (|z| > R). \tag{21}$$

Since $p(z) = z^2 - p$ is an even function of $z, f(z)$ must be an odd function of z in order to satisfy (21), and only the c_i with odd subscripts need be retained in (20). For all $n \geqq 3$, we have

$$\beta_n > \beta_2 = 1 + \frac{3}{2} r.$$

Moreover, since $r > 0$,

$$\begin{aligned} \beta_2^2 - |\rho| &= 1 + 3r + \frac{9}{4} r^2 - \tfrac{1}{4} r |r - 2| \\ &> 1 + \frac{3}{2} r = \beta_2. \end{aligned}$$

Define A as any number satisfying

$$1 < A < \frac{\beta_2^2 - |\rho|}{\beta_2} \tag{22}$$

and define R by (19). Then the last inequality implies $\beta_2 > R$, and hence $\beta_n > R$ for all $n \geqq 2$. Hence, for $n \geqq 2$, (17) and (21) imply

$$f(\beta_{n+1}) = f(p(\beta_n)) = \{f(\beta_n)\}^2$$

Therefore, if $n \geqq 2$,

$$f(\beta_n) = \theta^{2^n} \tag{23}$$

if θ is the positive number defined by $f(\beta_2) = \theta^4$. Since $\beta_n \to \infty$, we have $\theta > 1$.

The continuity of $\theta(r)$. If r ranges in any interval $r_0 \leqq r \leqq r_1$, then there is a constant A, independent of r, satisfying (22). Then the sequence of iterates $\varphi_n(z)$ defined by (10) converges uniformly in z and in r, and $\varphi_n(z)$ depends continuously on the parameter r. Hence, $\lim_{(n\to\infty)} \varphi_n(z)$ is continuous as a function of r; and $f(z)$ depends continuously on the parameter r. But we have just shown that

$$\theta(r) = \{f(\beta_2)\}^{1/4}$$
$$= \left\{f\left(1+\frac{3}{2}r\right)\right\}^{1/4}$$

Hence, $\theta(r)$ depends continuously on r.

For sufficiently large $|z|$, every function $f(z)$ of the form (20) has an inverse function of the form

$$F(w) = w + \gamma_0 + \frac{\gamma_1}{w} + \cdots$$

satisfying

$$F(f(z)) \equiv z.$$

For the case $p(z) = z^2 - \rho$, $f(z)$ is an odd function of z, and the inverse function of an odd function is an odd function. Applying F to (23), we find, for all sufficiently large n,

$$\beta_n = F(\theta^{2^n}) = \theta^{2^n} + \gamma_1\theta^{-2^n} + \gamma_3\theta^{-3\cdot 2^n} + \cdots. \tag{24}$$

The coefficients in the power series for $f(z)$ and $F(w)$ can be calculated recursively. Formula (21) yields

$$p(z) + \frac{c_1}{p(z)} + \frac{c_3}{p^3(z)} + \cdots = \left(z + \frac{c_1}{z} + \frac{c_3}{z^3}\cdots\right)^2$$

from which

$$z^2 - \rho + \frac{c_1}{z^2-\rho} + \cdots = z^2 + 2c_1 + (2c_3 + c_1^2)z^{-2} + \cdots.$$

Therefore,

$$c_1 = -\frac{\rho}{2}, \quad c_3 = -\frac{\rho(\rho+2)}{8}, \quad \text{etc.} \tag{25}$$

For the inverse function, we must have

$$z \equiv (z + c_1 z^{-1} + c_3 z^{-3} + \cdots) + \gamma_1(z + \cdots)^{-1} + \cdots.$$

Hence

$$\gamma_1 = -c_1 = \frac{\rho}{2} = \frac{r(r-2)}{8}$$

$$\gamma_3 = -c_3 - c_1^2 = -\frac{\rho(\rho-2)}{8} = -\frac{(r+2)r(r-2)(r-4)}{128} \tag{26}$$

etc.

Now (24) yields, for $n \to \infty$,

$$\beta_n = \theta^{2^n} + \frac{1}{8} r(r-2)\theta^{-2^n} + O(\theta^{-3\cdot 2^n}). \tag{27}$$

If $r = 2$ or $r = 4$, the preceding sections have shown that β_n has the *exact* form

$$\beta_n = \theta^{2^n} + \frac{r(r-2)}{8}\,\theta^{-2^n}$$

and

$$b_n = \theta^{2^n} + \frac{r}{2} + \frac{1}{8} r(r-2)\theta^{-2^n}.$$

Conversely, it is easily verified that, for $r > 0$, and expression of this exact form can satisfy (15) *only* if $r = 2$ or 4. In fact, the only *real* values of ρ which lead to this exact form are $\rho = 0$ and $\rho = 2$.

We remark that the inverse function, F, satisfies a functional equation. Letting $z = F(w)$ in the identity

$$f(p(z)) = c\{(z)\}^m$$

yields

$$f(p(F(w))) = cw^m.$$

Applying F to both sides now yields

$$p(F(w)) = F(cw^m).$$

In our application, we have the identity

$$F^2(w) - \rho = F(w^2). \tag{28}$$

The coefficients in the power series for $F(w)$ can be calculated directly from this equation.

The algebraic character of the constant, θ, depends on the nature of the function $f(z)$. If r is a positive integer, then β_2 is an integer or a half-integer, and

$$\theta = \{f(\beta_2)\}^{\frac{1}{2}}.$$

Thus, θ is algebraic if $f(z)$ is algebraic over the field of rationals. In the known cases,

$$f(z) = z \quad \text{if} \quad r = 2$$
$$f(z) = \tfrac{1}{2}[z + (z^2 - 4)^{\frac{1}{2}}] \quad \text{if} \quad r = 4$$

since these are the functions whose inverse functions are

$$F(w) = w \quad \text{if} \quad r = 2$$
$$F(w) = w + w^{-1} \quad \text{if} \quad r = 4.$$

Unfortunately, only $\rho = 0$ and $\rho = 2$ lead to functions $F(w)$ with finite expansions in w.

V. Coefficient relationships. In the previous section, we considered the functions

$$f(z) = z + c_1 z^{-1} + c_3 z^{-3} + c_5 z^{-5} + \cdots \tag{29}$$

and

$$F(w) = w + \gamma_1 w^{-1} + \gamma_3 w^{-3} + \gamma_5 w^{-5} + \cdots \tag{30}$$

where

$$\begin{gathered} F(f(z)) \equiv z, \\ f(z^2 - \rho) = f^2(z) \quad \text{and} \\ F^2(w) - \rho = F(w^2). \end{gathered} \tag{31}$$

In this section, we shall assume only that ρ is a real number, and derive certain identities involving the coefficient sequences $\{c_i\}$ and $\{\gamma_i\}$.

From (29) and (31), we obtain

$$\sum_{\text{odd}\, k=-1}^{\infty} \frac{c_k u^k}{(1-\rho u)^k} = \sum_{m=-1}^{\infty} \sum_{\text{odd}\, j=-1}^{2m+1} c_j c_{2m-j} u^m \tag{32}$$

where for convenience we have set $c_{-1} = 1$.

We rewrite (30) using $t = w^{-1}$ and $q_i = \gamma_{2i-1}$ to obtain

$$F(1/t) = \frac{1}{t} + q_1 t + q_2 t^3 + q_3 t^5 + q_4 t^7 + \cdots = \sum_{i=0}^{\infty} q_i t^{2i-1} \tag{33}$$

where we have set $q_0 = 1$.

Then from (31) we find

$$\begin{aligned}
q_1 &= \frac{\rho}{2} & q_5 &= -q_1 q_4 - q_2 q_3 \\
q_2 &= -\binom{q_1}{2} & q_6 &= -q_1 q_5 - q_2 q_4 - \binom{q_3}{2} \\
q_3 &= -q_1 q_2 & q_7 &= -q_1 q_6 - q_2 q_5 - q_3 q_4 \\
q_4 &= -q_1 q_3 - \binom{q_2}{2} & q_8 &= -q_1 q_7 - q_2 q_6 - q_3 q_5 - \binom{q_4}{2}
\end{aligned}$$

and in general

$$\begin{aligned}
q_{2l-1} &= -\sum_{i=1}^{l-1} q_i q_{2l-1-i}, \quad l > 1 \\
q_{2l} &= -\binom{q_l}{2} - \sum_{i=1}^{l-1} q_i q_{2l-i}, \quad l \geqq 1.
\end{aligned} \tag{34}$$

The assertion that the expansion for $F(w)$ is finite only in the cases $\rho = 0$ and $\rho = 2$ is proved as follows. Suppose

$$F(w) = w + \frac{q_1}{w} + \frac{q_2}{w^3} + \cdots + \frac{q_n}{w^{2n-1}}, \qquad n > 1, \tag{35}$$

where q_n is the last nonzero coefficient. From $F(w^2) + \rho = F^2(w)$, we see that $q_n^2 = q_n$, so that $q_n = 1$. Next, $q_{n-1} = 0$, because $2q_{n-1}$ is the coefficient of $w^{-(4n-4)}$ in $F^2(w)$, but the corresponding coefficient in $F(w^2)$ is 0. The next coefficient identity, examining $w^{-(4n-6)}$, is $q_{n-1} = 2q_{n-2} + q_{n-1}^2$, but since $q_{n-1} = 0$ we get $q_{n-2} = 0$, and proceeding inductively, we find $q_i = 0$ for all i, $1 \leqq i \leqq n-1$. However, $F(w) = w + 1/w^{2n-1}$ fails to satisfy the functional equation for all $n > 1$. The cases $F(w) = w$ and $F(w) = w + 1/w$ are the unique solutions of the functional equation for $n = 0$ and $n = 1$ respectively.

Other than for $\rho = 0$ and $\rho = 2$, no cases of "well-known" functions have been found as either $f(z)$ or $F(w)$. However, the case $\rho = -2$

turns out to be of considerable combinatorial interest. The recursion equations (34), starting with $\rho = -2$ and $q_1 = -1$, generate a sequence of integers $\{q_i\}$ such that $\{-q_i\} = \{1, 1, 1, 2, 3, 6, 11, 23, 46, 98, \cdots\}$, where it is easily seen (1) that $-q_i$ is the number of distinct "binary coding trees" with $i = 1$ interior nodes and i terminal nodes, where there are always two edges leading down from each interior node (see Figure (1); and (2) that $-q_i$ is the number of binary rooted trees with i nodes altogether, and at most two edges leading down from each node (see Figure 2). Indeed, the second set of trees is obtained by pruning away all branches leading to terminal nodes in the first set of trees. A recursion strikingly similar to (34) occurs in yet another tree enumeration problem [2].

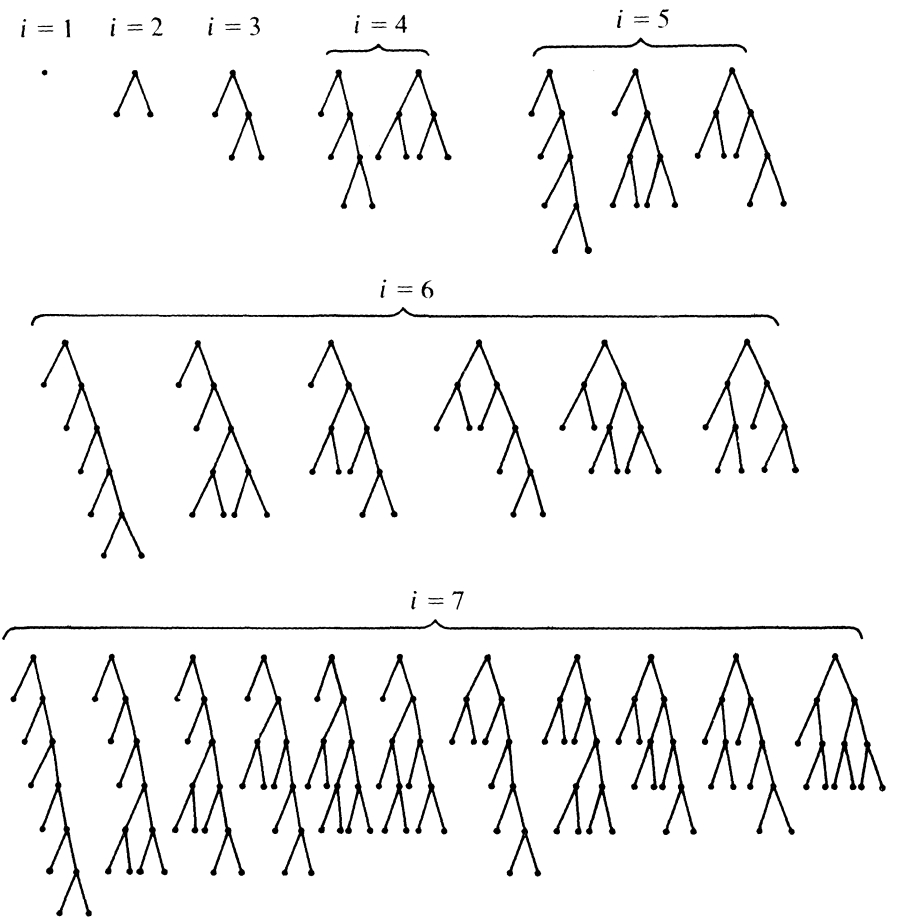

FIGURE 1. The "binary coding trees" of order i, $1 \leqq i \leqq 7$.

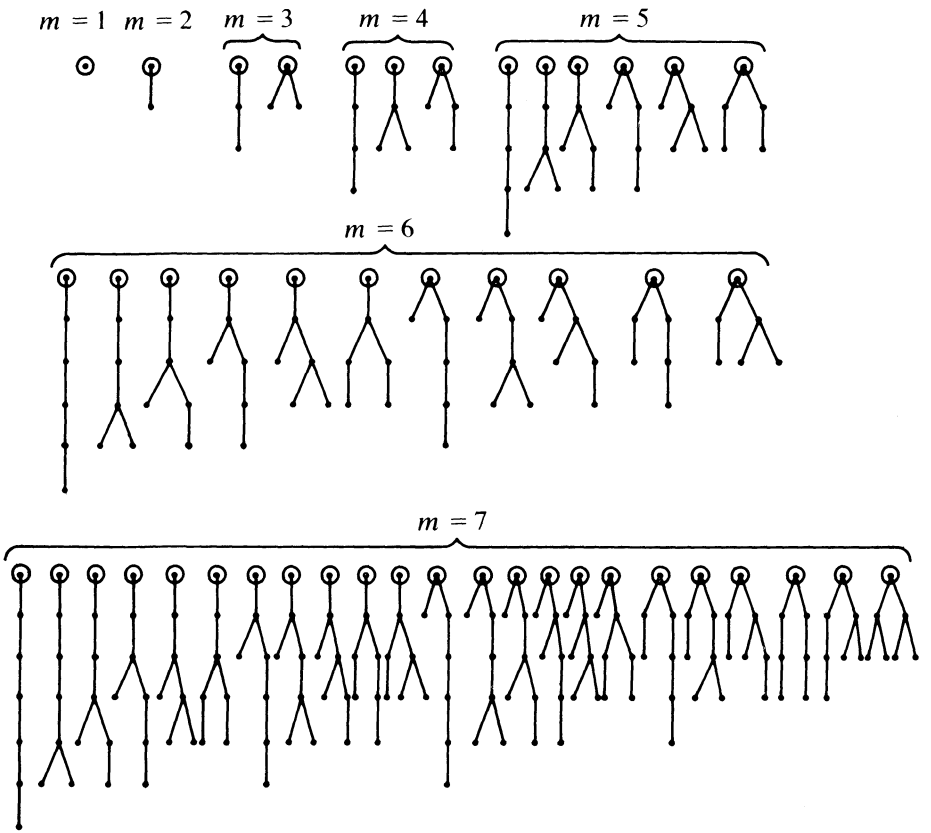

FIGURE 2. The "binary rooted trees" with $m = i - 1$ nodes, $1 \leqq m \leqq 7$.

Let $f(z)$ satisfy (29) and (31). Then equation (32) can be solved to obtain the following recursion relation for the coefficient sequence $\{c_{2i+1}\}$, starting with $c_{-1} = 1$:

$$2c_{2i+1} = -\sum_{\text{odd } j=1}^{2i-1} c_j c_{2i-j} + \sum_{\text{odd } j=1}^{i} c_j \binom{i-1}{j-1} \rho^{i-j}. \tag{36}$$

Thus, if i even,

$$c_{2i+1} = -\sum_{\text{odd } j=1}^{i-1} c_j c_{2i-j} + \tfrac{1}{2} \sum_{\text{odd } j=1}^{i-1} c_j \binom{i-1}{j-1} \rho^{i-j}, \tag{37}$$

while if i odd,

$$c_{2i+1} = -\sum_{\text{odd } j=1}^{i-2} c_j c_{2i-j} - \binom{c_i}{2} + \tfrac{1}{2} \sum_{\text{odd } j=1}^{i-2} c_j \binom{i-1}{j-1} \rho^{i-j}. \tag{38}$$

Equivalently:

$$2c_{2i+1} = -\sum_{\substack{\text{odd } j=1}}^{2i-1} c_j c_{2i-j} + \sum_{\substack{m=0 \\ (i-m \text{ odd})}}^{i-1} \binom{i-1}{m} c_{i-m}\rho^m. \tag{39}$$

Thus, if i even,

$$c_{2i+1} = -\sum_{\text{odd } j=1}^{i-1} c_j c_{2i-j} + \tfrac{1}{2} \sum_{\text{odd } m=1}^{i-1} \binom{i-1}{m} c_{i-m}\rho^m, \tag{40}$$

while if i odd,

$$c_{2i+1} = -\sum_{\text{odd } j=1}^{i-2} c_j c_{2i-j} - \binom{c_i}{2} + \tfrac{1}{2} \sum_{\text{even } m=2}^{i-1} \binom{i-1}{m} c_{i-m}\rho^m. \tag{41}$$

Explicitly:

$$\begin{aligned}
c_1 &= -\frac{\rho}{2} \\
c_3 &= -\binom{c_1}{2} \\
c_5 &= -c_1c_3 + \frac{c_1\rho}{2} \\
c_7 &= -c_1c_5 - \left(\frac{c_3}{2}\right) + \frac{c_1\rho^2}{2} \\
c_9 &= -c_1c_7 - c_3c_5 + \frac{c_1\rho^3 + 3c_3\rho}{2} \\
c_{11} &= -c_1c_9 - c_3c_7 - \left(\frac{c_5}{2}\right) + \frac{c_1\rho^4 + 6c_3\rho^2}{2} \\
c_{13} &= -c_1c_{11} - c_3c_9 - c_5c_7 + \frac{c_1\rho^5 + 10c_3\rho^3 + 5c_5\rho}{2} \\
c_{15} &= -c_1c_{13} - c_3c_{11} - c_5c_9 - \binom{c_7}{2} + \frac{c_1\rho^6 + 15c_3\rho^4 + 15c_5\rho^2}{2} \\
c_{17} &= -c_1c_{15} - c_3c_{13} - c_5c_{11} - c_7c_9 + \frac{c_1\rho^7 + 21c_3\rho^5 + 35c_5\rho^3 + 7c_7\rho}{2}
\end{aligned} \tag{42}$$

etc.

VI. Asymptotic ratios. Define $G(u)$ from (33) by

$$(43)\quad G(u) = u^{\frac{1}{2}}F(u^{-\frac{1}{2}}) = 1 + q_1u + q_2u^2 + q_3u^3 + q_4u^4 + \cdots = \sum_{i=0}^{\infty} q_iu^i.$$

Suppose there is a limiting ratio $R = \lim_{i\to\infty}(q_{i+1}/q_i)$. Then it is well known that the radius of convergence of the power series (43) for $G(u)$ is $1/|R|$. Therefore the series (30) for $F(w)$ converges for $|w| > |R^{\frac{1}{2}}|$, diverges for $|w| < |R^{\frac{1}{2}}|$, and may converge for some (or all) of the values of w with $|w| = |R^{\frac{1}{2}}|$. By (34), if ρ is real, then all the q_i are real, and R is real, $R = \pm|R|$.

From (34) we have

$$-q_{m+1} = q_1q_m + q_2q_{m-1} + q_3q_{m-2} + \cdots,$$

and therefore, for large m, dividing by q_m yields

$$-R = q_1 + \frac{q_2}{R} + \frac{q_3}{R^2} + \frac{q_4}{R^3} + \cdots$$

and in view of (30), R satisfies

$$(44)\qquad R + q_1 + \frac{q_2}{R} + \frac{q_3}{R^2} + \frac{q_4}{R^3} + \cdots = R^{\frac{1}{2}}F(R^{\frac{1}{2}}) = 0,$$

provided that $R^{\frac{1}{2}}$ is a point of convergence of the series (30) for $F(w)$. In this case, invoking (31), we find:

$$(45)\qquad \begin{aligned} F(R) &= F^2(R^{\frac{1}{2}}) - \rho = -\rho \\ R &= f(-\rho) = -f(\rho). \end{aligned}$$

It would be interesting to determine the domain of values of R for which (45) holds. Within that domain, (45) can be used in the practical computation of R. For example, when $\rho = -2$ in (42), we find:

$$c_1 = 1,\quad c_3 = 0,\quad c_5 = -1,\quad c_7 = 1,\quad c_9 = -5,\quad c_{11} = 12,$$
$$c_{13} = -22,\quad c_{15} = 19,\quad c_{17} = 68, \cdots$$

The limiting ratio R of the sequence

$$\{-q_i\} = \{1, 1, 1, 2, 3, 6, 11, 23, 46, 98, 207, 451, 983, 2179, 4850, 10905, \cdots\}$$

of the number of binary coding trees is then given by (45) as

$$R = f(2) = 2 + \frac{1}{2} - \frac{1}{32} + \frac{1}{127} - \frac{5}{512} + \frac{12}{2048} - \frac{22}{8192} \pm \cdots = 2{\cdot}470\cdots,$$

which is in good agreement with the empirical value.

From (34), for $\rho < 0$, all the q_i's are negative, while for $\rho > 2$, the q_i's precisely alternate in sign. For $0 < \rho < 2$, the sign behavior of the q_i's is considerably more intricate, and is not the same throughout this interval. The relation $F^2(w) - \rho = F(w^2)$ is sufficient to insure the convergence of $F(w)$ at $w = |R^{\frac{1}{2}}|$ provided that $|R| < 1$ and that the q_i's become and remain, for large i, either all of the same sign, or strictly alternating in sign. This sufficient condition can be restated entirely in terms of the value of ρ: $F(w)$ must converge at $w = |R^{\frac{1}{2}}|$ for all $\rho < \rho_1 \approx -0.43$, and for all $\rho > \rho_2 = 2$. Whether this condition is also necessary has not been determined.

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