AN OBSTRUCTION TO EXTENDING ISOTOPIES OF PIECEWISE LINEAR MANIFOLDS

Ewing L. Lusk
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EWING L. LUSK

Let $F: M \times I^n \to Q \times I^n$ be an $n$-isotopy (not necessarily PL) of a compact PL $m$-manifold $M$ in a PL $q$-manifold $Q$, and let $G: Q \times I^n \to Q \times I^n$ be an ambient isotopy of $Q$ which covers $F$ on $Q \times \partial I^n$. If $m \leq q - 3$ there is in $\pi_n \text{PL}(M, Q)$ an obstruction to finding an ambient isotopy of $Q$, isotopic to $G$, which covers $F$ and agrees with $G$ on $Q \times \partial I^n$.

1. Introduction. In the proof of the Hudson-Zeeman covering isotopy theorem [6], one has no control over the homeomorphism of the ambient manifold which one obtains at the end of the isotopy. In general, one might ask for sufficient conditions under which a given $n$-isotopy $F: M \times I^n \to Q \times I^n$ of one PL manifold in another, fixed on $\partial M$, can be covered by an ambient $n$-isotopy $H: Q \times I^n \to Q \times I^n$ fixed on $\partial Q$, in such a way that $H|Q \times \partial I^n$ is equal to some given level-preserving homeomorphism $G$ of $Q \times \partial I^n$ which covers $F|M \times \partial I^n$. Necessary conditions are that $F$ be level-preservingly locally flat and that $G$ have some extension to $Q \times I^n$ which is fixed on $\partial Q$. That these conditions are not sufficient can be seen by considering an isotopy $F: S^1 \times I \to I^2 \times I$ of a circle in the interior of $I^2$ which rotates the circle through $360^\circ$. Since $F$ can be chosen PL and locally flat, it follows from the ordinary covering isotopy theorem [6] that $F$ can be covered by an ambient isotopy $H$ of $I^2$ which is fixed on $\partial I^2$. But if $G: \partial(I^2 \times I) \to \partial(I^2 \times I)$ is the identity homeomorphism, then $H$ cannot be an extension of $G$. The difficulty here arises from the fact that the space of embeddings of $S^1$ into $I^2$ which is not simply connected. The theorem below extends results of Gluck, Husch, and Rushing [3,8]. Let $M$ and $Q$ be PL $m$- and $q$-manifolds respectively, with $M$ compact, and let $\text{PL}(M, Q; f)$ denote the semi-simplicial complex of proper PL embeddings of $M$ into $Q$, with base point $f$.

Theorem 1. Let $F: M \times I^n \to Q \times I^n$ be a proper level-preservingly locally flat $n$-isotopy (not necessarily PL) fixed on $\partial M$. Let $G: Q \times I^n \to Q \times I^n$ be an ambient $n$-isotopy of $Q$, fixed on $\partial Q$, such that $G \circ (F_0 \times 1)|M \times \partial I^n = F|M \times \partial I^n$. Suppose that $m \leq q - 3$. Then there is a homomorphism $h$ of $Q$ such that $hF_0$ is PL and an obstruction $\alpha$ in $\pi_n \text{PL}(M, Q; hF_0)$ such that $\alpha = 0$ if and only if there is a level-preserving isotopy $K$ of $Q \times I^n$, fixed on $\partial(Q \times I^n)$, such that $K_1 G \circ (F_0 \times 1) = F$; i.e. $K_1 G$ extends $G|Q \times \partial I^n$ and covers $F$. 

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Remark 1. If \( F \) and \( G \) are PL, then the local flatness condition on \( F \) need not be level-preserving, and \( K \) can be taken to be PL. The proof of Theorem 1 in this PL case is like the proof given in [8] for the case \( n = 1 \) and so is known. In the topological case, Theorem 1 follows straightforwardly from the fact that the inclusion \( \text{PL}(M, Q) \subset \text{TOP}(M, Q) \) is dense and a weak homotopy equivalence (See Theorem 2 below).

Remark 2. Various combinations of dimension and connectivity conditions are sufficient to ensure that \( \pi_n \text{PL}(M, Q; hF_0) = 0 \) and hence that the obstruction vanishes. We list some of them here. (See [7] and [9].)

(a) \( \pi_r(Q) = 0 \) for \( n \leq r \leq m + n \) and \( 2m + n \leq q - 2 \).

(b) \( M \) is \( (2m - q + n) \)-connected, \( Q \) is \( (2m - q + n + 1) \)-connected, \( \pi_r(Q) = 0 \) for \( n \leq r \leq m + n \), and \( m + n \leq q - 2 \).

(c) \( \pi_r(Q) = 0 \) for \( n \leq r \leq m + n \), \( F_0 \) is \( (2m - q + n + 1) \)-connected, and \( m + n \leq q - 2 \).

2. Definitions. Let \( I^n \) be the \( n \)-fold product of the unit interval \([0, 1]\). The point \((0,0, \cdots, 0)\) in \( I^n \) will be denoted by \( 0 \), and the subset \( I^{n-1} \times 0 \cup \partial I^{n-1} \times I \) of \( I^n = I^{n-1} \times I \) will be denoted by \( J^{n-1} \). An \( n \)-isotopy of \( M \) in \( Q \) is an embedding \( F: M \times I^n \to Q \times I^n \) which is level-preserving \((p \circ F = p \text{ where } p \text{ is projection onto } I^n)\). It is proper if \( F^{-1}(\partial Q \times I^n) = \partial M \times I^n \). An embedding \( F_i: M \to Q \) is defined for each \( t \in I^n \) by \( F_i(x, t) = (F_i(x), t) \). A 1-isotopy is called an isotopy, and \( F_0 \) and \( F_1 \) are said to be isotopic. An \( n \)-isotopy \( F \) is fixed on \( X \) if \( F|X \times I^n = F_0 \times 1|X \times I^n \), where 1 denotes the identity map. It is level-preservingly locally flat if for each \((x, t) \in M \times I^n \) there is a neighborhood \( N \) of \( t \) in \( I^n \), a level-preserving embedding \( H \) of either \( E^m \times N \) or \( E^n \times N \) into \( M \times N \) (depending on whether \( x \) is in \( \text{int } M \) or \( \partial M \)) with \( H(0, t) = (x, t) \), and a level preserving embedding \( G \) of either \( E^q \times N \) or \( E^q \times N \) into \( Q \times N \) depending on whether \( F_t(x) \) is in \( \text{int } Q \) or \( \partial Q \) with \( G(0, t) = F(x, t) \), such that \( G^{-1} FH \) is of the form \( i \times 1 \), where \( i \) is the natural inclusion of \( E^m \) into \( E^q \) or \( E^n \) into \( E^q \), as the case may be. An ambient \( n \)-isotopy of \( Q \) is a level-preserving homeomorphism \( H \) of \( Q \times I^n \) such that \( H_0 = 1 \). If \( A \subset X \), an \( \varepsilon \)-push of \((X, A)\) is an ambient isotopy of \( X \) which is fixed outside an \( \varepsilon \)-neighborhood of \( A \).

We make use of the semi-simplicial complexes \( \text{Aut}_{pl}(Q) \) and \( \text{PL}(M, Q) \), whose \( k \)-simplices are ambient \( k \)-isotopies of \( Q \) fixed on \( \partial Q \) and proper \( k \)-isotopies of \( M \) in \( Q \) fixed on \( \partial M \), respectively. The Hudson covering \( n \)-isotopy theorem [5] can be used to prove, as in [4], that if \( f: M \to Q \) is a given PL embedding then the simplicial map \( p: \text{Aut}_{pl}(Q) \to \text{PL}(M, Q) \) given by \( p(H) = H \circ (f \times 1) \) is a fibration, i.e.,
given level-preserving embeddings $K: Q \times I^{n-1} \rightarrow Q \times I^{n-1}$ and $L: M \times I^r \rightarrow Q \times I^r$ such that $p(K) = L \mid M \times I^{n-1}$, there is an $n$-isotropy $H: Q \times I^r \rightarrow Q \times I^r$ such that $p(H) = L$ and $H \mid Q \times I^{n-1} = K$. An element of $\pi_n^{PL}(M, Q; f)$ is represented by a level-preserving PL embedding $L: M \times \partial I^{n+1}$ such that $L_0 = f$.

3. Spaces of embeddings. In this section we consider the relationship between $PL(M, Q)$ and $TOP(M, Q)$, the semi-simplicial complex of topological embeddings of $M$ into $Q$. Recent work of Edwards and Miller [2, 12] has relaxed the dimension restrictions on the results in [10]. The key lemma is the following.

**Lemma 1.** Let $H: M \times I^n \rightarrow Q \times I^n$ be a level-preserving embedding. Suppose that $m \leq q - 3$ and $q \geq 5$. Then for any $\epsilon > 0$ there is a $\delta = \delta(\epsilon, H) > 0$ such that if $G_0, G_1: M \times I^n \rightarrow Q \times I^n$ are level-preserving PL embeddings with $d(G_0, H) < \delta$, then there is a level-preserving $\epsilon$-push $K$ of $(Q \times I^n, H(M \times I^n))$ such that $K \mid G_0 = G_1$. If $G_0$ and $G_1$ agree on $M \times \partial I^n$, then $K$ can be assumed fixed on $Q \times \partial I^n$.

**Proof.** If $H$ is of the form $h \times 1$ for some embedding $h: M \rightarrow Q$, then the lemma follows directly from Corollary 2 of [2] and Corollary 3 of [1]. Generalization to the case in which $H$ is not of this form can be carried out as in the second half of the proof of Theorem 4.2 $(m, s)$ in [10].

**Remark 3.** The above "local solvability" result is the basis for Theorems 2.1–2.5 of [10] which are stated there with more stringent dimension restrictions. We may now regard those results to be true for $m \leq q - 3$, $q \geq 5$. In particular, Theorems 2.1 and 2.4 give us

**Theorem 2.** If $m \leq q - 3$ and $q \geq 5$, then the inclusion $PL(M, Q) \subset TOP(M, Q)$ is dense and a weak homotopy equivalence; i.e., if $f: M \rightarrow Q$ is PL, then the homomorphism $i: \pi_n^{PL}(M, Q; f) \rightarrow \pi_n^{TOP}(M, Q; f)$ induced by inclusion is an isomorphism for all $n$.

4. **Proof of Theorem 1.** The following lemma, which is Theorem 2.3 of [10] with the new dimension conditions, makes possible the treatment of the non-PL case with PL techniques.

**Lemma 2.** Let $F: M \times I^n \rightarrow Q \times I^n$ be a level-preservingly locally flat proper $n$-isotopy which is PL on $\partial(M \times I^n)$. Suppose $m \leq q - 3$ and $q \geq 5$, and that $\epsilon > 0$ is given. Then there is a level-preserving
Proof of Theorem 1. By Lemma 2 with \( n = 0 \) (See [11]), there is a small homeomorphism \( h \) of \( Q \) such that \( hF_0: M \to Q \) is PL. Consider the embedding \( (h \times 1)G^{-1}F: M \times I^n \to Q \times I^n \). Since it is a level-preservingly locally flat \( n \)-isotopy and \( (h \times 1)G^{-1}F|\partial(M \times I^n) = (hF_0) \times 1 \), which is PL, there is by Lemma 2 a level-preserving isotopy \( T \) of \( Q \times I^n \), fixed on \( \partial(Q \times I^n) \), such that \( T_xF \) is PL.

Define \( L: M \times \partial I^{n+1} \to Q \times \partial I^{n+1} \) by considering \( I^{n+1} \) as \( I^n \times 1 \). Then \( L \) is PL and so represents an element \( \alpha \) of \( \pi_n \text{PL}(M,Q; hF_0) \). To say \( \alpha = 0 \) in \( \pi_n \text{PL}(M,Q; hF_0) \) is to say that there is a PL \((n+1)\)-isotopy \( H': M \times \partial I^{n+1} \to Q \times I^{n+1} \) such that \( H'|M \times \partial I^{n+1} = L \). Therefore we can use the lifting property of the fibration \( p: \text{Aut}_{\text{PL}}(Q) \to \text{PL}(M,Q) \) given by \( p(K) = K \circ (hF_0 \times 1) \) to find an ambient \((n+1)\)-isotopy \( H'': Q \times I^{n+1} \to Q \times I^{n+1} \) such that \( H''|Q \times J^n = 1 \) and \( H'' \circ (hF_0 \times 1) = H' \). Now we define

\[
K = (G \times 1)((h^{-1} \times 1 \times 1)T^{-1}H''((h \times 1 \times 1)(G^{-1} \times 1)):
(Q \times I^n) \times I \to (Q \times I^n) \times 1.
\]

Then \( K \cdot G \) covers \( F \) and extends \( G|Q \times I^n \), as desired.

Conversely, if \( K \) exists with the desired properties, then \( K': M \times \partial I^{n+1} \times I \to Q \times \partial I^{n+1} \times I \) defined by \( K' = T_1 \cdot (h \times 1)G^{-1}K_1 \cdot G(F_0 \times 1) \) on \( M \times J^n \times 1 \) and \( hF_0 \times 1 \) on \( M \times (I^{n-1} \times 1) \times I \) is a level-preserving isotopy taking \( L \) to \( hF_0 \times 1 \). Therefore \( \alpha \) is trivial as an element of \( \pi_n \text{TOP}(M,Q; hF_0) \), the semi-simplicial complex of embeddings of \( M \) into \( Q \). By Theorem 2, \( \alpha \) is trivial in \( \pi_n \text{PL}(M,Q; hF_0) \).

5. The obstruction \( \alpha \). In the construction above, \( \alpha \) appeared to depend on \( h, T, \) and \( G \). In this section we show that \( \alpha \) can be chosen in such a way that it depends only on \( F \).

In applying Lemma 2 to construct \( h \), above, we may choose \( h \) so that \( hF_0 \) is within \( \delta(F_0,1) \) of \( F_0 \), where \( \delta \) comes from Lemma 1. Any two such homeomorphisms \( h \) and \( h' \) will then be such that \( hF_0 \) and \( h'F_0 \) are ambient isotopic. Similarly we choose \( T \) to be a \( \delta((h \times 1)G^{-1}F,1) \)-push, so that if \( T' \) is another push which takes \( (h \times 1)G^{-1}F \) to a PL embedding, \( T_1(h \times 1)G^{-1}F \) and \( T_1'(h \times 1)G^{-1}F \) are PL ambient isotopic, and the \( \alpha \)'s constructed with them will be homotopic in \( \pi_n \text{PL}(M,Q; hF_0) \).

Now suppose that \( G \) and \( G' \) are level-preserving homeomorphisms of \( Q \times I^n \) satisfying the hypotheses of the theorem. Since \( G^{-1}F \) and
$G'^{-1}F$ are each isotopic to $F_0 \times 1$, they are isotopic. If we denote by $\alpha$ and $\alpha'$ the obstructions constructed as above from $G$ and $G'$, the isotopy of $G^{-1}F$ to $G'^{-1}F$ will induce a homotopy from $\alpha$ to $\alpha'$ in $\pi_n \text{TOP}(M,Q)$. By Theorem 2, $\alpha$ is homotopic to $\alpha'$ in $\pi_n \text{PL}(M,Q;hF_0)$, and so $\alpha$ does not depend on $G$.

REFERENCES


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