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**MUTUAL EXISTENCE OF SUM AND PRODUCT INTEGRALS**

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## MUTUAL EXISTENCE OF SUM AND PRODUCT INTEGRALS

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Functions are from  $R \times R$  to  $N$ , where  $R$  denotes the set of real numbers and  $N$  denotes a normed complete ring. If  $G$  has bounded variation on  $[a, b]$ , then  $\int_a^b G$  exists if and only if  ${}_x\Pi^\nu(1+G)$  exists for  $a \leq x < y \leq b$ . If each of  $\lim_{x \rightarrow p^+} H(p, x)$ ,  $\lim_{x \rightarrow p^-} H(x, p)$ ,  $\lim_{x, y \rightarrow p^+} H(x, y)$  and  $\lim_{x, y \rightarrow p^-} H(x, y)$  exists,  $G$  has bounded variation on  $[a, b]$  and either  $\int_a^b G$  exists or  ${}_x\Pi^\nu(1+G)$  exists for  $a \leq x < y \leq b$ , then  $\int_a^b HG$  and  $\int_a^b GH$  exist and  ${}_x\Pi^\nu(1+HG)$  and  ${}_x\Pi^\nu(1+GH)$  exist for  $a \leq x < y \leq b$ . If  $G$  has bounded variation on  $[a, b]$  and  $\nu$  is a nonnegative number, then  $\int_a^b G$  exists and  $\int_a^b \left| G - \int G \right| = \nu$  if and only if  ${}_x\Pi^\nu(1+G)$  exists for  $a \leq x < y \leq b$  and

$$\int_a^b |1 + G - \Pi(1 + G)| = \nu.$$

J. S. MacNerney [4] defines classes  $OA$  and  $OM$  of functions such that the integral-like formulas

$$V(a, b) = \int_a^b (W - 1) \quad \text{and} \quad W(a, b) = {}_a\Pi^b(1 + V)$$

are mutually reciprocal and establishes a one-to-one correspondence between the classes  $OA$  and  $OM$ . B. W. Helton [1] defines classes  $OA^\circ$  and  $OM^\circ$  of functions and shows that if  $G$  has bounded variation on  $[a, b]$ , then  $G \in OA^\circ$  on  $[a, b]$  if and only if  $G \in OM^\circ$  on  $[a, b]$ , where  $G \in OA^\circ$  on  $[a, b]$  only if  $\int_a^b G$  exists and  $\int_a^b \left| G - \int G \right| = 0$ , and  $G \in OM^\circ$  on  $[a, b]$  only if  ${}_x\Pi^\nu(1+G)$  exists for  $a \leq x < y \leq b$  and

$$\int_a^b |1 + G - \Pi(1 + G)| = 0.$$

The class  $OA$  is a proper subclass of  $OA^\circ$  and  $OM$  is closely related to the class  $OM^\circ$ . In the following, we establish a related result and show

that if  $G$  has bounded variation on  $[a, b]$ , then  $\int_a^b G$  exists if and only if  ${}_x\Pi^y(1+G)$  exists for  $a \leq x < y \leq b$ . This is not the same as the result of B. W. Helton since it is possible to construct a function  $G$  such that  $G$  has bounded variation on  $[a, b]$ ,  $\int_a^b G$  exists,  ${}_x\Pi^y(1+G)$  exists for  $a \leq x < y \leq b$ ,  $G \notin OA^\circ$  on  $[a, b]$  and  $G \notin OM^\circ$  on  $[a, b]$  [3]. We then use this result and ideas from another theorem of B. W. Helton [2, Theorem 2, p. 494] to establish that if each of  $\lim_{x \rightarrow p^+} H(p, x)$ ,  $\lim_{x \rightarrow p^-} H(x, p)$ ,  $\lim_{x, y \rightarrow p^+} H(x, y)$  and  $\lim_{x, y \rightarrow p^-} H(x, y)$  exists,  $G$  has bounded variation on  $[a, b]$  and either  $\int_a^b G$  exists or  ${}_x\Pi^y(1+G)$  exists for  $a \leq x < y \leq b$ , then  $\int_a^b HG$  and  $\int_a^b GH$  exist and  ${}_x\Pi^y(1+HG)$  and  ${}_x\Pi^y(1+GH)$  exist for  $a \leq x < y \leq b$ . Further, we show that if  $G$  has bounded variation on  $[a, b]$  and  $\nu$  is a nonnegative number, then  $G \in OA^\nu$  on  $[a, b]$  if and only if  $G \in OM^\nu$  on  $[a, b]$ , where  $G \in OA^\nu$  on  $[a, b]$  only if  $\int_a^b G$  exists and

$$\int_a^b \left| G - \int G \right| = \nu,$$

and  $G \in OM^\nu$  on  $[a, b]$  only if  ${}_x\Pi^y(1+G)$  exists for  $a \leq x < y \leq b$  and

$$\int_a^b |1 + G - \Pi(1 + G)| = \nu.$$

Finally, we show that if the norm used has the property that  $|AB| = |A||B|$  and if each of  $\lim_{x \rightarrow p^+} H(p, x)$ ,  $\lim_{x \rightarrow p^-} H(x, p)$ ,  $\lim_{x, y \rightarrow p^+} H(x, y)$  and  $\lim_{x, y \rightarrow p^-} H(x, y)$  exists,  $G$  has bounded variation on  $[a, b]$  and either  $G \in OA^\nu$  on  $[a, b]$  or  $G \in OM^\nu$  on  $[a, b]$ , then there exist nonnegative numbers  $\alpha$  and  $\beta$  such that  $HG$  is in  $OA^\alpha$  and  $OM^\alpha$  on  $[a, b]$  and  $GH$  is in  $OA^\beta$  and  $OM^\beta$  on  $[a, b]$ .

All integrals and definitions are of the subdivision-refinement type, and functions are from  $R \times R$  to  $N$ , where  $R$  denotes the set of real numbers and  $N$  denotes a ring which has a multiplicative identity element represented by 1 and has a norm  $|\cdot|$  with respect to which  $N$  is complete and  $|1| = 1$ . Unless noted otherwise, functions are assumed to be defined only for  $\{x, y\} \in R \times R$  such that  $x < y$ . The statement that  $G \in OB^\circ$  on  $[a, b]$  means that there exist a subdivision  $D$  of  $[a, b]$  and a number  $B$  such that if  $\{x_i\}_{i=0}^n$  is a refinement of  $D$ , then  $\sum_{i=1}^n |G_i| < B$ , where  $G_i$  denotes  $G(x_{i-1}, x_i)$ . When convenient, we use

$$\sum_{J(I)} G \quad \text{and} \quad \prod_{J(I)} (1 + G)$$

to denote

$$\sum_{i=1}^n G_i \quad \text{and} \quad \prod_{i=1}^n (1 + G_i),$$

respectively, where  $J = \{x_i\}_{i=0}^n$  represents a subdivision of some interval. The sets  $OA^\circ$ ,  $OM^\circ$ ,  $OA^\nu$  and  $OM^\nu$  have been defined previously, and  $G \in OA^+$  only if  $G$  is an additive function from  $R \times R$  to the nonnegative numbers. Also,  $G \in OM^*$  on  $[a, b]$  only if  ${}_x\Pi^\nu(1 + G)$  exists for  $a \leq x < y \leq b$  and if  $\epsilon > 0$  then there exists a subdivision  $D$  of  $[a, b]$  such that if  $\{x_i\}_{i=0}^n$  is a refinement of  $D$  and  $0 \leq p < q \leq n$ , then

$$\left| {}_{x_p}\Pi^{x_q}(1 + G) - \prod_{i=p+1}^q (1 + G_i) \right| < \epsilon.$$

The symbols  $G(p, p^+)$ ,  $G(p^-, p)$ ,  $G(p^+, p^+)$  and  $G(p^-, p^-)$  denote  $\lim_{x \rightarrow p^+} G(p, x)$ ,  $\lim_{x \rightarrow p^-} G(x, p)$ ,  $\lim_{x, y \rightarrow p^+} G(x, y)$  and  $\lim_{x, y \rightarrow p^-} G(x, y)$ , respectively, and  $G \in OL^\circ$  on  $[a, b]$  only if  $G(p, p^+)$ ,  $G(p^-, p)$ ,  $G(p^+, p^+)$  and  $G(p^-, p^-)$  exist for  $p \in [a, b]$ . Further,  $G \in S_2$  on  $[a, b]$  only if  $G(p, p^+)$  and  $G(p^-, p)$  exist for  $p \in [a, b]$ . Finally, statements of the form  $G > \beta$  should be interpreted in terms of subdivisions and refinements. See B. W. Helton [1] and J. S. MacNerney [4] for additional background.

We now establish an approximation theorem for product integrals. To do this, we initially develop a sequence of lemmas.

**LEMMA 1.1.** *If  $\beta > 0$ ,  $G$  is a function from  $R \times R$  to  $N$ ,  $|G| < 1 - \beta$  on  $[a, b]$ ,  $G \in OB^\circ$  on  $[a, b]$  and  ${}_x\Pi^\nu(1 + G)$  exists for  $a \leq x < y \leq b$ , then  $G \in OM^*$  on  $[a, b]$ .*

*Proof.* Let  $\epsilon > 0$ . There exist a subdivision  $D$  of  $[a, b]$  and a number  $B$  such that if  $\{x_i\}_{i=0}^n$  is a refinement of  $D$ , then

- (1)  $|G_i| < 1 - \beta$  for  $i = 1, 2, \dots, n$ ,
- (2)  $\Pi_{i=1}^n (1 + |G_i|) < B$ ,
- (3)  $\Pi_{i=1}^n (1 + \sum_{j=1}^\infty |(-1)^j G_i^j|) < B$ , and
- (4)  $|{}_a\Pi^b(1 + G) - \Pi_{i=1}^n (1 + G_i)| < \epsilon(3B)^{-1}$ .

Suppose  $\{x_i\}_{i=0}^n$  is a refinement of  $D$  and  $0 \leq p < q \leq n$ . Let  $Y = \{y_i\}_{i=0}^r$  and  $Z = \{z_i\}_{i=0}^s$  be refinements of  $\{x_i\}_{i=0}^p$  and  $\{x_i\}_{i=q}^n$ , respectively, such that

$$\left| \prod_{Y(I)} (1 + G) - {}_a\Pi^{x_p}(1 + G) \right| < \epsilon(3B^3)^{-1}$$

and

$$\left| - {}_{x_q}\Pi^b(1 + G) + \prod_{Z(I)} (1 + G) \right| < \epsilon(3B^2)^{-1}.$$

Further, let  $P$  and  $P'$  denote

$$\prod_{Y(I)} (1 + G) \quad \text{and} \quad {}_a\Pi^{x_p}(1 + G),$$

respectively, and let  $Q$  and  $Q'$  denote

$$\prod_{Z(I)} (1 + G) \quad \text{and} \quad {}_{x_q}\Pi^b(1 + G),$$

respectively. Note that  $P^{-1}$  and  $Q^{-1}$  exist and are

$$\prod_{i=1}^r \left[ 1 + \sum_{j=1}^{\infty} (-1)^j G^j (y_{r-i}, y_{r+1-i}) \right]$$

and

$$\prod_{i=1}^s \left[ 1 + \sum_{j=1}^{\infty} (-1)^j G^j (z_{s-i}, z_{s+1-i}) \right],$$

respectively.

Let  $W$  denote the subdivision  $D \cup Y \cup Z$  of  $[a, b]$ . Thus,

$$\begin{aligned} & \left| {}_{x_p}\Pi^{x_q}(1 + G) - \prod_{i=p+1}^q (1 + G_i) \right| \\ &= \left| P^{-1}P \left[ {}_{x_p}\Pi^{x_q}(1 + G) - \prod_{i=p+1}^q (1 + G_i) \right] Q Q^{-1} \right| \\ &\leq |P^{-1}| \left| P[{}_{x_p}\Pi^{x_q}(1 + G)]Q - P \left[ \prod_{i=p+1}^q (1 + G_i) \right] Q \right| |Q^{-1}| \\ &\leq B \left| P[{}_{x_p}\Pi^{x_q}(1 + G)]Q - \prod_{W(I)} (1 + G) \right| \\ &= B \left| [P - P' + P'] [{}_{x_p}\Pi^{x_q}(1 + G)] [Q' - Q' + Q] - \prod_{W(I)} (1 + G) \right| \end{aligned}$$

$$\begin{aligned}
&\leq B |P - P'| |_{x_p} \Pi^{x_a}(1 + G) | |Q| + B |_a \Pi^{x_a}(1 + G) | | - Q' + Q | \\
&\quad + B \left| {}_a \Pi^b(1 + G) - \prod_{\bar{w}(I)} (1 + G) \right| \\
&< B^3[\epsilon(3B^3)^{-1}] + B^2[\epsilon(3B^2)^{-1}] + B[\epsilon(3B)^{-1}] = \epsilon.
\end{aligned}$$

LEMMA 1.2. *If  $G$  is a function from  $R \times R$  to  $N$ ,  $G \in OB^\circ$  on  $[a, b]$  and  ${}_x \Pi^y(1 + G)$  exists for  $a \leq x < y \leq b$ , then  $G(a, a^+)$  and  $G(b^-, b)$  exist.*

*Proof.* We initially show that  $G(a, a^+)$  exists. Let  $\epsilon > 0$ . There exist numbers  $c$  and  $B$  such that  $a < c < b$  and if  $\{x_i\}_{i=0}^n$  is a subdivision of  $[a, c]$ , then

$$| -1 | \left[ \prod_{i=1}^n (1 + |G_i|) \right] < B \quad \text{and} \quad \sum_{i=2}^n |G_i| < \epsilon(4B^2)^{-1}.$$

Further, there exists a subdivision  $D = \{z_i\}_{i=0}^r$  of  $[a, c]$  such that if  $J$  and  $K$  are refinements of  $D$ , then

$$\left| \prod_{J(I)} (1 + G) - \prod_{K(I)} (1 + G) \right| < \epsilon/2.$$

We now suppose  $a < x < y < z_1$  and show that

$$|G(a, x) - G(a, y)| < \epsilon.$$

Let  $\{x_i\}_{i=0}^m$  and  $\{y_j\}_{j=0}^n$  denote  $D \cup \{x\}$  and  $D \cup \{y\}$ , respectively. Thus,

$$\begin{aligned}
\epsilon/2 &> \left| \prod_{i=1}^m (1 + G_i) - \prod_{j=1}^n (1 + G_j) \right| \\
&= \left| [1 + G(a, x)] \left[ \prod_{i=2}^m (1 + G_i) \right] - [1 + G(a, y)] \left[ \prod_{j=2}^n (1 + G_j) \right] \right| \\
&= \left| [1 + G(a, x)] \left[ 1 + \sum_{i=2}^m G_i \prod_{k=i+1}^m (1 + G_k) \right] \right. \\
&\quad \left. - [1 + G(a, y)] \left[ 1 + \sum_{j=2}^n G_j \prod_{k=j+1}^n (1 + G_k) \right] \right| \\
&\geq |G(a, x) - G(a, y)| - B \sum_{i=2}^m |G_i| \left| \prod_{k=i+1}^m (1 + G_k) \right| \\
&\quad - B \sum_{j=2}^n |G_j| \left| \prod_{k=j+1}^n (1 + G_k) \right|
\end{aligned}$$

$$> |G(a, x) - G(a, y)| - B^2[\epsilon(4B^2)^{-1}] + B^2[\epsilon(4B^2)^{-1}],$$

and hence,

$$\epsilon > |G(a, x) - G(a, y)|.$$

Since the existence of  $G(b^-, b)$  can be established in a similar manner, Lemma 1.2 follows.

LEMMA 1.3. *If  $\beta > 0$ ,  $G$  is a function from  $R \times R$  to  $N$ ,  $|G| < 1 - \beta$  on  $(a, b)$ ,  $G \in OB^\circ$  on  $[a, b]$  and  ${}_x\Pi^\beta(1 + G)$  exists for  $a \leq x < y \leq b$ , then  $G \in OM^*$  on  $[a, b]$ .*

*Proof.* Let  $\epsilon > 0$ . There exist a subdivision  $E_1$  of  $[a, b]$  and a number  $B > 1$  such that if  $\{x_i\}_{i=1}^m$  is a refinement of  $E_1$ , then

$$\prod_{i=1}^m (1 + |G_i|) < B$$

and

$$\left| {}_a\Pi^b(1 + G) - \prod_{i=1}^m (1 + G_i) \right| < \epsilon.$$

Let  $H$  be the function defined on  $[a, b]$  such that

$$H(x, y) = \begin{cases} G(x, y) & \text{if } x \neq a \text{ and } y \neq b \\ 0 & \text{if } x = a \text{ or } y = b. \end{cases}$$

Thus,  $H$  satisfies the hypothesis of Lemma 1.1, and hence, there exists a subdivision  $E_2$  of  $[a, b]$  such that if  $\{x_i\}_{i=0}^m$  is a refinement of  $E_2$  and  $0 \leq p < q \leq m$ , then

$$\left| {}_{x_p}\Pi^{x_q}(1 + H) - \prod_{i=p+1}^q (1 + H_i) \right| < \epsilon(3B)^{-1}.$$

It follows from Lemma 1.2 that  $G(a, a^+)$  and  $G(b^-, b)$  exist. Hence, there exists a point  $x$ , where  $a < x < b$ , such that if  $\{x_i\}_{i=0}^m$  and  $\{y_j\}_{j=0}^n$  are subdivisions of  $[a, x]$ ,  $1 \leq r \leq m$  and  $1 \leq s \leq n$ , then

$$\left| \prod_{i=1}^r (1 + G_i) - \prod_{j=1}^s (1 + G_j) \right| < \epsilon(3B)^{-1}.$$

Also, there exists a point  $y$ , where  $a < y < b$ , such that if  $\{x_i\}_{i=0}^m$  and  $\{y_j\}_{j=0}^n$  are subdivisions of  $[y, b]$ ,  $1 \leq r \leq m$  and  $1 \leq s \leq n$ , then

$$\left| \prod_{i=r}^m (1 + G_i) - \prod_{j=s}^n (1 + G_j) \right| < \epsilon (3B)^{-1}.$$

Let  $D$  denote the subdivision

$$E_1 \cup E_2 \cup \{x\} \cup \{y\}$$

of  $[a, b]$ . Further, suppose  $\{x_i\}_{i=0}^m$  is a refinement of  $D$  and  $0 \leq p < q \leq m$ . If  $p = 0$  and  $q = m$ , then the desired inequality follows from the existence of  ${}_a\Pi^b(1 + G)$ . If  $p \neq 0$  and  $q \neq m$ , then the inequality follows from the properties of the function  $H$ . Suppose  $p = 0$  and  $q \neq m$ . There exists a subdivision  $J$  of  $[a, x_1]$  such that

$$\left| {}_a\Pi^{x_1}(1 + G) - \prod_{J(J)} (1 + G) \right| < \epsilon (3B)^{-1}.$$

Thus,

$$\begin{aligned} & \left| {}_a\Pi^{x_q}(1 + G) - \prod_{i=1}^q (1 + G_i) \right| \\ & < \left| {}_a\Pi^{x_1}(1 + G) - (1 + G_1) \right| \left| {}_{x_1}\Pi^{x_q}(1 + G) \right| + B[\epsilon(3B)^{-1}] \\ & < B \left| \prod_{J(J)} (1 + G) - (1 + G_1) \right| + B[\epsilon(3B)^{-1}] + \epsilon/3 \\ & < B[\epsilon(3B)^{-1}] + 2\epsilon/3 = \epsilon. \end{aligned}$$

If  $p \neq 0$  and  $q = n$ , then a similar argument establishes the inequality. Therefore, Lemma 1.3 follows.

**THEOREM 1.** *If  $G$  is a function from  $R \times R$  to  $N$ ,  $G \in OB^\circ$  on  $[a, b]$  and  ${}_x\Pi^y(1 + G)$  exists for  $a \leq x < y \leq b$ , then  $G \in OM^*$  on  $[a, b]$ .*

*Proof.* Since  $G \in OB^\circ$  on  $[a, b]$ , there exists a subdivision  $\{x_i\}_{i=0}^m$  of  $[a, b]$  such that if  $1 \leq i \leq m$  and  $x_{i-1} < x < y < x_i$ , then  $|G(x, y)| < 1/2$ . Hence, this theorem can be established by using Lemma 1.3 and the identity

$$\prod_{i=1}^n a_i - \prod_{i=1}^n b_i = \sum_{i=1}^n \left( \prod_{j=1}^{i-1} b_j \right) (a_i - b_i) \left( \prod_{k=i+1}^n a_k \right),$$

where  $\prod_{j=1}^0 b_j = \prod_{k=n+1}^n a_k = 1$ .

We now use the approximation theorem to establish an existence theorem for sum integrals. In particular, we show that if  $G$  has



bounded variation on  $[a, b]$  and  ${}_x\Pi^y(1+G)$  exists for  $a \leq x < y \leq b$ , then  $\int_a^b G$  exists. Several lemmas are required.

LEMMA 2.1. *If  $G$  is a function from  $R \times R$  to  $N$ ,  $G \in OB^\circ$  on  $[a, b]$  and  ${}_x\Pi^y(1+G)$  exists for  $a \leq x < y \leq b$ , then*

$$\int_a^b G(u, v) {}_v\Pi^b(1+G)$$

*exists and is  $-1 + {}_a\Pi^b(1+G)$ .*

*Proof.* Let  $\epsilon > 0$ . There exist a subdivision  $E_1$  of  $[a, b]$  and a number  $B$  such that if  $\{x_i\}_{i=0}^m$  is a refinement of  $E_1$ , then

- (1)  $\sum_{i=1}^m |G_i| < B$ , and
- (2)  $|\Pi_{i=1}^m(1+G_i) - {}_a\Pi^b(1+G)| < \epsilon/2$ .

Theorem 1 implies that  $G \in OM^*$  on  $[a, b]$ , and hence, there exists a subdivision  $E_2$  of  $[a, b]$  such that if  $\{x_i\}_{i=0}^m$  is a refinement of  $E_2$  and  $0 \leq p < q \leq m$ , then

$$\left| {}_{x_p}\Pi^{x_q}(1+G) - \prod_{i=p+1}^q (1+G_i) \right| < \epsilon(2B)^{-1}.$$

Let  $D$  denote the subdivision  $E_1 \cup E_2$  of  $[a, b]$  and suppose  $\{x_i\}_{i=0}^m$  is a refinement of  $D$ . Thus,

$$\begin{aligned} & \left| \sum_{i=1}^m G_i [{}_{{x_i}}\Pi^b(1+G)] - [-1 + {}_a\Pi^b(1+G)] \right| \\ & < \left| \sum_{i=1}^m G_i [{}_{{x_i}}\Pi^b(1+G)] + 1 - \prod_{i=1}^m (1+G_i) \right| + \epsilon/2 \\ & = \left| \sum_{i=1}^m G_i [{}_{{x_i}}\Pi^b(1+G)] + 1 - \left[ 1 + \sum_{i=1}^m G_i \prod_{k=i+1}^m (1+G_k) \right] \right| + \epsilon/2 \\ & \leq \sum_{i=1}^m |G_i| \left| {}_{x_i}\Pi^b(1+G) - \prod_{k=i+1}^m (1+G_k) \right| + \epsilon/2 \\ & < B[\epsilon(2B)^{-1}] + \epsilon/2 = \epsilon. \end{aligned}$$

LEMMA 2.2. *If  $H$  and  $G$  are functions from  $R \times R$  to  $N$ ,  $H \in OL^\circ$  on  $[a, b]$ ,  $G \in OB^\circ$  on  $[a, b]$  and  $\int_a^b G$  exists, then  $\int_a^b HG$  exists and  $\int_a^b GH$  exists.*

*Proof.* B. W. Helton [2, Theorem 2, p. 494] proves that  $HG$  and  $GH$  are in  $OA^\circ$  on  $[a, b]$  with the hypothesis of Lemma 2.2 and the additional restriction that  $G \in OA^\circ$  on  $[a, b]$ . This lemma follows by essentially the same argument.

Observe that weakening the hypothesis of Helton's result by requiring only the existence of  $\int_a^b G$  produces a corresponding weakening of the conclusion since we now have that  $\int_a^b HG$  and  $\int_a^b GH$  exist rather than that  $HG$  and  $GH$  are in  $OA^\circ$  on  $[a, b]$ .

Lemma 2.2 is not true for functions defined on a linearly ordered set [4, p. 149]. For example, consider

$$S = [0, 1) \cup (1, 2],$$

with the usual ordering for the real numbers. Let  $G$  be the function defined on  $S \times S$  such that

$$G(x, y) = \begin{cases} 1 & \text{if } x < 1 \text{ and } y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus,  $G \in OA^\circ \cap OB^\circ$  on  $S \times S$ . Let  $H$  be the function defined on  $S \times S$  such that

$$H(x, y) = \begin{cases} 1 & \text{if } x < 1, y > 1 \text{ and } x \text{ rational} \\ -1 & \text{if } x < 1, y > 1 \text{ and } x \text{ irrational} \\ 0 & \text{otherwise.} \end{cases}$$

Thus,  $H \in OL^\circ$  on  $S \times S$ . However,  $\int_a^b HG$  does not exist.

LEMMA 2.3. *If  $\beta > 0$ ,  $G$  is a function from  $R \times R$  to  $N$ ,  $|G| < 1 - \beta$  on  $[a, b]$ ,  $G \in OB^\circ$  on  $[a, b]$  and  ${}_a\Pi^b(1 + G)$  exists, then  ${}_b\Pi^a(1 + H)$  exists and is  $[_a\Pi^b(1 + G)]^{-1}$ , where*

$$H(y, x) = \sum_{j=1}^{\infty} (-1)^j G^j(x, y)$$

for  $a \leq x < y \leq b$ .

*Proof.* We initially show that  ${}_b\Pi^a(1 + H)$  exists. Let  $\epsilon > 0$ . There exist a subdivision  $D$  of  $[a, b]$  and a number  $B$  such that if  $\{x_i\}_{i=0}^m$  and  $\{y_j\}_{j=0}^n$  are refinements of  $D$ , then

- (1)  $|G_i| < 1 - \beta$  for  $i = 1, 2, \dots, m$ ,
- (2)  $|\prod_{i=1}^m (1 + H_{m+1-i})| < B$ , and
- (3)  $|\prod_{i=1}^m (1 + G_i) - \prod_{j=1}^n (1 + G_j)| < \epsilon B^{-2}$ .

Note that we are using  $H_{m+1-i}$  to denote  $H(x_{m+1-i}, x_{m-i})$ . Suppose  $\{x_i\}_{i=0}^m$  and  $\{y_j\}_{j=0}^n$  are refinements of  $D$ . Thus,

$$\begin{aligned}
 & \left| \prod_{i=1}^m (1 + H_{m+1-i}) - \prod_{j=1}^n (1 + H_{n+1-j}) \right| \\
 & \leq \left| \prod_{i=1}^m (1 + H_{m+1-i}) \right| \left| 1 - \left[ \prod_{i=1}^m (1 + H_{m+1-i}) \right]^{-1} \left[ \prod_{j=1}^n (1 + H_{n+1-j}) \right] \right| \\
 & \leq B \left| 1 - \left[ \prod_{i=1}^m (1 + G_i) \right] \left[ \prod_{j=1}^n (1 + H_{n+1-j}) \right] \right| \\
 & \leq B \left| \prod_{j=1}^n (1 + G_j) - \prod_{i=1}^m (1 + G_i) \right| \left| \prod_{j=1}^n (1 + H_{n+1-j}) \right| \\
 & \quad + B \left| 1 - \left[ \prod_{j=1}^n (1 + G_j) \right] \left[ \prod_{j=1}^n (1 + H_{n+1-j}) \right] \right| \\
 & < B^2(\epsilon B^{-2}) + B(0) = \epsilon.
 \end{aligned}$$

We now show that  $[_a \Pi^b(1 + G)]^{-1}$  exists and is  $_b \Pi^a(1 + H)$ . Let  $\epsilon > 0$ . There exists a subdivision  $\{x_i\}_{i=0}^m$  of  $[a, b]$  such that

$$\left| [_a \Pi^b(1 + G)][_b \Pi^a(1 + H)] - \left[ \prod_{i=1}^m (1 + G_i) \right] \left[ \prod_{i=1}^m (1 + H_{m+1-i}) \right] \right| < \epsilon.$$

Hence,

$$\begin{aligned}
 & |[_a \Pi^b(1 + G)][_b \Pi^a(1 + H)] - 1| \\
 & < \left| \left[ \prod_{i=1}^m (1 + G_i) \right] \left[ \prod_{i=1}^m (1 + H_{m+1-i}) \right] - 1 \right| + \epsilon \\
 & = 0 + \epsilon = \epsilon.
 \end{aligned}$$

**LEMMA 2.4.** *If  $\beta > 0$ ,  $G$  is a function from  $R \times R$  to  $N$ ,  $|G| < 1 - \beta$  on  $[a, b]$ ,  $G \in OB^\circ$  on  $[a, b]$  and  $_x \Pi^y(1 + G)$  exists for  $a \leq x < y \leq b$ , then  $\int_a^b G$  exists.*

*Proof.* It follows from Lemma 2.1 that

$$\int_a^b G(u, v) {}_v \Pi^b(1 + G)$$

exists. Let  $H$  be the function defined on  $[a, b]$  such that

$$H(u, v) = [{}_v\Pi^b(1 + G)]^{-1}.$$

The existence of  $H$  follows from Lemma 2.3. Further,  $H \in OL^\circ$  on  $[a, b]$ . Hence, the existence of  $\int_a^b G$  can be established by using Lemma 2.2.

LEMMA 2.5. *If  $\beta > 0$ ,  $G$  is a function from  $R \times R$  to  $N$ ,  $|G| < 1 - \beta$  on  $(a, b)$ ,  $G \in OB^\circ$  on  $[a, b]$  and  ${}_x\Pi^y(1 + G)$  exists for  $a \leq x < y \leq b$ , then  $\int_a^b G$  exists.*

*Proof.* Lemma 2.5 follows by using Lemma 1.2 and Lemma 2.4.

THEOREM 2. *If  $G$  is a function from  $R \times R$  to  $N$ ,  $G \in OB^\circ$  on  $[a, b]$  and  ${}_x\Pi^y(1 + G)$  exists for  $a \leq x < y \leq b$ , then  $\int_a^b G$  exists.*

*Proof.* There exists a subdivision  $\{x_i\}_{i=0}^m$  of  $[a, b]$  such that if  $1 \leq i \leq m$  and  $x_{i-1} < x < y < x_i$ , then  $|G(x, y)| < 1/2$ . Hence, the theorem follows from Lemma 2.5.

An existence theorem for product integrals is now established. In particular, we show that if  $G$  has bounded variation on  $[a, b]$  and  $\int_a^b G$  exists, then  ${}_x\Pi^y(1 + G)$  exists for  $a \leq x < y \leq b$ .

LEMMA 3.1. *If  $G$  is a function from  $R \times R$  to  $N$  such that  $G \in OB^\circ$  on  $[a, b]$ , then there exists  $\alpha \in OA^+$  on  $[a, b]$  such that*

$$|G(x, y)| \leq \alpha(x, y)$$

for  $a \leq x < y \leq b$ .

*Proof.* There exist a subdivision  $\{x_i\}_{i=0}^n$  of  $[a, b]$  and a number  $B$  such that if  $H$  is a refinement of  $\{x_i\}_{i=0}^n$ , then  $\sum_{H(t)} |G| < B$ . Let  $g$  be the function such that for  $x_{p-1} < x \leq x_p$ ,  $g(x) = \text{lub } \sum_{H(t)} |G|$  for all refinements  $H$  of  $\{x_i\}_{i=0}^{p-1} \cup \{x\}$ . Let  $\alpha(x, y) = \int_x^y dg$ . This produces the desired function.

THEOREM 3. *If  $G$  is a function from  $R \times R$  to  $N$ ,  $G \in OB^\circ$  on  $[a, b]$  and  $\int_a^b G$  exists, then  ${}_x\Pi^y(1 + G)$  exists for  $a \leq x < y \leq b$ .*

*Proof.* Suppose  $a \leq x < y \leq b$ . In the following we show that  ${}_x\Pi^y(1+G)$  exists and is  $\sum_{p=0}^{\infty} G_p(x, y)$ , where  $G_0(x, y) = 1$  and

$$G_p(x, y) = (R) \int_x^y G \cdot G_{p-1}(\quad, y)$$

for  $p = 1, 2, \dots$ . The existence of these integrals follows from Lemma 2.2.

It follows from Lemma 3.1 that there exists  $\alpha \in OA^+$  such that if  $x \leq r < s \leq y$ , then

$$|G(r, s)| \leq \alpha(r, s).$$

Further, from a result of MacNerney [4, Theorem 6.2, p. 160],  $\sum_{p=0}^{\infty} g_p(x, y)$  exists, where  $g_0(x, y) = 1$  and

$$g_p(x, y) = (R) \int_x^y \alpha \cdot g_{p-1}(\quad, y)$$

for  $p = 1, 2, \dots$ .

It can be established by induction that if  $\{x_i\}_{i=0}^n$  is a subdivision of  $[x, y]$ , then

$$\begin{aligned} \prod_{i=1}^n (1 + G_i) &= 1 + \sum_{k_1=1}^n G_{k_1} + \sum_{k_1=1}^n \sum_{k_2=k_1+1}^n G_{k_1} G_{k_2} + \dots \\ &\quad + \sum_{k_1=1}^n \sum_{k_2=k_1+1}^n \dots \sum_{k_n=k_{n-1}+1}^n G_{k_1} G_{k_2} \dots G_{k_n}, \end{aligned}$$

where  $\sum_{i=p}^q G_i = 0$  if  $p > q$ . Further, it can also be established by induction that

$$\left| \sum_{k_1=1}^n \sum_{k_2=k_1+1}^n \dots \sum_{k_p=k_{p-1}+1}^n G_{k_1} G_{k_2} \dots G_{k_p} \right| \leq g_p(x, y)$$

for  $p = 1, 2, \dots$ .

Let  $\epsilon > 0$ . There exists a positive integer  $N$  such that

$$\sum_{p=N+1}^{\infty} g_p(x, y) < \epsilon/3.$$

Further, there exists a subdivision  $D$  of  $[x, y]$  such that if  $\{x_i\}_{i=0}^n$  is a refinement of  $D$ , then

$$\left| \left[ 1 + \sum_{k_1=1}^n G_{k_1} + \sum_{k_1=1}^n \sum_{k_2=k_1+1}^n G_{k_1} G_{k_2} + \cdots \right. \right. \\ \left. \left. + \sum_{k_1=1}^n \sum_{k_2=k_1+1}^n \cdots \sum_{k_N=k_{N-1}+1}^n G_{k_1} G_{k_2} \cdots G_{k_N} \right] - \sum_{p=0}^N G_p(x, y) \right| < \epsilon/3.$$

Suppose  $\{x_i\}_{i=0}^n$  is a refinement of  $D$ . Thus,

$$\left| \prod_{i=1}^n (1 + G_i) - \sum_{p=0}^{\infty} G_p(x, y) \right| \\ = \left| \left[ 1 + \sum_{k_1=1}^n G_{k_1} + \sum_{k_1=1}^n \sum_{k_2=k_1+1}^n G_{k_1} G_{k_2} + \cdots \right. \right. \\ \left. \left. + \sum_{k_1=1}^n \sum_{k_2=k_1+1}^n \cdots \sum_{k_n=k_{n-1}+1}^n G_{k_1} G_{k_2} \cdots G_{k_n} \right] - \sum_{p=0}^{\infty} G_p(x, y) \right| \\ < \left| \left[ 1 + \sum_{k_1=1}^n G_{k_1} + \sum_{k_1=1}^n \sum_{k_2=k_1+1}^n G_{k_1} G_{k_2} + \cdots \right. \right. \\ \left. \left. + \sum_{k_1=1}^n \sum_{k_2=k_1+1}^n \cdots \sum_{k_N=k_{N-1}+1}^n G_{k_1} G_{k_2} \cdots G_{k_N} \right] - \sum_{p=0}^N G_p(x, y) \right| \\ + \epsilon/3 + \epsilon/3 \\ < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

**THEOREM 4.** If  $G$  is a function from  $R \times R$  to  $N$  and  $G \in OB^\circ$  on  $[a, b]$ , then  $\int_a^b G$  exists if and only if  ${}_x\Pi^v(1 + G)$  exists for  $a \leq x < y \leq b$ .

*Proof.* This theorem follows as a corollary to Theorems 2 and 3.

**THEOREM 5.** If  $H$  and  $G$  are functions from  $R \times R$  to  $N$ ,  $H \in OL^\circ$  on  $[a, b]$ ,  $G \in OB^\circ$  on  $[a, b]$  and either  $\int_a^b G$  exists or  ${}_x\Pi^v(1 + G)$  exists for  $a \leq x < y \leq b$ , then  $\int_a^b HG$  and  $\int_a^b GH$  exist and  ${}_x\Pi^v(1 + HG)$  and  ${}_x\Pi^v(1 + GH)$  exist for  $a \leq x < y \leq b$ .

*Proof.* This theorem follows as a corollary to Theorem 4 and Lemma 2.2.

We now show that if  $G$  has bounded variation on  $[a, b]$ , then  $G \in OA^v$  on  $[a, b]$  if and only if  $G \in OM^v$  on  $[a, b]$ . This is a generalization of a result of B. W. Helton [1, Theorem 3.4, p. 301].

LEMMA 6.1. *If  $\epsilon > 0$  and  $G$  is a function from  $R \times R$  to  $N$  such that  $G \in OB^\circ$  and  $S_2$  on  $[a, b]$ , then there exists a subdivision  $D$  of  $[a, b]$  such that if  $\{x_i\}_{i=0}^n$  is a refinement of  $D$ ,  $1 \leq i \leq n$  and  $\{x_{ij}\}_{j=0}^{n(i)}$  is a subdivision of  $[x_{i-1}, x_i]$ , then*

$$\left| \prod_{j=1}^{n(i)} (1 + G_{ij}) - \left( 1 + \sum_{j=1}^{n(i)} G_{ij} \right) \right| < \epsilon.$$

*Proof.* Since  $G \in OB^\circ \cap S_2$  on  $[a, b]$ , this lemma can be established by applying the covering theorem.

LEMMA 6.2. *If  $\epsilon > 0$  and  $G$  is a function from  $R \times R$  to  $N$  such that  $G \in OB^\circ$  and  $S_2$  on  $[a, b]$ , then there exists a subdivision  $D$  of  $[a, b]$  such that if  $\{x_i\}_{i=0}^n$  is a refinement of  $D$  and  $\{x_{ij}\}_{j=0}^{n(i)}$  is a subdivision of  $[x_{i-1}, x_i]$  for  $1 \leq i \leq n$ , then*

$$\sum_{i=1}^n \left| \prod_{j=1}^{n(i)} (1 + G_{ij}) - \left( 1 + \sum_{j=1}^{n(i)} G_{ij} \right) \right| < \epsilon.$$

*Proof.* There exist a subdivision  $\{r_i\}_{i=0}^r$  of  $[a, b]$  and a number  $B$  such that if  $\{y_i\}_{i=0}^m$  is a refinement of  $\{r_i\}_{i=0}^r$ , then

- (1)  $\sum_{i=1}^m |G_i| < B$ , and
- (2)  $\prod_{i=1}^m (1 + |G_i|) < B$ .

It follows by applying the covering theorem that there exists a subdivision  $\{s_i\}_{i=0}^s$  of  $[a, b]$  such that if  $1 \leq i \leq s$  and  $\{x_{ij}\}_{j=0}^{s(i)}$  is a subdivision of  $[s_{i-1}, s_i]$ , then

$$\sum_{j=2}^{s(i)-1} |G_{ij}| < \epsilon (2B^2)^{-1}.$$

Further, it follows from Lemma 6.1 that there exists a subdivision  $\{t_i\}_{i=0}^t$  of  $[a, b]$  such that if  $\{x_i\}_{i=0}^n$  is a refinement of  $\{t_i\}_{i=0}^t$ ,  $1 \leq i \leq n$  and  $\{x_{ij}\}_{j=0}^{n(i)}$  is a subdivision of  $[x_{i-1}, x_i]$ , then

$$\left| \prod_{j=1}^{n(i)} (1 + G_{ij}) - \left( 1 + \sum_{j=1}^{n(i)} G_{ij} \right) \right| < \epsilon (4s)^{-1}.$$

Let  $D$  denote the subdivision

$$\{r_i\}_{i=0}^r \cup \{s_i\}_{i=0}^s \cup \{t_i\}_{i=0}^t$$

of  $[a, b]$  and suppose  $\{x_i\}_{i=0}^n$  is a refinement of  $D$ . Further, suppose  $\{x_{ij}\}_{j=0}^{n(i)}$  is a subdivision of  $[x_{i-1}, x_i]$  for  $1 \leq i \leq n$ . Let  $P$  be the subset of  $\{i\}_{i=1}^n$  such that  $i \in P$  only if  $x_i \in \{s_i\}_{i=0}^s$  or  $x_{i-1} \in \{s_i\}_{i=0}^s$ . Finally, let

$$Q = \{i\}_{i=1}^n - P.$$

In the following manipulations, we use the identity

$$\prod_{i=1}^n (1 + b_i) = 1 + \sum_{i=1}^n b_i + \sum_{i=1}^n b_i \left\{ \sum_{j=i+1}^n b_j \left[ \prod_{k=j+1}^n (1 + b_k) \right] \right\},$$

where  $\sum_{j=n+1}^n b_j = 0$  and  $\prod_{k=n+1}^n (1 + b_k) = 1$ . This result can be established by induction.

We now establish the desired inequality:

$$\begin{aligned} & \sum_{i=1}^n \left| \prod_{j=1}^{n(i)} (1 + G_{ij}) - \left( 1 + \sum_{j=1}^{n(i)} G_{ij} \right) \right| \\ &= \sum_{i \in Q} \left| \prod_{j=1}^{n(i)} (1 + G_{ij}) - \left( 1 + \sum_{j=1}^{n(i)} G_{ij} \right) \right| \\ & \quad + \sum_{i \in P} \left| \prod_{j=1}^{n(i)} (1 + G_{ij}) - \left( 1 + \sum_{j=1}^{n(i)} G_{ij} \right) \right| \\ &< \sum_{i \in Q} \left| 1 + \sum_{j=1}^{n(i)} G_{ij} + \sum_{j=1}^{n(i)} G_{ij} \left\{ \sum_{u=j+1}^{n(i)} G_{iu} \left[ \prod_{v=u+1}^{n(i)} (1 + G_{iv}) \right] \right\} \right. \\ & \quad \left. - \left( 1 + \sum_{j=1}^{n(i)} G_{ij} \right) \right| + 2s[\epsilon(4s)^{-1}] \\ &= \sum_{i \in Q} \left| \sum_{j=1}^{n(i)} G_{ij} \left\{ \sum_{u=j+1}^{n(i)} G_{iu} \left[ \prod_{v=u+1}^{n(i)} (1 + G_{iv}) \right] \right\} \right| + \epsilon/2 \\ &\leq \sum_{i \in Q} \sum_{j=1}^{n(i)} |G_{ij}| \left\{ \sum_{u=j+1}^{n(i)} |G_{iu}| \left[ \prod_{v=u+1}^{n(i)} (1 + |G_{iv}|) \right] \right\} + \epsilon/2 \\ &\leq B \sum_{i \in Q} \sum_{j=1}^{n(i)} |G_{ij}| \left\{ \sum_{u=j+1}^{n(i)} |G_{iu}| \right\} + \epsilon/2 \\ &\leq B[\epsilon(2B^2)^{-1}] \sum_{i \in Q} \sum_{j=1}^{n(i)} |G_{ij}| + \epsilon/2 \\ &< B[\epsilon(2B^2)^{-1}]B + \epsilon/2 = \epsilon. \end{aligned}$$

LEMMA 6.3. *If  $G$  is a function from  $R \times R$  to  $N$ ,  $G \in OB^\circ$  on  $[a, b]$  and  $\int_a^b G$  exists, then*

$$\int_a^b \left| \Pi(1 + G) - \left( 1 + \int G \right) \right| = 0.$$



*Proof.* The existence of  ${}_x\Pi^v(1+G)$  for  $a \leq x < y \leq b$  follows from Theorem 3. Also, since  $G \in OB^\circ$  on  $[a, b]$  and  $\int_a^b G$  exists,  $G \in S_2$  on  $[a, b]$ .

Let  $\epsilon > 0$ . It follows from Lemma 6.2 that there exists a subdivision  $D$  of  $[a, b]$  such that if  $\{x_i\}_{i=0}^n$  is a refinement of  $D$  and  $\{x_{ij}\}_{j=0}^{n(i)}$  is a subdivision of  $[x_{i-1}, x_i]$  for  $1 \leq i \leq n$ , then

$$\sum_{i=1}^n \left| \prod_{j=1}^{n(i)} (1 + G_{ij}) - \left( 1 + \sum_{j=1}^{n(i)} G_{ij} \right) \right| < \epsilon/3.$$

Suppose  $\{x_i\}_{i=0}^n$  is a refinement of  $D$ . For  $1 \leq i \leq n$ , let  $\{x_{ij}\}_{j=0}^{n(i)}$  be a subdivision of  $[x_{i-1}, x_i]$  such that

$$\left| {}_{x_{i-1}}\Pi^{x_i}(1+G) - \prod_{j=1}^{n(i)} (1 + G_{ij}) \right| < \epsilon/3n$$

and

$$\left| \sum_{j=1}^{n(i)} G_{ij} - \int_{x_{i-1}}^{x_i} G \right| < \epsilon/3n.$$

Thus,

$$\begin{aligned} & \sum_{i=1}^n \left| {}_{x_{i-1}}\Pi^{x_i}(1+G) - \left( 1 + \int_{x_{i-1}}^{x_i} G \right) \right| \\ & \leq \sum_{i=1}^n \left| {}_{x_{i-1}}\Pi^{x_i}(1+G) - \prod_{j=1}^{n(i)} (1 + G_{ij}) \right| \\ & \quad + \sum_{i=1}^n \left| \prod_{j=1}^{n(i)} (1 + G_{ij}) - \left( 1 + \sum_{j=1}^{n(i)} G_{ij} \right) \right| \\ & \quad + \sum_{i=1}^n \left| \sum_{j=1}^{n(i)} G_{ij} - \int_{x_{i-1}}^{x_i} G \right| \\ & < n(\epsilon/3n) + \epsilon/3 + n(\epsilon/3n) = \epsilon. \end{aligned}$$

**THEOREM 6.** *If  $v$  is a nonnegative number,  $G$  is a function from  $R \times R$  to  $N$  and  $G \in OB^\circ$  on  $[a, b]$ , then  $G \in OA^v$  on  $[a, b]$  if and only if  $G \in OM^v$  on  $[a, b]$ .*

*Proof.* Suppose  $G \in OM^v$  on  $[a, b]$ . It follows from Theorem 2 that  $\int_a^b G$  exists. Hence, it is only necessary to show that

$$\int_a^b \left| G - \int G \right| = \nu.$$

Let  $\epsilon > 0$ . There exists a subdivision  $D_1$  of  $[a, b]$  such that if  $\{x_i\}_{i=0}^n$  is a refinement of  $D_1$ , then

$$\nu - \epsilon/2 < \sum_{i=1}^n \left| 1 + G_i - \prod_{x_{i-1}}^{x_i} (1 + G) \right| < \nu + \epsilon/2.$$

Further, it follows from Lemma 6.3 that there exists a subdivision  $D_2$  of  $[a, b]$  such that if  $\{x_i\}_{i=0}^n$  is a refinement of  $D_2$ , then

$$\sum_{i=1}^n \left| \prod_{x_{i-1}}^{x_i} (1 + G) - \left( 1 + \int_{x_{i-1}}^{x_i} G \right) \right| < \epsilon(2| - 1|)^{-1}.$$

Let  $D = D_1 \cup D_2$ . Suppose  $\{x_i\}_{i=0}^n$  is a refinement of  $D$ . Now,

$$\begin{aligned} \sum_{i=1}^n \left| G_i - \int_{x_{i-1}}^{x_i} G \right| \\ = \sum_{i=1}^n \left| [1 + G_i - \prod_{x_{i-1}}^{x_i} (1 + G)] \right. \\ \left. + \left[ \prod_{x_{i-1}}^{x_i} (1 + G) - \left( 1 + \int_{x_{i-1}}^{x_i} G \right) \right] \right|. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{i=1}^n \left| G_i - \int_{x_{i-1}}^{x_i} G \right| \\ \leq \sum_{i=1}^n \left| 1 + G_i - \prod_{x_{i-1}}^{x_i} (1 + G) \right| \\ + \sum_{i=1}^n \left| \prod_{x_{i-1}}^{x_i} (1 + G) - \left( 1 + \int_{x_{i-1}}^{x_i} G \right) \right| \\ < \nu + \epsilon/2 + \epsilon/2 = \nu + \epsilon. \end{aligned}$$

Further,

$$\begin{aligned} \sum_{i=1}^n \left| G_i - \int_{x_{i-1}}^{x_i} G \right| \\ \geq \sum_{i=1}^n \left| 1 + G_i - \prod_{x_{i-1}}^{x_i} (1 + G) \right| \end{aligned}$$

$$\begin{aligned}
& - \left| -1 \right| \sum_{i=1}^n \left| G_i - \int_{x_{i-1}}^{x_i} G \right| \\
& > \nu - \epsilon/2 - \epsilon/2 = \nu - \epsilon.
\end{aligned}$$

Hence,

$$\nu - \epsilon < \sum_{i=1}^n \left| G_i - \int_{x_{i-1}}^{x_i} G \right| < \nu + \epsilon.$$

Therefore,  $G \in OA^\nu$  on  $[a, b]$ .

Suppose  $G \in OA^\nu$  on  $[a, b]$ . It follows from Theorem 3 that  ${}_x\Pi^\nu(1+G)$  exists for  $a \leq x < y \leq b$ . Hence, it is only necessary to show that

$$\int_a^b |1+G - \Pi(1+G)| = \nu.$$

Let  $\epsilon > 0$ . There exists a subdivision  $D_1$  of  $[a, b]$  such that if  $\{x_i\}_{i=0}^n$  is a refinement of  $D_1$ , then

$$\nu - \epsilon/2 < \sum_{i=1}^n \left| G_i - \int_{x_{i-1}}^{x_i} G \right| < \nu + \epsilon/2.$$

Further, it follows from Lemma 6.3 that there exists a subdivision  $D_2$  of  $[a, b]$  such that if  $\{x_i\}_{i=0}^n$  is a refinement of  $D_2$ , then

$$\sum_{i=1}^n \left| 1 + \int_{x_{i-1}}^{x_i} G - {}_{x_{i-1}}\Pi^{x_i}(1+G) \right| < \epsilon(2|-1|)^{-1}.$$

Let  $D = D_1 \cup D_2$ . Suppose  $\{x_i\}_{i=0}^n$  is a refinement of  $D$ . Now,

$$\begin{aligned}
& \sum_{i=1}^n \left| 1 + G_i - {}_{x_{i-1}}\Pi^{x_i}(1+G) \right| \\
& = \sum_{i=1}^n \left| \left[ G_i - \int_{x_{i-1}}^{x_i} G \right] + \left[ 1 + \int_{x_{i-1}}^{x_i} G - {}_{x_{i-1}}\Pi^{x_i}(1+G) \right] \right|.
\end{aligned}$$

It follows as in the preceding argument that

$$\nu - \epsilon < \sum_{i=1}^n \left| 1 + G_i - {}_{x_{i-1}}\Pi^{x_i}(1+G) \right| < \nu + \epsilon.$$

Therefore,  $G \in OM^\nu$  on  $[a, b]$ .

We now prove a theorem on the existence of integrals of products of functions. This result is related to a theorem by B. W. Helton [2, Theorem 2, p. 494].

LEMMA 7.1. *If  $\epsilon > 0$ ,  $H$  is a function from  $R \times R$  to  $N$  and  $H \in OL^\circ$  on  $[a, b]$ , then there exist a subdivision  $\{t_i\}_{i=0}^t$  of  $[a, b]$  and a sequence  $\{k_i\}_{i=1}^t$  such that if  $1 \leq i \leq t$  and  $t_{i-1} < x < y < t_i$ , then*

$$|H(x, y) - k_i| < \epsilon.$$

*Proof.* This lemma is a variation of a lemma used by B. W. Helton [2, Lemma, p. 498]. The proof presented there can be used to establish the lemma as we have stated it.

LEMMA 7.2. *Suppose  $|AB| = |A||B|$  for  $A, B \in N$ . If  $\nu$  is a nonnegative number,  $k \in N$ ,  $G$  is a function from  $R \times R$  to  $N$  and  $G \in OA^\nu$  on  $[a, b]$ , then  $kG \in OA^{[k]\nu}$  on  $[a, b]$ .*

*Proof.* Since  $|AB| = |A||B|$ , the proof is readily constructed. If the preceding equality did not hold, the lemma would not necessarily follow. An example of such a situation is presented after the proof of Theorem 7.

THEOREM 7. *Suppose  $|AB| = |A||B|$  for  $A, B \in N$ . If  $\nu$  is a nonnegative number,  $H$  and  $G$  are functions from  $R \times R$  to  $N$ ,  $H \in OL^\circ$  on  $[a, b]$ ,  $G \in OB^\circ$  on  $[a, b]$  and either  $G \in OA^\nu$  on  $[a, b]$  or  $G \in OM^\nu$  on  $[a, b]$ , then there exist nonnegative numbers  $\alpha$  and  $\beta$  such that  $HG$  is in  $OA^\alpha$  and  $OM^\alpha$  on  $[a, b]$  and  $GH$  is in  $OA^\beta$  and  $OM^\beta$  on  $[a, b]$ .*

*Proof.* We initially establish that there exists a nonnegative number  $\alpha$  such that  $HG \in OA^\alpha$  on  $[a, b]$ . It follows from Theorem 6 that  $G \in OA^\nu$  on  $[a, b]$ . Hence, the existence of  $\int_a^b HG$  follows from Theorem 5. We use the Cauchy criterion to establish the existence of

$$\int_a^b \left| HG - \int HG \right|.$$

Let  $\epsilon > 0$ . There exist a subdivision  $E_1$  of  $[a, b]$  and a number  $B$  such that if  $\{x_i\}_{i=0}^n$  is a refinement of  $E_1$ , then

$$\sum_{i=1}^n |G_i| < B.$$

It follows from Lemma 7.1 that there exist a subdivision  $E_2 = \{t_i\}_{i=0}^t$  of  $[a, b]$  and a sequence  $\{k_i\}_{i=1}^t$  such that if  $1 \leq i \leq t$  and  $t_{i-1} < x < y < t_i$ , then

$$|H(x, y) - k_i| < \epsilon(8| - 1|B)^{-1}.$$

Since  $G \in OB^\circ \cap OA^\nu$  on  $[a, b]$ , it follows that there exist subdivisions  $\{r_i\}_{i=0}^{t+1}$  and  $\{s_i\}_{i=0}^{t+1}$  of  $[a, b]$  such that

$$(1) \quad t_{i-1} < r_i < s_i < t_i \text{ for } 1 \leq i \leq t, \text{ and}$$

$$(2) \quad \sum_{j=1}^n \left| H_j G_i - \int_{x_{j-1}}^{x_j} HG \right| < \epsilon[8(t+1)]^{-1} \text{ for } 1 \leq i \leq t+1 \text{ and each}$$

refinement  $\{x_j\}_{j=0}^n$  of  $\{s_{i-1}, t_{i-1}, r_i\}$ .

It follows from Lemma 7.2 that  $k_i G \in OA^{k_i \nu}$  on  $[r_i, s_i]$  for  $1 \leq i \leq t$ . Hence, for each  $i$  there exists a subdivision  $D_i$  of  $[r_i, s_i]$  such that if  $J$  and  $K$  are refinements of  $D_i$ , then

$$\left| \sum_{J(I)} \left| k_i G - \int k_i G \right| - \sum_{K(I)} \left| k_i G - \int k_i G \right| \right| < \epsilon(4t)^{-1}.$$

Let  $D$  denote the subdivision  $\cup_{i=1}^t E_i \cup_{i=1}^t D_i$  of  $[a, b]$ . Suppose  $J_1$  and  $J_2$  are refinements of  $D$ ,  $P_{1i}$  and  $P_{2i}$  are subdivisions of  $[s_{i-1}, r_i]$  for  $1 \leq i \leq t+1$ ,  $Q_{1i}$  and  $Q_{2i}$  are subdivisions of  $[r_i, s_i]$  for  $1 \leq i \leq t$  and  $J_1$  and  $J_2$  are equal to

$$\bigcup_{i=1}^{t+1} P_{1i} \bigcup_{i=1}^t Q_{1i} \quad \text{and} \quad \bigcup_{i=1}^{t+1} P_{2i} \bigcup_{i=1}^t Q_{2i},$$

respectively. For convenience, suppose

$$\sum_{J_1(I)} \left| HG - \int HG \right| \geq \sum_{J_2(I)} \left| HG - \int HG \right|.$$

Thus,

$$\begin{aligned} & \left| \sum_{J_1(I)} \left| HG - \int HG \right| - \sum_{J_2(I)} \left| HG - \int HG \right| \right| \\ &= \sum_{J_1(I)} \left| HG - \int HG \right| - \sum_{J_2(I)} \left| HG - \int HG \right| \\ &= \sum_{i=1}^{t+1} \sum_{P_{1i}(I)} \left| HG - \int HG \right| + \sum_{i=1}^t \sum_{Q_{1i}(I)} \left| HG - \int HG \right| \\ & \quad - \sum_{i=1}^{t+1} \sum_{P_{2i}(I)} \left| HG - \int HG \right| - \sum_{i=1}^t \sum_{Q_{2i}(I)} \left| HG - \int HG \right| \end{aligned}$$

$$\begin{aligned}
& < (t+1)\{\epsilon[8(t+1)]^{-1}\} + \sum_{i=1}^t \sum_{Q_{1i}(t)} \left| HG - \int HG \right| \\
& \quad + (t+1)\{\epsilon[8(t+1)]^{-1}\} - \sum_{i=1}^t \sum_{Q_{2i}(t)} \left| HG - \int HG \right| \\
& = \sum_{i=1}^t \sum_{Q_{1i}(t)} \left| (H - k_i + k_i)G - \int (H - k_i + k_i)G \right| \\
& \quad - \sum_{i=1}^t \sum_{Q_{2i}(t)} \left| (H - k_i + k_i)G - \int (H - k_i + k_i)G \right| + \epsilon/4 \\
& \leq | -1 | \sum_{j=1}^2 \sum_{i=1}^t \sum_{Q_{ji}(t)} | (H - k_i)G | \\
& \quad + \sum_{j=1}^2 \sum_{i=1}^t \sum_{Q_{ji}(t)} \left| \int (H - k_i)G \right| \\
& \quad + \sum_{i=1}^t \sum_{Q_{1i}(t)} \left| k_i G - \int k_i G \right| \\
& \quad - \sum_{i=1}^t \sum_{Q_{2i}(t)} \left| k_i G - \int k_i G \right| + \epsilon/4 \\
& < 2B | -1 | [\epsilon(8 | -1 | B)^{-1}] + 2B[\epsilon(8 | -1 | B)^{-1}] + t[\epsilon(4t)^{-1}] + \epsilon/4 \\
& \leq \epsilon.
\end{aligned}$$

Therefore,  $\int_a^b \left| HG - \int HG \right|$  exists. Hence, there exists a nonnegative number  $\alpha$  such that  $G \in OA^\alpha$  on  $[a, b]$ . Thus, it follows from Theorem 6 that  $G \in OM^\alpha$  on  $[a, b]$ .

A similar argument can be used to establish the existence of  $\beta$ . Therefore, the theorem follows.

Theorem 7 does not remain true if the requirement that  $|AB| = |A||B|$  is removed. In the following we establish this assertion by constructing a function  $G$  and a constant  $K$  such that  $\int_0^1 G$  exists,  $\int_0^1 \left| G - \int G \right|$  exists and  $\int_0^1 \left| KG - \int KG \right|$  does not exist.

We consider the set of infinite diagonal matrices with bounded elements and  $|M| = \text{lub } |m_{ij}|$ . For  $p = 1, 2, \dots$ , let  $A_p$  be the infinite diagonal matrix such that  $a_{pp} = 1$  and  $a_{qq} = 0$  if  $q \neq p$ . Let  $A = \{A_p | p = 1, 2, \dots\}$ . There exists a reversible function  $f$  from the rational numbers in  $[0, 1]$  to  $A$ . Let  $G$  be an interval function defined on  $[0, 1]$  such that

$$G(u, v) = \begin{cases} (v - u)f(v) & \text{if } v \text{ is rational} \\ (v - u)f(r) & \text{where } r \text{ is a rational number in} \\ & (u, v) \text{ if } v \text{ is irrational.} \end{cases}$$

For each rational number  $r$  in  $[0, 1]$ , let  $p(r)$  be the positive integer such that  $f(r) = A_{p(r)}$ . Let  $K$  be the infinite diagonal matrix such that if  $r = m/n$  is a rational number contained in  $[0, 1]$  and  $m$  and  $n$  have no common integral factors other than 1, then

$$k_{p(r), p(r)} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

We have now constructed a function  $G$  and a constant  $K$  such that  $\int_0^1 G = 0$ ,  $\int_0^1 |G - \int G| = 1$  and  $\int_0^1 |KG - \int KG|$  does not exist. This example was suggested by an example in a previous paper by the author [3].

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