MUTUAL EXISTENCE OF SUM AND PRODUCT INTEGRALS

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Functions are from $R \times R$ to $N$, where $R$ denotes the set of real numbers and $N$ denotes a normed complete ring. If $G$ has bounded variation on $[a, b]$, then $\int_a^b G$ exists if and only if $\int_a^b (1 + G)$ exists for $a \leq x < y \leq b$. If each of $\lim_{x \to p} H(p, x)$, $\lim_{x \to p} H(x, p)$, $\lim_{x, y \to p} H(x, y)$ and $\lim_{x, y \to p} H(x, y)$ exists, $G$ has bounded variation on $[a, b]$ and either $\int_a^b G$ exists or $\int_a^b (1 + G)$ exists for $a \leq x < y \leq b$, then $\int_a^b HG$ and $\int_a^b GH$ exist and $\int_a^b (1 + HG)$ and $\int_a^b (1 + GH)$ exist for $a \leq x < y \leq b$. If $G$ has bounded variation on $[a, b]$ and $\nu$ is a nonnegative number, then $\int_a^b G$ exists and $\int_a^b |G - \int G| = \nu$ if and only if $\int_a^b (1 + G)$ exists for $a \leq x < y \leq b$ and

$$\int_a^b |1 + G - \Pi(1 + G)| = \nu.$$

J. S. MacNerney [4] defines classes $OA$ and $OM$ of functions such that the integral-like formulas

$$V(a, b) = \int_a^b (W - 1) \quad \text{and} \quad W(a, b) = a^b(1 + V)$$

are mutually reciprocal and establishes a one-to-one correspondence between the classes $OA$ and $OM$. B. W. Helton [1] defines classes $OA^\circ$ and $OM^\circ$ of functions and shows that if $G$ has bounded variation on $[a, b]$, then $G \in OA^\circ$ on $[a, b]$ if and only if $G \in OM^\circ$ on $[a, b]$, where $G \in OA^\circ$ on $[a, b]$ only if $\int_a^b G$ exists and $\int_a^b |G - \int G| = 0$, and $G \in OM^\circ$ on $[a, b]$ only if $\int_a^b (1 + G)$ exists for $a \leq x < y \leq b$ and

$$\int_a^b |1 + G - \Pi(1 + G)| = 0.$$

The class $OA$ is a proper subclass of $OA^\circ$ and $OM$ is closely related to the class $OM^\circ$. In the following, we establish a related result and show
that if $G$ has bounded variation on $[a, b]$, then $\int_a^b G$ exists if and only if $\int_a^b \Pi \alpha (1 + G)$ exists for $a \leq x < y \leq b$. This is not the same as the result of B. W. Helton since it is possible to construct a function $G$ such that $G$ has bounded variation on $[a, b]$, $\int_a^b G$ exists, $\int_a^b x \Pi y (1 + G)$ exists for $a \leq x < y \leq b$, $G \not\in OA^\circ$ on $[a, b]$ and $G \not\in OM^\circ$ on $[a, b]$ [3]. We then use this result and ideas from another theorem of B. W. Helton [2, Theorem 2, p. 494] to establish that if each of $\lim_{x \rightarrow p} H(p, x)$, $\lim_{x \rightarrow p} H(x, p)$, $\lim_{x, y \rightarrow p} H(x, y)$ and $\lim_{x, y \rightarrow p} H(x, y)$ exists, $G$ has bounded variation on $[a, b]$ and either $\int_a^b G$ exists or $\int_a^b \Pi \alpha (1 + G)$ exists for $a \leq x < y \leq b$, then $\int_a^b HG$ and $\int_a^b GH$ exist and $\int_a^b \Pi \alpha (1 + HG)$ and $\int_a^b \Pi \alpha (1 + GH)$ exist for $a \leq x < y \leq b$. Further, we show that if $G$ has bounded variation on $[a, b]$ and $\nu$ is a nonnegative number, then $G \in OA^\nu$ on $[a, b]$ if and only if $G \in OM^\nu$ on $[a, b]$, where $G \in OA^\nu$ on $[a, b]$ only if $\int_a^b G$ exists and

$$\int_a^b |G - \int_a^b G| = \nu,$$

and $G \in OM^\nu$ on $[a, b]$ only if $\int_a^b \Pi \alpha (1 + G)$ exists for $a \leq x < y \leq b$ and

$$\int_a^b |1 + G - \Pi (1 + G)| = \nu.$$

Finally, we show that if the norm used has the property that $|AB| = |A| |B|$ and if each of $\lim_{x \rightarrow p} H(p, x)$, $\lim_{x \rightarrow p} H(x, p)$, $\lim_{x, y \rightarrow p} H(x, y)$ and $\lim_{x, y \rightarrow p} H(x, y)$ exists, $G$ has bounded variation on $[a, b]$ and either $G \in OA^\nu$ on $[a, b]$ or $G \in OM^\nu$ on $[a, b]$, then there exist nonnegative numbers $\alpha$ and $\beta$ such that $HG$ is in $OA^\alpha$ and $OM^\alpha$ on $[a, b]$ and $GH$ is in $OA^\beta$ and $OM^\beta$ on $[a, b]$.

All integrals and definitions are of the subdivision-refinement type, and functions are from $R \times R$ to $N$, where $R$ denotes the set of real numbers and $N$ denotes a ring which has a multiplicative identity element represented by 1 and has a norm $| \cdot |$ with respect to which $N$ is complete and $|1| = 1$. Unless noted otherwise, functions are assumed to be defined only for $\{x, y\} \in R \times R$ such that $x < y$. The statement that $G \in OB^\circ$ on $[a, b]$ means that there exist a subdivision $D$ of $[a, b]$ and a number $B$ such that if $\{x_i\}^{n}_{i=0}$ is a refinement of $D$, then $\Sigma_{i=1}^n |G_i| < B$, where $G_i$ denotes $G(x_{i-1}, x_i)$. When convenient, we use
\[
\sum_{i=1}^{n} G_i \quad \text{and} \quad \prod_{i=1}^{n} (1 + G_i),
\]
to denote
\[
\sum_{i=1}^{n} G_i \quad \text{and} \quad \prod_{i=1}^{n} (1 + G_i),
\]
respectively, where \( J = \{x_i\}_{i=0}^{n} \) represents a subdivision of some interval. The sets \( OA^\circ, OM^\circ, OA^r \) and \( OM^r \) have been defined previously, and \( G \in OA^+ \) only if \( G \) is an additive function from \( R \times R \) to the nonnegative numbers. Also, \( G \in OM^r \) on \([a, b]\) only if \( x, \Pi^r(1 + G) \) exists for \( a \leq x < y \leq b \) and if \( \epsilon > 0 \) then there exists a subdivision \( D \) of \([a, b]\) such that if \( \{x_i\}_{i=0}^{n} \) is a refinement of \( D \) and \( 0 \leq p < q \leq n \), then
\[
\left| x_p \Pi^q (1 + G) - \prod_{i=p+1}^{q} (1 + G_i) \right| < \epsilon.
\]
The symbols \( G(p, p^+), G(p^-, p), G(p^+, p^+) \) and \( G(p^-, p^-) \) denote \( \lim_{x \to p^+} G(x, p) \), \( \lim_{x \to p^-} G(x, p) \), \( \lim_{x,y \to p^-} G(x, y) \) and \( \lim_{x,y \to p^+} G(x, y) \), respectively, and \( G \in OL^\circ \) on \([a, b]\) only if \( G(p, p^+), G(p^-, p), G(p^+, p^+) \) and \( G(p^-, p^-) \) exist for \( p \in [a, b] \). Further, \( G \in S_2 \) on \([a, b]\) only if \( G(p, p^+) \) and \( G(p^-, p^-) \) exist for \( p \in [a, b] \). Finally, statements of the form \( G > \beta \) should be interpreted in terms of subdivisions and refinements. See B. W. Helton [1] and J. S. MacNerney [4] for additional background.

We now establish an approximation theorem for product integrals. To do this, we initially develop a sequence of lemmas.

**Lemma 1.1.** If \( \beta > 0 \), \( G \) is a function from \( R \times R \) to \( N \), \( |G| < 1 - \beta \) on \([a, b]\), \( G \in OB^\circ \) on \([a, b]\) and \( x, \Pi^r(1 + G) \) exists for \( a \leq x < y \leq b \), then \( G \in OM^r \) on \([a, b]\).

**Proof.** Let \( \epsilon > 0 \). There exist a subdivision \( D \) of \([a, b]\) and a number \( B \) such that if \( \{x_i\}_{i=0}^{n} \) is a refinement of \( D \), then
\[
\begin{align*}
1 & \leq 1 - \beta \\
2 & \Pi_{i=1}^{n} (1 + |G_i|) < B, \\
3 & \Pi_{i=1}^{n} (1 + \sum_{i=1}^{\infty} (-1)^{j}G_{i}^{j}) < B, \text{ and} \\
4 & \left| a \Pi^{p} (1 + G) - \Pi_{i=p+1}^{q} (1 + G_i) \right| < \epsilon (3B)^{-1}.
\end{align*}
\]

Suppose \( \{x_i\}_{i=0}^{n} \) is a refinement of \( D \) and \( 0 \leq p < q \leq n \). Let \( Y = \{y_i\}_{i=0}^{p} \) and \( Z = \{z_i\}_{i=0}^{q} \) be refinements of \( \{x_i\}_{i=0}^{p} \) and \( \{x_i\}_{i=0}^{q} \), respectively, such that

\[
\sum_{i=1}^{n} G_i \quad \text{and} \quad \prod_{i=1}^{n} (1 + G_i)
\]
Further, let $P$ and $P'$ denote
\[
\prod_{Y(i)} (1 + G) \quad \text{and} \quad a \prod^* (1 + G),
\]
respectively, and let $Q$ and $Q'$ denote
\[
\prod_{Z(i)} (1 + G) \quad \text{and} \quad x \prod^b (1 + G),
\]
respectively. Note that $P^{-1}$ and $Q^{-1}$ exist and are
\[
\prod_{i=1}^r \left[ 1 + \sum_{j=1}^e (-1)^j G^i (y_{r-i}, y_{r+1-i}) \right]
\]
and
\[
\prod_{i=1}^s \left[ 1 + \sum_{j=1}^e (-1)^j G^i (z_{s-i}, z_{s+1-i}) \right],
\]
respectively.

Let $W$ denote the subdivision $D \cup Y \cup Z$ of $[a, b]$. Thus,
\[
\left| x \prod^s (1 + G) - \prod_{i=p+1}^q (1 + G_i) \right|
\]
\[
= \left| P^{-1} P \left[ x \prod^s (1 + G) - \prod_{i=p+1}^q (1 + G_i) \right] Q Q^{-1} \right|
\]
\[
\leq |P^{-1}| \left| P \left[ x \prod^s (1 + G) \right] Q - P \left[ \prod_{i=p+1}^q (1 + G_i) \right] Q \right| |Q^{-1}|
\]
\[
\leq B \left| P \left[ x \prod^s (1 + G) \right] Q - \prod_{w(i)} (1 + G) \right|
\]
\[
= B \left| [P - P' + P'] \left[ x \prod^s (1 + G) \right] [Q' - Q + Q] - \prod_{w(i)} (1 + G) \right|
\]
LEMMA 1.2. If $G$ is a function from $R \times R$ to $N, G \in OB^\circ$ on $[a, b]$ and , $\Pi^1(1 + G)$ exists for $a \leq x < y \leq b$, then $G(a, a^*)$ and $G(b^-, b)$ exist.

Proof. We initially show that $G(a, a^*)$ exists. Let $\epsilon > 0$. There exist numbers $c$ and $B$ such that $a < c < b$ and if $\{x_i\}^n$ is a subdivision of $[a, c]$, then

$$| - 1| \prod_{i=1}^n (1 + |G_i|) < B \quad \text{and} \quad \sum_{i=2}^n |G_i| < \epsilon(4B^2)^{-1}.$$ 

Further, there exists a subdivision $D = \{z_i\}^\circ$ of $[a, c]$ such that if $J$ and $K$ are refinements of $D$, then

$$\left| \prod_{J(I)} (1 + G) - \prod_{K(I)} (1 + G) \right| < \epsilon/2.$$ 

We now suppose $a < x < y < z_1$ and show that

$$|G(a, x) - G(a, y)| < \epsilon.$$ 

Let $\{x_i\}^m$ and $\{y_j\}^n$ denote $D \cup \{x\}$ and $D \cup \{y\}$, respectively. Thus,

$$\epsilon/2 > \left| \prod_{i=1}^m (1 + G_i) - \prod_{j=1}^n (1 + G_j) \right|$$

$$= \left| [1 + G(a, x)] \left[ \prod_{i=2}^m (1 + G_i) \right] - [1 + G(a, y)] \left[ \prod_{j=2}^n (1 + G_j) \right] \right|$$

$$= \left| [1 + G(a, x)] \left[ 1 + \sum_{i=2}^m G_i \prod_{k=i+1}^m (1 + G_k) \right] \right.$$ 

$$- [1 + G(a, y)] \left[ 1 + \sum_{j=2}^n G_j \prod_{k=j+1}^n (1 + G_k) \right] \right|$$

$$\leq |G(a, x) - G(a, y)| - B \sum_{i=2}^m |G_i| \left| \prod_{k=i+1}^m (1 + G_k) \right|$$

$$- B \sum_{j=2}^n |G_j| \left| \prod_{k=j+1}^n (1 + G_k) \right|$$
> |G(a, x) − G(a, y)| − B^2[ε(4B^2)^{-1}] + B^2[ε(4B^2)^{-1}],

and hence,

\[ \epsilon > |G(a, x) - G(a, y)|. \]

Since the existence of \( G(b^-, b) \) can be established in a similar manner, Lemma 1.2 follows.

**Lemma 1.3.** If \( \beta > 0 \), \( G \) is a function from \( R \times R \) to \( N \), \( |G| < 1 - \beta \) on \( (a, b) \), \( G \in OB^∞ \) on \( [a, b] \) and \( \pi^x(1 + G) \) exists for \( a \leq x < y \leq b \), then \( G \in OM^* \) on \( [a, b] \).

**Proof.** Let \( \epsilon > 0 \). There exist a subdivision \( E_1 \) of \( [a, b] \) and a number \( B > 1 \) such that if \( \{x_i\}_{i=1}^m \) is a refinement of \( E_1 \), then

\[ \prod_{i=1}^m (1 + |G_i|) < B \]

and

\[ \left| \pi^b(1 + G) - \prod_{i=1}^m (1 + G_i) \right| < \epsilon. \]

Let \( H \) be the function defined on \( [a, b] \) such that

\[ H(x, y) = \begin{cases} G(x, y) & \text{if } x \neq a \text{ and } y \neq b \\ 0 & \text{if } x = a \text{ or } y = b. \end{cases} \]

Thus, \( H \) satisfies the hypothesis of Lemma 1.1, and hence, there exists a subdivision \( E_2 \) of \( [a, b] \) such that if \( \{x_i\}_{i=0}^m \) is a refinement of \( E_2 \) and \( 0 \leq p < q \leq m \), then

\[ \left| \prod_{i=p+1}^q (1 + H_i) \right| < \epsilon(3B)^{-1}. \]

It follows from Lemma 1.2 that \( G(a, a^+) \) and \( G(b^-, b) \) exist. Hence, there exists a point \( x \), where \( a < x < b \), such that if \( \{x_i\}_{i=0}^m \) and \( \{y_j\}_{j=0}^n \) are subdivisions of \( [a, x], 1 \leq r \leq m \) and \( 1 \leq s \leq n \), then

\[ \left| \prod_{i=1}^r (1 + G_i) - \prod_{j=1}^s (1 + G_j) \right| < \epsilon(3B)^{-1}. \]

Also, there exists a point \( y \), where \( a < y < b \), such that if \( \{x_i\}_{i=0}^m \) and \( \{y_j\}_{j=0}^n \) are subdivisions of \( [y, b], 1 \leq r \leq m \) and \( 1 \leq s \leq n \), then
Let $D$ denote the subdivision

$$E_1 \cup E_2 \cup \{x\} \cup \{y\}$$

of $[a, b]$. Further, suppose $\{x_i\}_{i=0}^m$ is a refinement of $D$ and $0 \leq p < q \leq m$. If $p = 0$ and $q = m$, then the desired inequality follows from the existence of $x_1 \prod^b (1 + G)$. If $p \neq 0$ and $q \neq m$, then the inequality follows from the properties of the function $H$. Suppose $p = 0$ and $q \neq m$. There exists a subdivision $J$ of $[a, x_1]$ such that

$$\left| a_1 \prod^y (1 + G) - \prod_{j=1}^q (1 + G_i) \right| < \varepsilon(3B)^{-1}.$$ 

Thus,

$$\left| a_1 \prod^y (1 + G) - \prod_{i=1}^q (1 + G_i) \right| < \left| a_1 \prod^y (1 + G) - (1 + G_i) \right| + \varepsilon(3B)^{-1} + B[\varepsilon(3B)^{-1}] + \varepsilon / 3$$

$$< B \left| \prod_{j=1}^q (1 + G) - (1 + G_i) \right| + B[\varepsilon(3B)^{-1}] + \varepsilon / 3$$

$$< B[\varepsilon(3B)^{-1}] + 2\varepsilon / 3 = \varepsilon.$$ 

If $p \neq 0$ and $q = n$, then a similar argument establishes the inequality. Therefore, Lemma 1.3 follows.

**THEOREM 1.** If $G$ is a function from $R \times R$ to $N$, $G \in OB^\circ$ on $[a, b]$ and $x_1 \prod^b (1 + G)$ exists for $a \leq x < y \leq b$, then $G \in OM^*$ on $[a, b]$.

**Proof.** Since $G \in OB^\circ$ on $[a, b]$, there exists a subdivision $\{x_i\}_{i=0}^m$ of $[a, b]$ such that if $1 \leq i \leq m$ and $x_{i-1} < x < y < x_i$, then $|G(x, y)| < 1 / 2$. Hence, this theorem can be established by using Lemma 1.3 and the identity

$$\prod_{i=1}^n a_i - \prod_{i=1}^n b_i = \sum_{i=1}^n \left( \prod_{j=1}^{i-1} b_j \right) (a_i - b_i) \left( \prod_{k=i+1}^n a_k \right),$$

where $\prod_{i=1}^n b_i = \prod_{k=n+1}^n a_k = 1$.

We now use the approximation theorem to establish an existence theorem for sum integrals. In particular, we show that if $G$ has
bounded variation on \([a, b]\) and \(\alpha \Pi^y(1 + G)\) exists for \(a \leq x < y \leq b\), then \(\int_a^b G\) exists. Several lemmas are required.

**Lemma 2.1.** If \(G\) is a function from \(R \times R\) to \(N\), \(G \in OB^\circ\) on \([a, b]\) and \(\alpha \Pi^y(1 + G)\) exists for \(a \leq x < y \leq b\), then

\[
\int_a^b G(u, v) \alpha \Pi^y(1 + G)
\]

exists and is \(-1 + a \Pi^b(1 + G)\).

**Proof.** Let \(\epsilon > 0\). There exist a subdivision \(E_1\) of \([a, b]\) and a number \(B\) such that if \(\{x_i\}_{i=0}^m\) is a refinement of \(E_1\), then

1. \(\sum_{i=1}^m |G_i| < B\), and
2. \(|\prod_{i=1}^m (1 + G_i) - a \Pi^b(1 + G)| < \epsilon/2\).

Theorem 1 implies that \(G \in OM^*\) on \([a, b]\), and hence, there exists a subdivision \(E_2\) of \([a, b]\) such that if \(\{x_i\}_{i=0}^m\) is a refinement of \(E_2\) and \(0 \leq p < q \leq m\), then

\[
\left| \prod_{i=p+1}^q (1 + G_i) \right| < \epsilon(2B)^{-1}.
\]

Let \(D\) denote the subdivision \(E_1 \cup E_2\) of \([a, b]\) and suppose \(\{x_i\}_{i=0}^m\) is a refinement of \(D\). Thus,

\[
\left| \sum_{i=1}^m G_i[\alpha \Pi^b(1 + G)] - [-1 + a \Pi^b(1 + G)] \right|
\leq \left| \sum_{i=1}^m G_i[\alpha \Pi^b(1 + G)] + \prod_{i=1}^m (1 + G_i) \right| + \epsilon/2
\leq \left| \sum_{i=1}^m G_i[\alpha \Pi^b(1 + G)] + 1 \prod_{i=1}^m (1 + G_i) \right| + \epsilon/2
\leq \sum_{i=1}^m |G_i| \left| \alpha \Pi^b(1 + G) - \prod_{k=i+1}^m (1 + G_k) \right| + \epsilon/2
\leq B[\epsilon(2B)^{-1}] + \epsilon/2 = \epsilon.
\]

**Lemma 2.2.** If \(H\) and \(G\) are functions from \(R \times R\) to \(N\), \(H \in OL^\circ\) on \([a, b]\), \(G \in OB^\circ\) on \([a, b]\) and \(\int_a^b G\) exists, then \(\int_a^b HG\) exists and \(\int_a^b GH\) exists.
Proof. B. W. Helton [2, Theorem 2, p. 494] proves that $HG$ and $GH$ are in $OA^\circ \cap OB^\circ$ on $[a, b]$ with the hypothesis of Lemma 2.2 and the additional restriction that $G \in OA^\circ$ on $[a, b]$. This lemma follows by essentially the same argument.

Observe that weakening the hypothesis of Helton's result by requiring only the existence of $\int_a^b G$ produces a corresponding weakening of the conclusion since we now have that $\int_a^b HG$ and $\int_a^b GH$ exist rather than that $HG$ and $GH$ are in $OA^\circ$ on $[a, b]$.

Lemma 2.2 is not true for functions defined on a linearly ordered set [4, p. 149]. For example, consider

$$S = [0, 1) \cup (1, 2],$$

with the usual ordering for the real numbers. Let $G$ be the function defined on $S \times S$ such that

$$G(x, y) = \begin{cases} 1 & \text{if } x < 1 \text{ and } y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $G \in OA^\circ \cap OB^\circ$ on $S \times S$. Let $H$ be the function defined on $S \times S$ such that

$$H(x, y) = \begin{cases} 1 & \text{if } x < 1, y > 1 \text{ and } x \text{ rational} \\ -1 & \text{if } x < 1, y > 1 \text{ and } x \text{ irrational} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $H \in OL^\circ$ on $S \times S$. However, $\int_a^b HG$ does not exist.

Lemma 2.3. If $\beta > 0$, $G$ is a function from $R \times R$ to $N$, $|G| < 1 - \beta$ on $[a, b]$, $G \in OB^\circ$ on $[a, b]$ and $a \Pi^a(1 + G)$ exists, then $b \Pi^b(1 + H)$ exists and is $[a \Pi^a(1 + G)]^{-1}$, where

$$H(y, x) = \sum_{i=1}^{\infty} (-1)^i G^i(x, y)$$

for $a \leq x < y \leq b$.

Proof. We initially show that $b \Pi^b(1 + H)$ exists. Let $\epsilon > 0$. There exist a subdivision $D$ of $[a, b]$ and a number $B$ such that if $\{x_i\}_{i=0}^n$ and $\{y_j\}_{j=0}^n$ are refinements of $D$, then
(1) \(|G_i| < 1 - \beta\) for \(i = 1, 2, \cdots, m\),
(2) \(|\prod_{i=1}^{m} (1 + H_{m+1-i})| < B\), and
(3) \(|\prod_{i=1}^{m} (1 + G_i) - \prod_{i=1}^{n} (1 + G_i)| < \varepsilon B^{-2}\).

Note that we are using \(H_{m+1-i}\) to denote \(H(x_{m+1-i}, x_{m-i})\). Suppose \(\{x_i\}_{i=0}^{m}\) and \(\{y_j\}_{j=0}^{n}\) are refinements of \(D\). Thus,

\[
\left| \prod_{i=1}^{m} (1 + H_{m+1-i}) - \prod_{j=1}^{n} (1 + H_{n+1-j}) \right|
\leq \left| \prod_{i=1}^{m} (1 + H_{m+1-i}) \right| \left| 1 - \left[ \prod_{i=1}^{m} (1 + H_{m+1-i}) \right]^{-1} \left[ \prod_{j=1}^{n} (1 + H_{n+1-j}) \right] \right|
\leq B \left| 1 - \left[ \prod_{i=1}^{m} (1 + G_i) \right] \left[ \prod_{j=1}^{n} (1 + H_{n+1-j}) \right] \right|
\leq B \left| \prod_{j=1}^{n} (1 + G_j) - \prod_{i=1}^{m} (1 + G_i) \right| \left| \prod_{j=1}^{n} (1 + H_{n+1-j}) \right|
+ B \left| 1 - \left[ \prod_{j=1}^{n} (1 + G_j) \right] \left[ \prod_{j=1}^{n} (1 + H_{n+1-j}) \right] \right|
< B^2(\varepsilon B^{-2}) + B(0) = \varepsilon.

We now show that \([a, \Pi^b(1 + G)]^{-1}\) exists and is \(\varepsilon \Pi^a(1 + H)\). Let \(\varepsilon > 0\). There exists a subdivision \(\{x_i\}_{i=0}^{m}\) of \([a, b]\) such that

\[
\left| \left[ a, \Pi^b(1 + G) \right] \left[ b, \Pi^a(1 + H) \right] - \left[ \prod_{i=1}^{m} (1 + G_i) \right] \left[ \prod_{i=1}^{m} (1 + H_{m+1-i}) \right] \right| < \varepsilon.
\]

Hence,

\[
\left| \left[ a, \Pi^b(1 + G) \right] \left[ a, \Pi^a(1 + H) \right] - 1 \right|
< \left| \left[ \prod_{i=1}^{m} (1 + G_i) \right] \left[ \prod_{i=1}^{m} (1 + H_{m+1-i}) \right] - 1 \right| + \varepsilon
= 0 + \varepsilon = \varepsilon.
\]

**Lemma 2.4.** If \(\beta > 0\), \(G\) is a function from \(R \times R\) to \(N\), \(|G| < 1 - \beta\) on \([a, b]\), \(G \in OB^\circ\) on \([a, b]\) and \(\Pi^a(1 + G)\) exists for \(a \leq x < y \leq b\), then \(\int_a^b G\) exists.

**Proof.** It follows from Lemma 2.1 that

\[
\int_a^b G(u, v), \Pi^b(1 + G)
\]
exists. Let $H$ be the function defined on $[a, b]$ such that

$$H(u, v) = [\int_a^b (1 + G)]^{-1}.$$

The existence of $H$ follows from Lemma 2.3. Further, $H \in O\ell^\circ$ on $[a, b]$. Hence, the existence of $\int_a^b G$ can be established by using Lemma 2.2.

**Lemma 2.5.** If $\beta > 0$, $G$ is a function from $R \times R$ to $N$, $|G| < 1 - \beta$ on $(a, b)$, $G \in O\ell^\circ$ on $[a, b]$ and $\int x^n (1 + G)$ exists for $a \leq x < y \leq b$, then $\int_a^b G$ exists.

**Proof.** Lemma 2.5 follows by using Lemma 1.2 and Lemma 2.4.

**Theorem 2.** If $G$ is a function from $R \times R$ to $N, G \in O\ell^\circ$ on $[a, b]$ and $\int x^n (1 + G)$ exists for $a \leq x < y \leq b$, then $\int_a^b G$ exists.

**Proof.** There exists a subdivision $\{x_i\}_{i=0}^m$ of $[a, b]$ such that if $1 \leq i \leq m$ and $x_{i-1} < x < y < x_i$, then $|G(x, y)| < 1/2$. Hence, the theorem follows from Lemma 2.5.

An existence theorem for product integrals is now established. In particular, we show that if $G$ has bounded variation on $[a, b]$ and $\int_a^b G$ exists, then $\int x^n (1 + G)$ exists for $a \leq x < y \leq b$.

**Lemma 3.1.** If $G$ is a function from $R \times R$ to $N$ such that $G \in O\ell^\circ$ on $[a, b]$, then there exists $\alpha \in O\ell^\circ$ on $[a, b]$ such that

$$|G(x, y)| \leq \alpha(x, y)$$

for $a \leq x < y \leq b$.

**Proof.** There exist a subdivision $\{x_i\}_{i=0}^m$ of $[a, b]$ and a number $B$ such that if $H$ is a refinement of $\{x_i\}_{i=0}^m$, then $\Sigma_{H(i)} |G| < B$. Let $g$ be the function such that for $x_{p-1} < x \leq x_p$, $g(x) = \text{lub} \, \Sigma_{H(i)} |G|$ for all refinements $H$ of $\{x_i\}_{i=0}^{p-1} \cup \{x\}$. Let $\alpha(x, y) = \int_x^y dg$. This produces the desired function.

**Theorem 3.** If $G$ is a function from $R \times R$ to $N, G \in O\ell^\circ$ on $[a, b]$ and $\int_a^b G$ exists, then $\int x^n (1 + G)$ exists for $a \leq x < y \leq b$. 
Proof. Suppose \( a \leq x < y \leq b \). In the following we show that \( \prod_i (1 + G) \) exists and is \( \sum_{p=0}^n G_p(x, y) \), where \( G_0(x, y) = 1 \) and

\[
G_p(x, y) = (R) \int_x^y G \cdot G_{p-1}(x, y)
\]

for \( p = 1, 2, \cdots \). The existence of these integrals follows from Lemma 2.2.

It follows from Lemma 3.1 that there exists \( \alpha \in OA^+ \) such that if \( x \leq r < s \leq y \), then

\[
|G(r, s)| \leq \alpha(r, s).
\]

Further, from a result of MacNerney [4, Theorem 6.2, p. 160], \( \sum_{p=0}^n g_p(x, y) \) exists, where \( g_0(x, y) = 1 \) and

\[
g_p(x, y) = (R) \int_x^y \alpha \cdot g_{p-1}(x, y)
\]

for \( p = 1, 2, \cdots \).

It can be established by induction that if \( \{x_i\}_{i=0}^n \) is a subdivision of \([x, y]\), then

\[
\prod_{i=1}^n (1 + G_i) = 1 + \sum_{k_1=1}^n G_{k_1} + \sum_{k_1=1}^n \sum_{k_2=k_1+1}^n G_{k_1} G_{k_2} + \cdots
\]

\[
+ \sum_{k_1=1}^n \sum_{k_2=k_1+1}^n \cdots \sum_{k_n=k_{n-1}+1}^n G_{k_1} G_{k_2} \cdots G_{k_n},
\]

where \( \sum_{i=p}^q G_i = 0 \) if \( p > q \). Further, it can also be established by induction that

\[
\left| \sum_{k_1=1}^n \sum_{k_2=k_1+1}^n \cdots \sum_{k_p=k_{p-1}+1}^n G_{k_1} G_{k_2} \cdots G_{k_p} \right| \leq g_p(x, y)
\]

for \( p = 1, 2, \cdots \).

Let \( \epsilon > 0 \). There exists a positive integer \( N \) such that

\[
\sum_{p=N+1}^n g_p(x, y) < \epsilon / 3.
\]

Further, there exists a subdivision \( D \) of \([x, y]\) such that if \( \{x_i\}_{i=0}^n \) is a refinement of \( D \), then
\[
\left| \left[ 1 + \sum_{k_1=1}^{n} G_{k_1} + \sum_{k_1=1}^{n} \sum_{k_2=k_1+1}^{n} G_{k_1} G_{k_2} + \cdots \\
+ \sum_{k_1=1}^{n} \sum_{k_2=k_1+1}^{n} \cdots \sum_{k_N=k_{N-1}+1}^{n} G_{k_1} G_{k_2} \cdots G_{k_N} \right] - \sum_{p=0}^{N} G_p(x, y) \right| < \epsilon/3.
\]

Suppose \(\{x_i\}_{i=0}^{n}\) is a refinement of \(D\). Thus,

\[
\left| \prod_{i=1}^{n} (1 + G_i) - \sum_{p=0}^{N} G_p(x, y) \right|
\]

\[
= \left| \left[ 1 + \sum_{k_1=1}^{n} G_{k_1} + \sum_{k_1=1}^{n} \sum_{k_2=k_1+1}^{n} G_{k_1} G_{k_2} + \cdots \\
+ \sum_{k_1=1}^{n} \sum_{k_2=k_1+1}^{n} \cdots \sum_{k_N=k_{N-1}+1}^{n} G_{k_1} G_{k_2} \cdots G_{k_N} \right] - \sum_{p=0}^{N} G_p(x, y) \right|
\]

\[
< \left| \left[ 1 + \sum_{k_1=1}^{n} G_{k_1} + \sum_{k_1=1}^{n} \sum_{k_2=k_1+1}^{n} G_{k_1} G_{k_2} + \cdots \\
+ \sum_{k_1=1}^{n} \sum_{k_2=k_1+1}^{n} \cdots \sum_{k_N=k_{N-1}+1}^{n} G_{k_1} G_{k_2} \cdots G_{k_N} \right] - \sum_{p=0}^{N} G_p(x, y) \right|
\]

\[
+ \epsilon/3 + \epsilon/3
\]

\[
< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.
\]

**Theorem 4.** If \(G\) is a function from \(R \times R\) to \(N\) and \(G \in OB^\circ\) on \([a, b]\), then \(\int_a^b G\) exists if and only if \(\int_a^b \Pi^\circ(1 + G)\) exists for \(a \leq x < y \leq b\).

**Proof.** This theorem follows as a corollary to Theorems 2 and 3.

**Theorem 5.** If \(H\) and \(G\) are functions from \(R \times R\) to \(N\), \(H \in OL^\circ\) on \([a, b]\), \(G \in OB^\circ\) on \([a, b]\) and either \(\int_a^b G\) exists or \(\int_a^b \Pi^\circ(1 + G)\) exists for \(a \leq x < y \leq b\), then \(\int_a^b HG\) and \(\int_a^b GH\) exist and \(\int_a^b \Pi^\circ(1 + HG)\) and \(\int_a^b \Pi^\circ(1 + GH)\) exist for \(a \leq x < y \leq b\).

**Proof.** This theorem follows as a corollary to Theorem 4 and Lemma 2.2.

We now show that if \(G\) has bounded variation on \([a, b]\), then \(G \in OA^\circ\) on \([a, b]\) if and only if \(G \in OM^\circ\) on \([a, b]\). This is a generalization of a result of B. W. Helton [1, Theorem 3.4, p. 301].
LEMMA 6.1. If \( \epsilon > 0 \) and \( G \) is a function from \( \mathbb{R} \times \mathbb{R} \) to \( \mathbb{N} \) such that \( G \in OB^\circ \) and \( S_2 \) on \([a, b]\), then there exists a subdivision \( D \) of \([a, b]\) such that if \( \{x_i\}_{i=0}^n \) is a refinement of \( D \), \( 1 \leq i \leq n \) and \( \{x_{ij}\}_{j=0}^{n(i)} \) is a subdivision of \([x_{i-1}, x_i]\), then

\[
\left| \prod_{j=1}^{n(i)} (1 + G_{ij}) - \left(1 + \sum_{j=1}^{n(i)} G_{ij}\right) \right| < \epsilon.
\]

Proof. Since \( G \in OB^\circ \cap S_2 \) on \([a, b]\), this lemma can be established by applying the covering theorem.

LEMMA 6.2. If \( \epsilon > 0 \) and \( G \) is a function from \( \mathbb{R} \times \mathbb{R} \) to \( \mathbb{N} \) such that \( G \in OB^\circ \) and \( S_2 \) on \([a, b]\), then there exists a subdivision \( D \) of \([a, b]\) such that if \( \{x_i\}_{i=0}^n \) is a refinement of \( D \) and \( \{x_{ij}\}_{j=0}^{n(i)} \) is a subdivision of \([x_{i-1}, x_i]\) for \( 1 \leq i \leq n \), then

\[
\sum_{i=1}^n \left| \prod_{j=1}^{n(i)} (1 + G_{ij}) - \left(1 + \sum_{j=1}^{n(i)} G_{ij}\right) \right| < \epsilon.
\]

Proof. There exist a subdivision \( \{r_i\}_{i=0}^n \) of \([a, b]\) and a number \( B \) such that if \( \{y_i\}_{i=0}^m \) is a refinement of \( \{r_i\}_{i=0}^n \), then

1. \( \sum_{i=1}^m |G_{i}| < B \), and
2. \( \prod_{i=1}^m (1 + |G_{i}|) < B \).

It follows by applying the covering theorem that there exists a subdivision \( \{s_i\}_{i=0}^s \) of \([a, b]\) such that if \( 1 \leq i \leq s \) and \( \{x_{ij}\}_{j=0}^{n(i)} \) is a subdivision of \([s_{i-1}, s_i]\), then

\[
\sum_{i=2}^{s+1} |G_{ij}| < \epsilon(2B^2)^{-1}.
\]

Further, it follows from Lemma 6.1 that there exists a subdivision \( \{t_i\}_{i=0}^l \) of \([a, b]\) such that if \( \{x_i\}_{i=0}^n \) is a refinement of \( \{t_i\}_{i=0}^l \), \( 1 \leq i \leq n \) and \( \{x_{ij}\}_{j=0}^{n(i)} \) is a subdivision of \([x_{i-1}, x_i]\), then

\[
\left| \prod_{j=1}^{n(i)} (1 + G_{ij}) - \left(1 + \sum_{j=1}^{n(i)} G_{ij}\right) \right| < \epsilon(4s)^{-1}.
\]

Let \( D \) denote the subdivision

\[\{r_i\}_{i=0}^n \cup \{s_i\}_{i=0}^s \cup \{t_i\}_{i=0}^l\]

of \([a, b]\) and suppose \( \{x_i\}_{i=0}^n \) is a refinement of \( D \). Further, suppose \( \{x_{ij}\}_{j=0}^{n(i)} \) is a subdivision of \([x_{i-1}, x_i]\) for \( 1 \leq i \leq n \). Let \( P \) be the subset of \( \{i\}_{i=1}^n \) such that \( i \in P \) only if \( x_i \in \{s_i\}_{i=0}^s \) or \( x_{i-1} \in \{s_i\}_{i=0}^s \). Finally, let
\[ Q = \{i\}_{i=1}^n - P. \]

In the following manipulations, we use the identity
\[
\prod_{i=1}^n (1 + b_i) = 1 + \sum_{i=1}^n b_i + \sum_{i=1}^n b_i \left\{ \sum_{j=i+1}^n b_j \left[ \prod_{k=j+1}^n (1 + b_k) \right] \right\},
\]
where \( \sum_{i=n+1}^n b_i = 0 \) and \( \prod_{k=n+1}^n (1 + b_k) = 1 \). This result can be established by induction.

We now establish the desired inequality:

\[
\sum_{i=1}^n \left| \prod_{j=1}^n (1 + G_{ij}) - \left( 1 + \sum_{j=1}^n G_{ij} \right) \right|
\]

\[
= \sum_{i \in Q} \left| \prod_{j=1}^n (1 + G_{ij}) - \left( 1 + \sum_{j=1}^n G_{ij} \right) \right|
\]

\[
+ \sum_{i \in P} \left| \prod_{j=1}^n (1 + G_{ij}) - \left( 1 + \sum_{j=1}^n G_{ij} \right) \right|
\]

\[
< \sum_{i \in Q} \left| 1 + \sum_{j=1}^n G_{ij} + \sum_{j=1}^n G_{ij} \left\{ \sum_{u=j+1}^n G_{iu} \left[ \prod_{v=u+1}^n (1 + G_{iv}) \right] \right\} \right|
\]

\[
- \left( 1 + \sum_{j=1}^n G_{ij} \right) + 2s [\epsilon (4s)^{-1}] \]

\[
= \sum_{i \in Q} \left| \sum_{j=1}^n G_{ij} \left\{ \sum_{u=j+1}^n G_{iu} \left[ \prod_{v=u+1}^n (1 + G_{iv}) \right] \right\} \right| + \epsilon / 2
\]

\[
\leq B \sum_{i \in Q} \sum_{j=1}^n \left| G_{ij} \right| \left\{ \sum_{u=j+1}^n \left| G_{iu} \right| \left[ \prod_{v=u+1}^n (1 + |G_{iv}|) \right] \right\} + \epsilon / 2
\]

\[
\leq B \left[ \epsilon (2B^2)^{-1} \right] \sum_{i \in Q} \sum_{j=1}^n \left| G_{ij} \right| + \epsilon / 2
\]

\[
< B \left[ \epsilon (2B^2)^{-1} \right] B + \epsilon / 2 = \epsilon.
\]

**Lemma 6.3.** If \( G \) is a function from \( R \times R \) to \( N, G \in OB^\circ \) on \([a, b]\) and \( \int_a^b G \) exists, then

\[
\int_a^b \left| \Pi(1 + G) - \left( 1 + \int G \right) \right| = 0.
\]
Proof. The existence of \( \int \Pi'(1 + G) \) for \( a \leq x < y \leq b \) follows from Theorem 3. Also, since \( G \in OB^\circ \) on \([a, b]\) and \( \int_a^b G \) exists, \( G \in S_2 \) on \([a, b]\).

Let \( \epsilon > 0 \). It follows from Lemma 6.2 that there exists a subdivision \( D \) of \([a, b]\) such that if \( \{x_i\}_{i=0}^n \) is a refinement of \( D \) and \( \{x_{ij}\}_{j=0}^{(i)} \) is a subdivision of \([x_{i-1}, x_i]\) for \( 1 \leq i \leq n \), then

\[
\sum_{i=1}^{n} \left| \prod_{j=1}^{(i)} (1 + G_{ij}) - \left( 1 + \sum_{j=1}^{(i)} G_{ij} \right) \right| < \epsilon / 3.
\]

Suppose \( \{x_i\}_{i=0}^n \) is a refinement of \( D \). For \( 1 \leq i \leq n \), let \( \{x_{ij}\}_{j=0}^{(i)} \) be a subdivision of \([x_{i-1}, x_i]\) such that

\[
\left| x_{i-1}, \Pi^{x_i}(1 + G) - \prod_{j=1}^{(i)} (1 + G_{ij}) \right| < \epsilon / 3n
\]

and

\[
\left| \sum_{j=1}^{(i)} G_{ij} - \int_{x_{i-1}}^{x_i} G \right| < \epsilon / 3n.
\]

Thus,

\[
\sum_{i=1}^{n} \left| x_{i-1}, \Pi^{x_i}(1 + G) - \left( 1 + \int_{x_{i-1}}^{x_i} G \right) \right| \\
\leq \sum_{i=1}^{n} \left| x_{i-1}, \Pi^{x_i}(1 + G) - \prod_{j=1}^{(i)} (1 + G_{ij}) \right| \\
+ \sum_{i=1}^{n} \left| \prod_{j=1}^{(i)} (1 + G_{ij}) - \left( 1 + \sum_{j=1}^{(i)} G_{ij} \right) \right| \\
+ \sum_{i=1}^{n} \left| \sum_{j=1}^{(i)} G_{ij} - \int_{x_{i-1}}^{x_i} G \right| \\
< n(\epsilon / 3n) + \epsilon / 3 + n(\epsilon / 3n) = \epsilon.
\]

**Theorem 6.** If \( \nu \) is a nonnegative number, \( G \) is a function from \( \mathbb{R} \times \mathbb{R} \) to \( \mathbb{N} \) and \( G \in OB^\circ \) on \([a, b]\), then \( G \in OA^\nu \) on \([a, b]\) if and only if \( G \in OM^\nu \) on \([a, b]\).

**Proof.** Suppose \( G \in OM^\nu \) on \([a, b]\). It follows from Theorem 2 that \( \int_a^b G \) exists. Hence, it is only necessary to show that
Let $\epsilon > 0$. There exists a subdivision $D_1$ of $[a, b]$ such that if $\{x_i\}_{i=0}^n$ is a refinement of $D_1$, then

$$\nu - \epsilon/2 < \sum_{i=1}^{n} |1 + G_i - x_{i-1} \Pi^\alpha (1 + G)| < \nu + \epsilon/2.$$  

Further, it follows from Lemma 6.3 that there exists a subdivision $D_2$ of $[a, b]$ such that if $\{x_i\}_{i=0}^n$ is a refinement of $D_2$, then

$$\sum_{i=1}^{n} \left| x_{i-1} \Pi^\alpha (1 + G) - \left(1 + \int_{x_{i-1}}^{x_i} G\right) \right| < \epsilon (2| - 1|)^{-1}.$$

Let $D = D_1 \cup D_2$. Suppose $\{x_i\}_{i=0}^n$ is a refinement of $D$. Now,

$$\sum_{i=1}^{n} \left| G_i - \int_{x_{i-1}}^{x_i} G \right|$$

$$= \sum_{i=1}^{n} \left[ 1 + G_i - x_{i-1} \Pi^\alpha (1 + G) \right]$$

$$+ \left[ x_{i-1} \Pi^\alpha (1 + G) - \left(1 + \int_{x_{i-1}}^{x_i} G\right) \right].$$

Thus,

$$\sum_{i=1}^{n} \left| G_i - \int_{x_{i-1}}^{x_i} G \right|$$

$$\leq \sum_{i=1}^{n} \left| 1 + G_i - x_{i-1} \Pi^\alpha (1 + G) \right|$$

$$+ \sum_{i=1}^{n} \left| x_{i-1} \Pi^\alpha (1 + G) - \left(1 + \int_{x_{i-1}}^{x_i} G\right) \right|$$

$$< \nu + \epsilon/2 + \epsilon/2 = \nu + \epsilon.$$  

Further,

$$\sum_{i=1}^{n} \left| G_i - \int_{x_{i-1}}^{x_i} G \right|$$

$$\leq \sum_{i=1}^{n} \left| 1 + G_i - x_{i-1} \Pi^\alpha (1 + G) \right|$$
\[-1 - 1 \left| \sum_{i=1}^{n} \left| x_{i-1} \Pi^i (1 + G) - \left(1 + \int_{x_{i-1}}^{x_i} G \right) \right| \right.\]
\[> v - \epsilon/2 - \epsilon/2 = v - \epsilon.\]

Hence,
\[v - \epsilon < \sum_{i=1}^{n} \left| G_i - \int_{x_{i-1}}^{x_i} G \right| < v + \epsilon.\]

Therefore, \(G \in OA^*\) on \([a, b]\).

Suppose \(G \in OA^*\) on \([a, b]\). It follows from Theorem 3 that \(\Pi^r (1 + G)\) exists for \(a \leq x < y \leq b\). Hence, it is only necessary to show that
\[\int_a^b |1 + G - \Pi (1 + G)| = v.\]

Let \(\epsilon > 0\). There exists a subdivision \(D_1\) of \([a, b]\) such that if \(\{x_i\}_{i=0}^{n}\) is a refinement of \(D_1\), then
\[v - \epsilon/2 < \sum_{i=1}^{n} \left| G_i - \int_{x_{i-1}}^{x_i} G \right| < v + \epsilon/2.\]

Further, it follows from Lemma 6.3 that there exists a subdivision \(D_2\) of \([a, b]\) such that if \(\{x_i\}_{i=0}^{n}\) is a refinement of \(D_2\), then
\[\sum_{i=1}^{n} \left| 1 + \int_{x_{i-1}}^{x_i} G - x_{i-1} \Pi^i (1 + G) \right| < \epsilon (2|1|)^{-1}.\]

Let \(D = D_1 \cup D_2\). Suppose \(\{x_i\}_{i=0}^{n}\) is a refinement of \(D\). Now,
\[\sum_{i=1}^{n} \left| 1 + G_i - x_{i-1} \Pi^i (1 + G) \right| \]
\[= \sum_{i=1}^{n} \left| \left[ G_i - \int_{x_{i-1}}^{x_i} G \right] + \left[ 1 + \int_{x_{i-1}}^{x_i} G - x_{i-1} \Pi^i (1 + G) \right] \right|.\]

It follows as in the preceding argument that
\[v - \epsilon < \sum_{i=1}^{n} \left| 1 + G_i - x_{i-1} \Pi^i (1 + G) \right| < v + \epsilon.\]

Therefore, \(G \in OM^*\) on \([a, b]\).
We now prove a theorem on the existence of integrals of products of functions. This result is related to a theorem by B. W. Helton [2, Theorem 2, p. 494].

**Lemma 7.1.** If $\epsilon > 0$, $H$ is a function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{N}$ and $H \in \text{OL}^\circ$ on $[a, b]$, then there exist a subdivision $\{t_i\}_{i=0}^n$ of $[a, b]$ and a sequence $\{k_i\}_{i=1}^n$ such that if $1 \leq i \leq n$ and $t_{i-1} < x < y < t_i$, then

$$|H(x, y) - k_i| < \epsilon.$$  

**Proof.** This lemma is a variation of a lemma used by B. W. Helton [2, Lemma, p. 498]. The proof presented there can be used to establish the lemma as we have stated it.

**Lemma 7.2.** Suppose $|AB| = |A||B|$ for $A, B \in \mathbb{N}$. If $\nu$ is a nonnegative number, $k \in \mathbb{N}, G$ is a function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{N}$ and $G \in \text{OA}^\nu$ on $[a, b]$, then $kG \in \text{OA}^{k\nu}$ on $[a, b]$.

**Proof.** Since $|AB| = |A||B|$, the proof is readily constructed. If the preceding equality did not hold, the lemma would not necessarily follow. An example of such a situation is presented after the proof of Theorem 7.

**Theorem 7.** Suppose $|AB| = |A||B|$ for $A, B \in \mathbb{N}$. If $\nu$ is a nonnegative number, $H$ and $G$ are functions from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{N}, H \in \text{OL}^\circ$ on $[a, b], G \in \text{OB}^\circ$ on $[a, b]$ and either $G \in \text{OA}^\nu$ on $[a, b]$ or $G \in \text{OM}^\nu$ on $[a, b]$, then there exist nonnegative numbers $\alpha$ and $\beta$ such that $HG$ is in $\text{OA}^\alpha$ and $\text{OM}^\alpha$ on $[a, b]$ and $GH$ is in $\text{OA}^\beta$ and $\text{OM}^\beta$ on $[a, b]$.

**Proof.** We initially establish that there exists a nonnegative number $\alpha$ such that $HG \in \text{OA}^\alpha$ on $[a, b]$. It follows from Theorem 6 that $G \in \text{OA}^\nu$ on $[a, b]$. Hence, the existence of $\int_a^b HG$ follows from Theorem 5. We use the Cauchy criterion to establish the existence of

$$\int_a^b |HG - \int_a^b HG|.$$  

Let $\epsilon > 0$. There exist a subdivision $E_1$ of $[a, b]$ and a number $B$ such that if $\{x_i\}_{i=0}^n$ is a refinement of $E_1$, then

$$\sum_{i=1}^n |G_i| < B.$$
It follows from Lemma 7.1 that there exist a subdivision \( E_2 = \{ t_i \}_{i=0}^t \) of \([a, b]\) and a sequence \( \{ k_i \}_{i=1}^t \) such that if \( 1 \leq i \leq t \) and \( t_{i-1} < x < y < t_i \), then

\[
|H(x, y) - k_i| < \epsilon (8 - 1|B|^{-1}).
\]

Since \( G \in OB^* \cap OA^* \) on \([a, b]\), it follows that there exist subdivisions \( \{ r_i \}_{i=0}^{t+1} \) and \( \{ s_i \}_{i=0}^{t+1} \) of \([a, b]\) such that

1. \( t_{i-1} < r_i < s_i < t_i \) for \( 1 \leq i \leq t \), and
2. \( \sum_{j=1}^n \left| H_j G_j - \int_{s_{j-1}}^{s_j} HG \right| < \epsilon [8(t + 1)]^{-1} \) for \( 1 \leq i \leq t + 1 \) and each refinement \( \{ x_j \}_{j=0}^n \) of \( \{ s_{i-1}, t_{i-1}, r_i \} \).

It follows from Lemma 7.2 that \( k_i G \in OA^{|k_i|} \) on \([r_i, s_i]\) for \( 1 \leq i \leq t \). Hence, for each \( i \) there exists a subdivision \( D_i \) of \([r_i, s_i]\) such that if \( J \) and \( K \) are refinements of \( D_i \), then

\[
\left| \sum_{J(\ell)} k_G - \int k_G \right| - \sum_{K(\ell)} k_G - \int k_G \right| < \epsilon (4t)^{-1}.
\]

Let \( D \) denote the subdivision \( \cup_{i=1}^{t+1} E_i \cup \bigcup_{i=1}^{t} D_i \) of \([a, b]\). Suppose \( J_1 \) and \( J_2 \) are refinements of \( D \), \( P_{1i} \) and \( P_{2i} \) are subdivisions of \([s_{i-1}, r_i]\) for \( 1 \leq i \leq t + 1 \), \( Q_{1i} \) and \( Q_{2i} \) are subdivisions of \([r_i, s_i]\) for \( 1 \leq i \leq t \) and \( J_1 \) and \( J_2 \) are equal to

\[
\bigcup_{i=1}^{t+1} P_{1i} \bigcup_{i=1}^{t} Q_{1i} \quad \text{and} \quad \bigcup_{i=1}^{t+1} P_{2i} \bigcup_{i=1}^{t} Q_{2i},
\]

respectively. For convenience, suppose

\[
\sum_{J(\ell)} |HG - \int HG| \geq \sum_{J(\ell)} |HG - \int HG|.
\]

Thus,

\[
\left| \sum_{J(\ell)} |HG - \int HG| - \sum_{J(\ell)} |HG - \int HG| \right|
\]

\[= \sum_{J(\ell)} |HG - \int HG| - \sum_{J(\ell)} |HG - \int HG| \]

\[= \sum_{i=1}^{t+1} \sum_{P_{\ell}(i)} |HG - \int HG| + \sum_{i=1}^{t} \sum_{Q_{\ell}(i)} |HG - \int HG| \]

\[- \sum_{i=1}^{t+1} \sum_{P_{\ell}(i)} |HG - \int HG| - \sum_{i=1}^{t} \sum_{Q_{\ell}(i)} |HG - \int HG| \]

\[< \epsilon (4t)^{-1}.
\]
Therefore, $\int_a^b \left| HG - \int HG \right|$ exists. Hence, there exists a nonnegative number $\alpha$ such that $G \in OA^\alpha$ on $[a, b]$. Thus, it follows from Theorem 6 that $G \in OM^\alpha$ on $[a, b]$. A similar argument can be used to establish the existence of $\beta$. Therefore, the theorem follows.

Theorem 7 does not remain true if the requirement that $|AB| = |A| |B|$ is removed. In the following we establish this assertion by constructing a function $G$ and a constant $K$ such that $\int_0^1 G$ exists, $\int_0^1 |G - \int G|$ exists and $\int_0^1 |KG - \int KG|$ does not exist.

We consider the set of infinite diagonal matrices with bounded elements and $|M| = \text{lub} \, |m_{ij}|$. For $p = 1, 2, \cdots$, let $A_p$ be the infinite diagonal matrix such that $a_{pp} = 1$ and $a_{qq} = 0$ if $q \neq p$. Let $A = \{A_p \mid p = 1, 2, \cdots\}$. There exists a reversible function $f$ from the rational numbers in $[0, 1]$ to $A$. Let $G$ be an interval function defined on $[0, 1]$ such that
\[
G(u, v) = \begin{cases} 
(v - u)f(v) & \text{if } v \text{ is rational} \\
(v - u)f(r) & \text{where } r \text{ is a rational number in } (u, v) \text{ if } v \text{ is irrational.}
\end{cases}
\]

For each rational number \( r \) in \([0, 1]\), let \( p(r) \) be the positive integer such that \( f(r) = A_{p(r)} \). Let \( K \) be the infinite diagonal matrix such that if \( r = m/n \) is a rational number contained in \([0, 1]\) and \( m \) and \( n \) have no common integral factors other than 1, then

\[
k_{p(r), p(r)} = \begin{cases} 
0 & \text{if } n \text{ is odd} \\
1 & \text{if } n \text{ is even.}
\end{cases}
\]

We have now constructed a function \( G \) and a constant \( K \) such that \( \int_0^1 G = 0 \), \( \int_0^1 |G - \int G| = 1 \) and \( \int_0^1 |KG - \int KG| \) does not exist. This example was suggested by an example in a previous paper by the author [3].

**References**


Received November 1, 1973.
Pacific Journal of Mathematics
Vol. 56, No. 2 December, 1975

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