ON THE FIRST AND THE SECOND CONJUGATE POINTS

W. J. Kim
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Three properties of conjugate points and extremal solutions of an \( n \)th-order linear ordinary differential equation are discussed. Also, a connection between the zero distribution and the factorization of an \( n \)th-order differential operator in the interval \((a, \eta_2(a))\) is established.

1. Introduction. We shall be concerned with the \( n \)th-order differential equation

\[
Ly = \sum_{k=0}^{n} p_k(x) y^{(k)} = 0,
\]

where the coefficients are real-valued functions which are continuous on an interval \( I \) and \( p_n(x) \not= 0, x \in I \). A differential equation of the form (1.1) is called nonsingular on \( I \). A solution \( y \) of (1.1) is said to have a zero of order \( k \) at \( c \in I \) if \( y(c) = y'(c) = \cdots = y^{(k-1)}(c) = 0 \); if in addition \( y^{(k)}(c) \not= 0 \), we say that \( y \) has a zero of order exactly \( k \) at \( c \). A zero of order exactly one is called simple. The \( m \)th conjugate point \( \eta_m(a) \) of a point \( a \in I \) is the smallest number \( b > a, b \in I \), such that there exists a nontrivial solution of (1.1) which vanishes at \( a \) and has \( n + m - 1 \) zeros (counting multiplicities) on \( [a, b] \) [6]. Obviously, we have the relation \( \eta_1(a) \leq \eta_2(a) \leq \cdots \). A nontrivial solution of (1.1) which has \( n \) zeros on \( [a, \eta_i(a)] \) is called an extremal solution for the interval \([a, \eta_i(a)]\). A nontrivial solution of (1.1) is said to have an \( i_1 - i_2 - \cdots - i_j \) distribution of zeros on \( I \) if it has a zero of order \( i_k \) at \( x_k \in I, x_1 < x_2 < \cdots < x_j, k = 1, 2, \ldots, j \).

So far as the study of zero distribution of solutions [1–5, 8–11, 14] is concerned, it is convenient to divide the problem into two cases: \( \eta_1(a) = \eta_2(a) \) and \( \eta_1(a) < \eta_2(a) \). In a recent paper, Gustafson [2] obtained an interesting result for the case \( \eta_1(a) = \eta_2(a) \). Evidently, \( \eta_1(a) < \eta_2(a) \) for any second-order differential equation of the form (1.1). However, for higher-order equations both cases \( \eta_1(a) = \eta_2(a) \) and \( \eta_1(a) < \eta_2(a) \) occur. For example, \( \eta_1(a) = \eta_2(a) = \eta_3(a) \) for the equation \( y^{(10)} + 10y^{''} + 9y = 0 [1], \) while \( \eta_1(a) < \eta_2(a) \) for

\[
(ry^{''})'' - py = 0, \quad r > 0, \quad p > 0, \quad r \in C'', \quad p \in C,
\]
Other equations with the property

\[(P_1) \quad \eta_1(a) < \eta_2(a)\]

have been observed by Peterson [10].

Suppose Eq. (1.1) has an extremal solution for \([\alpha, \eta_1(\alpha)]\). Then it is well-known that (1.1) has an extremal solution for \([\alpha, \eta_2(\alpha)]\) which does not vanish on \((\alpha, \eta_1(\alpha))\) [12]. Of particular interest is the equation which has the property

\[(P_2) \quad \text{No extremal solution for } [\alpha, \eta_1(\alpha)] \text{ vanishes on } (\alpha, \eta_1(\alpha)).\]

For example, it can be easily shown that \(y''' + y = 0\) and \(y''' - y = 0\) have the property \((P_2)\). In fact, every extremal solution of \(y''' + y = 0\) has a 2–1 distribution of zeros. On the other hand, every extremal solution of \(y''' - y = 0\) has a 1–2 distribution of zeros. These two equations also have the property \((P_1)\).

As it turns out, closely connected with \((P_1)\) and \((P_2)\) is the property

\[(P_3) \quad \text{There do not exist two (not necessarily distinct) extremal solutions for } [\alpha, \eta_1(\alpha)] \text{ with zero distributions } (n - k) - k \text{ and } (n - k - 1) - (k + 1), \text{ respectively, where } k \text{ is a fixed number, } 1 \leq k \leq n - 2.\]

In §2 we prove that \((P_3)\) implies \((P_1)\) and \((P_2)\). Conversely, \((P_1)\) and \((P_2)\) taken together imply \((P_3)\). Moreover, we shall show that in general \((P_1)\) neither implies nor is implied by \((P_2)\). As the last result of this section we shall exhibit a class of differential equations which has the properties \((P_1)\), \((P_2)\), and \((P_3)\).

In §3 we assume \((P_1)\) and investigate the zero distribution of solutions on the intervals \([\alpha, \eta_1(\alpha)]\) and \((\alpha, \eta_2(\alpha))\), and their consequences. In particular, we discuss a connection between the zero distribution and the factorization of (1.1) on the interval \((\alpha, \eta_2(\alpha))\).

In the sequel we shall have an occasion to use the function \(w(x; x^{[k_1]}, x^{[k_2]}, \ldots, x^{[k_p]})\) defined and used in [5]. Let \(y_1, y_2, \ldots, y_n\) be \(n\) linearly independent solutions of (1.1). Then the function \(w\) is defined by
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\[ w(x; x_1^{[k_1]}, x_2^{[k_2]}, \ldots, x_p^{[k_p]}) \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1(x_1) & y_2(x_1) & \cdots & y_n(x_1) \\ y_1'(x_1) & y_2'(x_1) & \cdots & y_n'(x_1) \\ \vdots & \vdots & & \vdots \\ y_1^{(k_1-1)}(x_1) & y_2^{(k_1-1)}(x_1) & \cdots & y_n^{(k_1-1)}(x_1) \\ y_1(x_2) & y_2(x_2) & \cdots & y_n(x_2) \\ \vdots & \vdots & & \vdots \\ y_1(x_p) & y_2(x_p) & \cdots & y_n(x_p) \\ \vdots & \vdots & & \vdots \\ y_1^{(k_p-1)}(x_p) & y_2^{(k_p-1)}(x_p) & \cdots & y_n^{(k_p-1)}(x_p) \end{vmatrix} \]

1 \leq p \leq n - 1, \ k_1 + k_2 + \cdots + k_p = n - 1. Obviously, this function \( w \) is a solution of (1.1) with a zero of order \( k_i \) at \( x_i, i = 1, 2, \ldots, p \). Moreover, it is continuous function of \( x_1, x_2, \ldots, x_p \).

2. Properties \( (P_1), (P_2) \) and \( (P_3) \). Suppose (1.1) has an extremal solution \( Y \) for \([\alpha, \eta_\alpha(a)]\) with an \( i_1 - i_2 - \cdots - i_j \) distribution of zeros, i.e., \( Y \) has a zero of order \( i_k \) at \( x_k, k = 1, 2, \ldots, j \), \( i_1 + i_2 + \cdots + i_j = n \), \( a = x_1 < x_2 < \cdots < x_j = \eta_\alpha(a) \). Numerous results have been obtained for the zero distribution of \( Y \) \([2, 5, 9, 10, 12]\). Of particular importance in this section is the following result which will be frequently referred to in the proofs.

**Theorem 2.1** \([5]\). If \( Y \) has a zero of order exactly \( i_m \) at \( x_m \), \( 2 \leq m \leq j - 1 \), then (1.1) has an extremal solution for \([\alpha, \eta_\alpha(a)]\) with an \( i_1 - \cdots - i_{m-1} - (i_m - 1) - i_{m+1} - \cdots - i_j \) distribution of zeros and an additional zero at an arbitrary point \( \xi \in [\alpha, \eta_\alpha(a)] \).

A simple application of this theorem shows that \( (P_3) \) implies \( (P_2) \). This result can then be used to prove that \( (P_3) \) also implies \( (P_1) \). On the other hand, if (1.1) does not satisfy \( (P_3) \), it is easily confirmed that (1.1) must violate either \( (P_1) \) or \( (P_2) \).

**Theorem 2.2.** Eq. (1.1) has the property \( (P_3) \) if and only if it satisfies \( (P_1) \) and \( (P_2) \).

We shall illustrate by means of examples that in general \( (P_1) \) neither implies nor is implied by \( (P_2) \). The nonsingular equation
\[ y''' - \frac{6 \sin x \cos x (\cos^2 x - \sin^2 x)}{3 \sin^2 x \cos^2 x - 2} y'' \]

(2.1)

\[-\frac{9 \sin^2 x \cos^2 x + 14}{3 \sin^2 x \cos^2 x - 2} y' = 0\]

has as a fundamental set of solutions \(\sin^2 x \cos x, \cos^2 x \sin x,\) and 1 [1, 13]. The Wronskian \(W\) of these three solutions is given by \(W = 3 \sin^2 x \cos^2 x - 2 < 0\), and the corresponding adjoint equation

\[ v''' + \left( \frac{6 \sin x \cos x (\cos^2 x - \sin^2 x)}{3 \sin^2 x \cos^2 x - 2} v \right)'' \]

(2.1)*

\[-\left( \frac{9 \sin^2 x \cos^2 x + 14}{3 \sin^2 x \cos^2 x - 2} v \right)' = 0\]

has a fundamental set of solutions \(\sin^2 2x/W, (2 \sin x - 3 \sin^3 x)/W,\) and \((3 \cos^3 x - 2 \cos x)/W\). It is easily confirmed that \(\eta_i(0) = \eta_2(0) = \pi/2\) for (2.1)* and no extremal solution for \([0, \eta_i(0)]\) of (2.1)* vanishes in \((0, \eta_i(0))\). This shows that \((P_2)\) does not in general imply \((P_1)\).

To see that \((P_1)\) does not in general imply \((P_2)\), consider the nonsingular equation

(2.2) \((6x^2 - 8x + 3)y^{(iv)} - (12x - 8)y''' + 12y'' = 0,\)

for which \(1, x, x(1 - x)^2,\) and \(x(1 - x)^3\) form a fundamental set of solutions, \(\eta_i(0) = 1,\) and of which no extremal solution for \([0, 1]\) has a 3–1 distribution of zeros [4]. Moreover, no extremal solution for \([0, 1]\) can vanish more than once in \((0, 1)\). It is easily verified that (2.2) has no nontrivial solution with zeros of order 2 and 3 at \(x = 0\) and \(x = 1,\) respectively. From these facts we can readily deduce \(\eta_i(0) < \eta_3(0)\). On the other hand, \(x(\lambda - x)(1 - x)^2, 0 < \lambda < 1,\) is an extremal solution for \([0, 1]\) which vanishes at \(\lambda, 0 < \lambda < 1.\)

An obvious consequence of these examples is that \((P_3)\) is not in general implied by either \((P_1)\) or \((P_2)\) alone.

In view of Theorem 2.2, it is clear that any differential equation which satisfies \((P_3)\) will also satisfy \((P_1)\) and \((P_2)\). Consider a differential equation of the form

(2.3) \(L_n y + py = 0,\)

where the operator \(L_n\) is successively defined by
The functions $\rho_0, \rho_1, \ldots, \rho_n$ are assumed to be positive, $\rho_k \in C^{n-k}$, $k = 0, 1, \ldots, n$, and $p$ is assumed not to vanish. Eq. (2.3) was extensively studied by Nehari [7], who established the following result: If a nontrivial solution of (2.3) has zeros of order $k$ and $n-k$ at $x = a$ and $x = b$, respectively ($a < b$), then $n-k$ is even or odd, according as $p < 0$ or $p > 0$. Evidently, this result implies that Eq. (2.3) satisfies (P$_3$). Hence, we have the following theorem.

**Theorem 2.3.** Eq. (2.3) has the properties (P$_1$), (P$_2$), and (P$_3$).

3. Zero distribution and factorization. In this section we exclusively consider a differential equation of the form (1.1) with property (P$_0$). Let $Y$ be an extremal solution of (1.1) for $[a, \eta_1(a)]$ with an $i_1 - i_2 - \cdots - i_j$ distribution of zeros, $a = x_1 < x_2 < \cdots < x_j = \eta_1(a)$, $i_1 + i_2 + \cdots + i_j = n$. Then $Y$ has a zero of order exactly $i_k$ at $x_k$, $k = 1, 2, \ldots, j$. This is because $\eta_1(a) < \eta_2(a)$. Therefore, by a repeated application of Theorem 2.1, we obtain

**Theorem 3.1.** Suppose (1.1) has the property $\eta_1(a) < \eta_2(a)$ and has an extremal solution for $[a, \eta_1(a)]$ with an $i_1 - i_2 - \cdots - i_j$ distribution of zeros, $i_1 + i_2 + \cdots + i_j = n$. Let $k_1, k_2, \ldots, k_p$ be arbitrary positive integers such that $k_1 + k_2 + \cdots + k_p = n$, and let $a = \xi_1, \xi_2, \ldots, \xi_p = \eta_1(a)$ be distinct points in $[a, \eta_1(a)]$. If $i_1 \leq k_1$ and $i_j \leq k_p$, then (1.1) has an extremal solution for $[a, \eta_1(a)]$ which has a zero of order exactly $k_m$ at $\xi_m, m = 1, 2, \ldots, p$.

As is clear from Theorem 3.1, the zeros of solutions in $(a, \eta_1(a))$ can be moved to an arbitrary point in $(a, \eta_1(a))$, or can be separated into lower-order zeros in $(a, \eta_1(a))$. However, no such statements can be made in general for the zeros at the end points $a$ and $\eta_1(a)$. On the other hand, the zeros of an extremal solution for $[a, \eta_1(a)]$ can be simultaneously separated into simple zeros in $[a, \eta_1(a) + \epsilon], \epsilon > 0 [4, 14]$. By using a slight modification of the arguments given in the proof of Theorem 1 in [4], we shall establish the following result.

**Theorem 3.2.** If (1.1) has a nontrivial solution with an $(n-l) - l$ distribution of zeros in $(a, \eta_2(a))$, then (1.1) has a nontrivial solution with the zero distribution

$$1 - 1 - \cdots - 1 - j - 1 - \cdots - 1, \quad i + j + k = n,$$

in $(a, \eta_2(a))$, provided $i \geq n - l$ or $k \geq l$. 

The functions $L_0 y = \rho_0 y, L_k y = \rho_k (L_{k-1} y)'$, $k = 2, 3, \ldots, n$. 

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Loy = po y, Lky = ρk (Lk-xy), k = 2, 3, · · · , n.
Proof. Consider the case \( i \geq n - l \). Let \( y \) be a nontrivial solution of (1.1) which has zeros of order \( n - l \) and \( l \) at \( x = b \) and \( x = c \), respectively, \( a < b < c < \eta_2(a) \) and suppose \( l \) is maximal. Consider the function

\[
w(x) = \begin{cases} 
  w(x; c^{[n-l]}) & \text{if } l = n - 1, \\
  w(x; b^{[n-l-1]}, c^{[l]}), & \text{otherwise}
\end{cases}
\]

defined in §1. This function \( w \) cannot vanish identically; for if \( w \equiv 0 \) it would imply the existence of a nontrivial solution with a zero of order \( n - l - 1 \) at \( b \) and a zero of order \( l + 1 \) at \( c \), contrary to the assumption. Therefore, \( w \) is a nontrivial solution of (1.1) with a zero of order exactly \( n - l \) at \( b \) and a zero of order exactly \( l \) at \( c \). Consequently, the \( n - l \) zeros at \( b \) and \( l - j \) (out of \( l \) \( c \) can be separated into \( n - j \) simple zeros in such a way that there are \( i \) simple zeros to the left and \( k \) simple zeros to the right of the \( j \)-th order zero at \( c \). (Cf. The proof of Theorem 1 [4]). This proves the theorem for the case \( i \geq n - l \).

The proof for the case \( k \geq l \) is similar.

Remark. The above theorem can be restated as follows: If (1.1) does not have a nontrivial solution with the zero distribution (3.1) in \((a, \eta_2(a))\), then (1.1) does not have nontrivial solutions in \((a, \eta_2(a))\) with zero distributions \((n - 1) - 1, (n - 2) - 2, \ldots, (n - k) - k, (n - k - j) - (k + j), \ldots, 1 - (n - 1)\).

We shall see that this result provides a link between the zero distribution and the factorization of the differential operator \( L \) in (1.1).

Let \( y_1, y_2, \ldots, y_n \) be \( n \) linearly independent solutions of (1.1) and define

\[
W_k = \begin{vmatrix}
  y_1 & y_2 & \cdots & y_k \\
  y_1' & y_2' & \cdots & y_k' \\
  \cdots & \cdots & \cdots & \cdots \\
  y_1^{(k-1)} & y_2^{(k-1)} & \cdots & y_k^{(k-1)}
\end{vmatrix}, \quad k = 1, 2, \ldots, n.
\]

It is well-known that \( W_p > 0 \) if and only if the operator \( L \) can be written as \( L = L_1L_2 \), where \( L_1 \) and \( L_2 \) are nonsingular differential operators of order \( n - p \) and \( p \), respectively [15]. We require the following obvious extension of this result.
\textbf{Theorem 3.3.} Eq. (1.1) has \( k \) solutions \( y_1, y_2, \ldots, y_k \) such that \( W_{k_1} > 0, \ W_{k_2} > 0, \ldots, \ W_{k_i} > 0 \), \( k_1 < k_2 < \cdots < k_i = k \), if and only if the differential operator \( L \) in (1.1) can be written as the product of \( l + 1 \) nonsingular differential operators, i.e., \( L = L_{l+1} L_l \cdots L_1 \), where \( L_1 \) is of order \( k_1 \), \( L_i \) is of order \( k_i - k_{i-1} \), \( i = 2, 3, \ldots, l \), and \( L_{l+1} \) is of order \( n - k_l \).

Suppose (1.1) does not have a nontrivial solution with an \((n - p) - p\) distribution of zeros in \((a, b)\). Let \( y_1, y_2, \ldots, y_n \) be solutions of (1.1) such that \( y_i^{(n-j)}(a + \epsilon) = \delta_i, \ \epsilon > 0, \ i, j = 1, 2, \ldots, n. \) Then \( W_p > 0 \) in \((a + \epsilon, b)\). Since \( \epsilon > 0 \) is arbitrary, we may assume that \( W_p > 0 \) in \((a, b)\). Hence, we have \( L = L_1 L_2 \), where \( L_1 \) and \( L_2 \) are nonsingular differential operators of order \( n - p \) and \( p \), respectively.

Likewise, from Theorems 3.2, 3.3, and the above remark we deduce

\textbf{Theorem 3.4.} If (1.1) does not have a nontrivial solution with the zero distribution (3.1) in \((a, \eta_2(a))\), the differential operator \( L \) can be written as the product of nonsingular differential operators,

\[ L = L_{i+k+1} L_{i+k} \cdots L_1 \]

in \((a, \eta_2(a))\), where \( L_m, m \neq k + 1, \) is of first order and \( L_{k+1} \) is of \( j \)th order.

Let

\[ \mathcal{L} v = \sum_{k=0}^{n} q_k(\xi) v^{(k)} = 0 \]

be the differential equation obtained from \( Ly = 0 \) through the change of variable \( \xi = a + \eta_2(a) - x \). Clearly, \( Ly = 0 \) has a nontrivial solution with an \( i_1 - i_2 - \cdots - i_k \) distribution of zeros in \((a, \eta_2(a))\) if and only if \( \mathcal{L} v = 0 \) has a nontrivial solution with an \( i_k - i_{k-1} - \cdots - i_1 \) distribution of zeros in \((a, \eta_2(a))\). In particular, if \( Ly = 0 \) does not have a nontrivial solution with the zero distribution (3.1), then \( \mathcal{L} v = 0 \) does not have a nontrivial solution with the zero distribution

\[ 1 - \underbrace{1 - \cdots - 1}_{k+1} - \underbrace{j - 1 - \cdots - 1}_{i+1}, \quad i + j + k = n, \]

in \((a, \eta_2(a))\). Apply Theorem 3.4 to the nonsingular differential operator \( \mathcal{L} \): \( \mathcal{L} \) can be written as the product of nonsingular differential operators

\[ (3.2) \quad \mathcal{L} = \mathcal{L}_{i+k+1} \mathcal{L}_{i+k} \cdots \mathcal{L}_1 \]
in \((a, \eta_2(a))\), where \(\hat{\xi}_p, p \neq i + 1\), is of first order and \(\hat{\xi}_{i+1}\) is of \(j\)th order. Transform the equation \(\mathcal{L}v = \hat{\xi}_{i+k+1} \hat{\xi}_{i+k} \cdots \hat{\xi}_1 v = 0\) back to \(Ly = 0\) by substituting \(x = a + \eta_2(a) - \xi\). Under this transformation each differential operator \(\hat{\xi}_p, p = 1, 2, \ldots, i + k + 1\), in (3.2) remains nonsingular. Moreover, the order of each \(\hat{\xi}_p\) and the order in which these differential operators appear remain unchanged. We summarize this result in the following theorem.

**Theorem 3.5.** If (1.1) does not have a nontrivial solution with the zero distribution (3.1) in \((a, \eta_2(a))\), the differential operator \(L\) can be written as the product of nonsingular differential operators, \(L = \mathcal{L}_{i+k+1} \mathcal{L}_{i+k} \cdots \mathcal{L}_1\), in \((a, \eta_2(a))\), where \(\mathcal{L}_p, p \neq i + 1\), is of first order and \(\mathcal{L}_{i+1}\) is of \(j\)th order.

**References**


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