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**AN OBSTRUCTION TO EXTENDING ISOTOPIES OF
PIECEWISE LINEAR MANIFOLDS**

EWING L. LUSK

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Let $F: M \times I^n \rightarrow Q \times I^n$ be an n -isotopy (not necessarily PL) of a compact PL m -manifold M in a PL q -manifold Q , and let $G: Q \times I^n \rightarrow Q \times I^n$ be an ambient isotopy of Q which covers F on $Q \times \partial I^n$. If $m \leq q - 3$ there is in $\pi_n \text{PL}(M, Q)$ an obstruction to finding an ambient isotopy of Q , isotopic to G , which covers F and agrees with G on $Q \times \partial I^n$.

1. Introduction. In the proof of the Hudson-Zeeman covering isotopy theorem [6], one has no control over the homeomorphism of the ambient manifold which one obtains at the end of the isotopy. In general, one might ask for sufficient conditions under which a given n -isotopy $F: M \times I^n \rightarrow Q \times I^n$ of one PL manifold in another, fixed on ∂M , can be covered by an ambient n -isotopy $H: Q \times I^n \rightarrow Q \times I^n$ fixed on ∂Q , in such a way that $H|_{Q \times \partial I^n}$ is equal to some given level-preserving homeomorphism G of $Q \times \partial I^n$ which covers $F|M \times \partial I^n$. Necessary conditions are that F be level-preservingly locally flat and that G have some extension to $Q \times I^n$ which is fixed on ∂Q . That these conditions are not sufficient can be seen by considering an isotopy $F: S^1 \times I \rightarrow I^2 \times I$ of a circle in the interior of I^2 which rotates the circle through 360° . Since F can be chosen PL and locally flat, it follows from the ordinary covering isotopy theorem [6] that F can be covered by an ambient isotopy H of I^2 which is fixed on ∂I^2 . But if $G: \partial(I^2 \times I) \rightarrow \partial(I^2 \times I)$ is the identity homeomorphism, then H cannot be an extension of G . The difficulty here arises from the fact that the space of embeddings of S^1 into I^2 is not simply connected. The theorem below extends results of Gluck, Husch, and Rushing [3,8]. Let M and Q be PL m - and q -manifolds respectively, with M compact, and let $\text{PL}(M, Q; f)$ denote the semi-simplicial complex of proper PL embeddings of M into Q , with base point f .

THEOREM 1. *Let $F: M \times I^n \rightarrow Q \times I^n$ be a proper level-preservingly locally flat n -isotopy (not necessarily PL) fixed on ∂M . Let $G: Q \times I^n \rightarrow Q \times I^n$ be an ambient n -isotopy of Q , fixed on ∂Q , such that $G \circ (F_0 \times 1)|_{M \times \partial I^n} = F|M \times \partial I^n$. Suppose that $m \leq q - 3$. Then there is a homeomorphism h of Q such that hF_0 is PL and an obstruction α in $\pi_n \text{PL}(M, Q; hF_0)$ such that $\alpha = 0$ if and only if there is a level-preserving isotopy K of $Q \times I^n$, fixed on $\partial(Q \times I^n)$, such that $K_1 G \circ (F_0 \times 1) = F$; i.e. $K_1 G$ extends $G|_{Q \times \partial I^n}$ and covers F .*

REMARK 1. If F and G are PL, then the local flatness condition on F need not be level-preserving, and K can be taken to be PL. The proof of Theorem 1 in this PL case is like the proof given in [8] for the case $n = 1$ and so is known. In the topological case, Theorem 1 follows straightforwardly from the fact that the inclusion $\text{PL}(M, Q) \subset \text{TOP}(M, Q)$ is dense and a weak homotopy equivalence (See Theorem 2 below).

REMARK 2. Various combinations of dimension and connectivity conditions are sufficient to ensure that $\pi_n \text{PL}(M, Q; hF_0) = 0$ and hence that the obstruction vanishes. We list some of them here. (See [7] and [9].)

(a) $\pi_r(Q) = 0$ for $n \leq r \leq m + n$ and $2m + n \leq q - 2$.

(b) M is $(2m - q + n)$ -connected, Q is $(2m - q + n + 1)$ -connected, $\pi_r(Q) = 0$ for $n \leq r \leq m + n$, and $m + n \leq q - 2$.

(c) $\pi_r(Q) = 0$ for $n \leq r \leq m + n$, F_0 is $(2m - q + n + 1)$ -connected, and $m + n \leq q - 2$.

2. Definitions. Let I^n be the n -fold product of the unit interval $[0, 1]$. The point $(0, 0, \dots, 0)$ in I^n will be denoted by 0 , and the subset $I^{n-1} \times 0 \cup \partial I^{n-1} \times I$ of $I^n = I^{n-1} \times I$ will be denoted by J^{n-1} . An n -isotopy of M in Q is an embedding $F: M \times I^n \rightarrow Q \times I^n$ which is level-preserving ($p \circ F = p$ where p is projection onto I^n). It is *proper* if $F^{-1}(\partial Q \times I^n) = \partial M \times I^n$. An embedding $F_t: M \rightarrow Q$ is defined for each $t \in I^n$ by $F(x, t) = (F_t(x), t)$. A 1-isotopy is called an *isotopy*, and F_0 and F_1 are said to be *isotopic*. An n -isotopy F is *fixed on X* if $F|X \times I^n = F_0 \times 1|X \times I^n$, where 1 denotes the identity map. It is *level-preservingly locally flat* if for each $(x, t) \in M \times I^n$ there is a neighborhood N of t in I^n , a level-preserving embedding H of either $E^m \times N$ or $E^m_+ \times N$ into $M \times N$ (depending on whether x is in $\text{int } M$ or ∂M) with $H(0, t) = (x, t)$, and a level preserving embedding G of either $E^q \times N$ or $E^q_+ \times N$ into $Q \times N$ depending on whether $F_t(x)$ is in $\text{int } Q$ or ∂Q with $G(0, t) = F(x, t)$, such that $G^{-1}FH$ is of the form $i \times 1$, where i is the natural inclusion of E^m into E^q or E^m_+ into E^q_+ , as the case may be. An *ambient n -isotopy* of Q is a level-preserving homeomorphism H of $Q \times I^n$ such that $H_0 = 1$. If $A \subset X$, an ε -push of (X, A) is an ambient isotopy of X which is fixed outside an ε -neighborhood of A .

We make use of the semi-simplicial complexes $\text{Aut}_{\text{PL}}(Q)$ and $\text{PL}(M, Q)$, whose k -simplices are ambient k -isotopies of Q fixed on ∂Q and proper k -isotopies of M in Q fixed on ∂M , respectively. The Hudson covering n -isotopy theorem [5] can be used to prove, as in [4], that if $f: M \rightarrow Q$ is a given PL embedding then the simplicial map $p: \text{Aut}_{\text{PL}}(Q) \rightarrow \text{PL}(M, Q)$ given by $p(H) = H \circ (f \times 1)$ is a fibration, i.e.,

given level-preserving embeddings $K: Q \times J^{n-1} \rightarrow Q \times J^{n-1}$ and $L: M \times I^n \rightarrow Q \times I^n$ such that $p(K) = L|_{M \times J^{n-1}}$, there is an n -isotopy $H: Q \times I^n \rightarrow Q \times I^n$ such that $p(H) = L$ and $H|_{Q \times J^{n-1}} = K$. An element of $\pi_n \text{PL}(M, Q; f)$ is represented by a level-preserving PL embedding $L: M \times \partial I^{n+1}$ such that $L_0 = f$.

3. Spaces of embeddings. In this section we consider the relationship between $\text{PL}(M, Q)$ and $\text{TOP}(M, Q)$, the semi-simplicial complex of topological embeddings of M into Q . Recent work of Edwards and Miller [2, 12] has relaxed the dimension restrictions on the results in [10]. The key lemma is the following.

LEMMA 1. *Let $H: M \times I^n \rightarrow Q \times I^n$ be a level-preserving embedding. Suppose that $m \leq q - 3$ and $q \geq 5$. Then for any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon, H) > 0$ such that if $G_0, G_1: M \times I^n \rightarrow Q \times I^n$ are level-preserving PL embeddings with $d(G_i, H) < \delta$, then there is a level-preserving ε -push K of $(Q \times I^n, H(M \times I^n))$ such that $K_i G_0 = G_1$. If G_0 and G_1 agree on $M \times \partial I^n$, then K can be assumed fixed on $Q \times \partial I^n$.*

Proof. If H is of the form $h \times 1$ for some embedding $h: M \rightarrow Q$, then the lemma follows directly from Corollary 2 of [2] and Corollary 3 of [1]. Generalization to the case in which H is not of this form can be carried out as in the second half of the proof of Theorem 4.2 (m, s) in [10].

REMARK 3. The above “local solvability” result is the basis for Theorems 2.1–2.5 of [10] which are stated there with more stringent dimension restrictions. We may now regard those results to be true for $m \leq q - 3, q \geq 5$. In particular, Theorems 2.1 and 2.4 give us

THEOREM 2. *If $m \leq q - 3$ and $q \geq 5$, then the inclusion $\text{PL}(M, Q) \subset \text{TOP}(M, Q)$ is dense and a weak homotopy equivalence; i.e., if $f: M \rightarrow Q$ is PL, then the homomorphism $i_*: \pi_n \text{PL}(M, Q; f) \rightarrow \pi_n \text{TOP}(M, Q; f)$ induced by inclusion is an isomorphism for all n .*

4. Proof of Theorem 1. The following lemma, which is Theorem 2.3 of [10] with the new dimension conditions, makes possible the treatment of the non-PL case with PL techniques.

LEMMA 2. *Let $F: M \times I^n \rightarrow Q \times I^n$ be a level-preservingly locally flat proper n -isotopy which is PL on $\partial(M \times I^n)$. Suppose $m \leq q - 3$ and $q \geq 5$, and that $\varepsilon > 0$ is given. Then there is a level-preserving*

ε -push T of $(Q \times I^n, F(M \times I^n))$, fixed on $\partial(Q \times I^n)$, such that T_1F is PL.

Proof of Theorem 1. By Lemma 2 with $n = 0$ (See [11]), there is a small homeomorphism h of Q such that $hF_0: M \rightarrow Q$ is PL. Consider the embedding $(h \times 1)G^{-1}F: M \times I^n \rightarrow Q \times I^n$. Since it is a level-preservingly locally flat n -isotopy and $(h \times 1)G^{-1}F|_{\partial(M \times I^n)} = (hF_0) \times 1$, which is PL, there is by Lemma 2 a level-preserving isotopy T of $Q \times I^n$, fixed on $\partial(Q \times I^n)$, such that $T_1(h \times 1)G^{-1}F$ is PL. Now define $L: M \times \partial I^{n+1} \rightarrow Q \times \partial I^{n+1}$ by considering I^{n+1} as $I^n \times I$ and letting L be $T_1(h \times 1)G^{-1}F$ on $M \times I^n \times 1$ and $(hF_0) \times 1$ on $M \times J^n$. Then L is PL and so represents an element α of $\pi_n \text{PL}(M, Q; hF_0)$. To say $\alpha = 0$ in $\pi_n \text{PL}(M, Q; hF_0)$ is to say that there is a PL $(n + 1)$ -isotopy $H': M \times I^{n+1} \rightarrow Q \times I^{n+1}$ such that $H'|_{M \times \partial I^{n+1}} = L$. Therefore we can use the lifting property of the fibration $p: \text{Aut}_{\text{PL}}(Q) \rightarrow \text{PL}(M, Q)$ given by $p(K) = K \circ (hF_0 \times 1)$ to find an ambient $(n + 1)$ -isotopy $H'': Q \times I^{n+1} \rightarrow Q \times I^{n+1}$ such that $H''|_{Q \times J^n} = 1$ and $H'' \circ (hF_0 \times 1) = H'$. Now we define

$$K = (G \times 1)(h^{-1} \times 1 \times 1)T^{-1}H''(h \times 1 \times 1)(G^{-1} \times 1);$$

$$(Q \times I^n) \times I \rightarrow (Q \times I^n) \times 1.$$

Then K_1G covers F and extends $G|_{Q \times I^n}$, as desired.

Conversely, if K exists with the desired properties, then $K': M \times \partial I^{n+1} \times I \rightarrow Q \times \partial I^{n+1} \times I$ defined by $K'_t = T_{1-t}(h \times 1)G'K_{1-t}G(F_0 \times 1)$ on $M \times J^n \times I$ and $hF_0 \times 1$ on $M \times (I^{n-1} \times 1) \times I$ is a level-preserving isotopy taking L to $hF_0 \times 1$. Therefore α is trivial as an element of $\pi_n \text{TOP}(M, Q; hF_0)$, the semi-simplicial complex of embeddings of M into Q . By Theorem 2, α is trivial in $\pi_n \text{PL}(M, Q; hF_0)$.

5. The obstruction α . In the construction above, α appeared to depend on h, T , and G . In this section we show that α can be chosen in such a way that it depends only on F .

In applying Lemma 2 to construct h , above, we may choose h so that hF_0 is within $\delta(F_0, 1)$ of F_0 , where δ comes from Lemma 1. Any two such homeomorphisms h and h' will then be such that hF_0 and $h'F_0$ are ambient isotopic. Similarly we choose T to be a $\delta((h \times 1)G^{-1}F, 1)$ -push, so that if T' is another push which takes $(h \times 1)G^{-1}F$ to a PL embedding, $T_1(h \times 1)G^{-1}F$ and $T'_1(h \times 1)G^{-1}F$ are PL ambient isotopic, and the α 's constructed with them will be homotopic in $\pi_n \text{PL}(M, Q; hF_0)$.

Now suppose that G and G' are level-preserving homeomorphisms of $Q \times I^n$ satisfying the hypotheses of the theorem. Since $G^{-1}F$ and

$G'^{-1}F$ are each isotopic to $F_0 \times 1$, they are isotopic. If we denote by α and α' the obstructions constructed as above from G and G' , the isotopy of $G^{-1}F$ to $G'^{-1}F$ will induce a homotopy from α to α' in $\pi_n \text{TOP}(M, Q)$. By Theorem 2, α is homotopic to α' in $\pi_n \text{PL}(M, Q; hF_0)$, and so α does not depend on G .

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