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**THE CONVERSE TO THE SMITH THEOREM FOR
 Z_p -HOMOLOGY SPHERES**

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Let X be a finite CW complex with the Z_p homology of an n -sphere. Let Z_p act cellularly on X . The Smith theorem asserts that the fixed point set X^{Z_p} has the Z_p homology of an m -sphere for $-1 \leq m \leq n$. A converse to this Smith theorem is proved.

Suppose X is a finite CW complex, p is a prime, and $\alpha : X \rightarrow X$ is a homeomorphism of period p (i.e., α^p is the identity map). Let X^{Z_p} denote the set of points in X left fixed by α . The well-known Smith theorem states that, if X has the Z_p homology of a disk (respectively, an n -sphere), then X^{Z_p} has the Z_p homology of a disk (respectively, some m -sphere where $-1 \leq m \leq n$ and the (-1) -sphere is the empty set). The converse to this theorem for the case where X has the Z_p homology of a disk appears in a paper of Lowell Jones [2].

This current paper shows how to extend Jones' methods to obtain the converse for the case where X has the Z_p homology of an n -sphere. Specifically, we prove the following theorem:

THEOREM 1. *Let p be a prime integer and n a positive integer. Let K be a connected finite CW complex satisfying $H_n(K; Z_p) = Z_p$ and for which, if $i \neq n$ and $i \neq 0$, $H_i(K; Z)$ is a finite group of order prime to p .*

Then there exist a finite, simply connected, connected CW complex X containing K as a subcomplex and a cellular homeomorphism $\alpha : X \rightarrow X$ of period p so that

- (1) $X^{Z_p} = K$
- (2) For some $m > 0$, $H_i(X; Z) = 0$ if $i \neq 0$, $i \neq n + 2m$.
- (3) If $H_n(K; Z) = Z \oplus A$ where A is a finite abelian group of order prime to p , then $H_{n+2m}(X; Z) = Z$.
- (4) If $H_n(K; Z) = Z_{p^s} \oplus A$ where A is a finite abelian group of order prime to p , and $s \geq 1$, then $H_{n+2m}(X; Z) = Z_{p^s}$.

Here Z denotes the ring of integers and Z_{p^s} denotes the cyclic group of order p^s . It is well-known that $H_n(K; Z)$ must satisfy the hypotheses of either (3) or (4) since $H_n(K; Z_p) = Z_p$.

The proof is similar to Jones' proof of [2; Theorem 1.1], but utilizes some further algebraic lemmas. The algebraic lemmas are given in §1, and their topological analogues are given in §2. The proof of the theorem appears in §3.

If p is not prime, the methods still apply and yield a CW complex X possessing a semi-free Z_p action α with fixed point set K . The cases (3) and (4) are, however, not exhaustive.

I wish to thank the referee for strengthening the original version of Theorem I.

1. Algebraic lemmas. Let $R = Z[Z_p]$, the integral group ring for the group Z_p with generator g . Elements of R will be written $\sum a_i g^i$ where $a_i \in Z$. All summations run over $i = 0, \dots, p-1$. The element g^0 is the identity, often written e . In some formulas we shall use the identifications $a_p = a_0$, $a_{p-1} = a_{-1}$, $a_{p+1} = a_1$. Denote by σ the element $\sigma = \sum g^i$. If A and B are left R modules and $f: A \rightarrow B$ is a homomorphism, denote by $\text{Ker } f$ the kernel of f ; by $\text{Coker } f$ the cokernel of f ; by $\text{Image } f$ the image of f . A left R module M is said to be trivial provided $g^i m = m$ for $m \in M$ and $g^i \in Z_p$.

LEMMA 1. Let $\epsilon: R \rightarrow Z_p^s$ be the augmentation map which takes $\sum b_i g^i$ to $\sum b_i \pmod{p^s}$. View Z_p^s as a trivial left R module. There is an exact sequence of left R modules

$$R \oplus R \xrightarrow{\mu} R \xrightarrow{\epsilon} Z_p^s \rightarrow 0$$

and a homomorphism $\lambda: R \rightarrow \text{Ker } \mu$ such that

- (1) λ is monic;
- (2) $\text{Coker } \lambda = Z_p^s$.

Proof. Define μ , if $(a, b) \in R \oplus R$, by

$$\mu(a, b) = (e - g)a + p^{s-1} \sigma b$$

where

$$\sigma = e + g + g^2 + \dots + g^{p-1} \in R.$$

Define $\lambda: R \rightarrow R \oplus R$, if $a \in R$, by

$$\lambda(a) = (p^{s-1} \sigma a, (g - e)a).$$

We now verify that these maps have the properties asserted above:

Claim 1. $\epsilon\mu = 0$.

This follows since

$$\begin{aligned}\epsilon\mu(a, b) &= \epsilon((e - g)a + p^{s-1}\sigma b) = a\epsilon(e - g) + p^{s-1}b\epsilon(\sigma) \\ &= a \cdot 0 + p^{s-1}b \cdot p = 0,\end{aligned}$$

using the left R module structure of Z_p^s .

Claim 2. $\text{Ker } \epsilon \subset \text{Image } \mu$.

If $\epsilon(\sum a_i g^i) = 0$, then $\sum a_i \equiv 0 \pmod{p^s}$. Let

$$\sum b_i g^i = \sum a_i g^i - \left(\sum a_i\right) p^{-1} \sigma.$$

Then $\sum b_i = 0 \in Z$, and it is easy to see that $\sum b_i g^i = (e - g)c$ for some $c \in R$. Hence

$$\mu\left(c, \left(\sum a_i\right) p^{-s} e\right) = (e - g)c + \left(\sum a_i\right) p^{-1} \sigma = \sum a_i g^i.$$

Claim 3. $\text{Image } \lambda \subset \text{ker } \mu$.

To see this, if $a \in R$, note

$$\mu\lambda(a) = (e - g)p^{s-1}\sigma a + p^{s-1}\sigma(g - e)a = 0.$$

Claim 4. λ is monic.

To see this, note $\text{ker } \lambda = \text{ker}(g - e) \cap \text{ker}(p^{s-1}\sigma)$ where $(g - e)$ denotes the homomorphism of multiplication by $(g - e)$, and $p^{s-1}\sigma$ denotes multiplication by $p^{s-1}\sigma$. Then

$$\text{ker } \lambda = \{a\sigma : a \in Z\} \cap \left\{ \sum a_i g^i : p^{s-1} \left(\sum a_i\right) = 0 \in Z \right\} = 0.$$

Claim 5. $\text{Coker } \lambda = Z_p^s$.

To see this, note

$$\begin{aligned}\text{Ker } \mu &= \left\{ \left(\sum a_i g^i, \sum b_i g^i\right) : (e - g) \sum a_i g^i + p^{s-1} \sigma \sum b_i g^i = 0 \right\} \\ &= \left\{ \left(\sum a_i g^i, \sum b_i g^i\right) : a_i - a_{i-1} + p^{s-1} \left(\sum b_j\right) = 0 \text{ for all } i \right\}.\end{aligned}$$

Summing these latter conditions over i , we obtain $\sum a_i - \sum a_i + p^s \sum b_j = 0$. Hence $\sum b_j = 0$ and $a_i = a_{i-1}$ for all i . Thus

$$\text{Ker } \mu = \left\{ \left(a\sigma, \sum b_i g^i\right) : a \in Z, \sum b_i = 0 \right\}.$$

Define $\gamma : \text{Ker } \mu \rightarrow Z_p^s$ by

$$\gamma\left(a\sigma, \sum b_i g^i\right) = a + p^{s-1}[pb_0 + (p-1)b_1 + (p-2)b_2 + \cdots + b_{p-1}] \pmod{p^s}.$$

Then γ is surjective. Moreover, $\gamma\lambda = 0$, which may be seen as follows:

$$\begin{aligned}\gamma\lambda\left(\sum a_i g^i\right) &= \gamma\left(p^{s-1}\left(\sum a_i\right)\sigma, \sum (a_{i-1} - a_i)g^i\right) \\ &= p^{s-1}\left(\sum a_i\right) + p^{s-1}[p(a_{p-1} - a_0) + (p-1)(a_0 - a_1) + \cdots + (a_{p-2} - a_{p-1})] \\ &= p^{s-1}\left[\sum a_i + a_0(-p + p - 1) + a_1(-(p-1) + p - 2) + \cdots\right. \\ &\quad \left.+ a_{p-2}(1 - 2) + a_{p-1}(p - 1)\right] \\ &= p^{s-1}pa_{p-1} \equiv 0 \pmod{p^s}.\end{aligned}$$

Thus to prove Claim 5 there remains to show only that $\text{Ker } \gamma \subset \text{Image } \lambda$. But if $\gamma(a\sigma, \sum b_i g^i) = 0$, then

$$(1) \quad a + p^{s-1}[pb_0 + (p-1)b_1 + \cdots + b_{p-1}] \equiv 0 \pmod{p^s}.$$

For arbitrary $c_0 \in Z$, define $c_{i+1} = c_i - b_{i+1}$ for $i = 0, 1, \dots, p-2$. Then $b_0 = c_{p-1} - c_0$ since $\sum b_i = 0$, and

$$\begin{aligned}\lambda\left(\sum c_i g^i\right) &= \left(p^{s-1}\left(\sum c_i\right)\sigma, \sum (c_{i-1} - c_i)g^i\right) \\ &= \left(p^{s-1}[c_0 + (c_0 - b_1) + (c_0 - b_1 - b_2) + \cdots\right. \\ &\quad \left.+ (c_0 - b_1 - \cdots - b_{p-1})]\sigma, \sum b_i g^i\right) \\ &= \left(p^{s-1}[pc_0 - (p-1)b_1 - (p-2)b_2 - \cdots - b_{p-1}]\sigma, \sum b_i g^i\right).\end{aligned}$$

By (1) we may choose c_0 so

$$p^s c_0 = a + p^{s-1}[(p-1)b_1 + (p-2)b_2 + \cdots + b_{p-1}].$$

But then $\lambda(\sum c_i g^i) = (a\sigma, \sum b_i g^i)$.

LEMMA 2. Let $\epsilon: R \rightarrow Z$ be the augmentation map. There is a map $\lambda: R \rightarrow R$ so

$$0 \rightarrow Z \rightarrow R \xrightarrow{\lambda} R \xrightarrow{\epsilon} Z \rightarrow 0 \quad \text{is exact.}$$

Proof. Let $\lambda(a) = (e - g)a$ for $a \in R$. Then $\epsilon\lambda = 0$ and $\text{ker } \epsilon = \text{Image } \lambda$ easily. Moreover

$$\begin{aligned} \text{Ker } \lambda &= \left\{ \sum a_i g^i : \sum (a_i - a_{i-1}) g^i = 0 \right\} \\ &= \left\{ \sum a_i g^i : a_0 = a_1 = a_2 = \cdots = a_{p-1} \right\} \\ &= \{ b\sigma : b \in Z \} \cong Z. \end{aligned}$$

LEMMA 3. *If q is an integer prime to p , and $\epsilon: R \rightarrow Z_q$ is the augmentation map, then there is an exact sequence*

$$0 \rightarrow R \rightarrow R \xrightarrow{\epsilon} Z_q \rightarrow 0.$$

Proof. This is Lowell Jones' Lemma 1.1 [2; p. 53].

2. Topological lemmas. The major steps in the proof of Theorem I consist of applications of the following lemmas, which may be regarded as topological analogues of the lemmas of §1.

We shall let R be $Z[Z_p]$. Unless otherwise indicated, all homology groups have integer coefficients. Note that if X is a CW complex and $\alpha: X \rightarrow X$ is a homeomorphism of period p , then $H_i(X; Z)$ inherits the structure of a left R -module.

LEMMA A. *Suppose X is a connected, simply connected, finite CW complex with a cellular Z_p action given by $\alpha: X \rightarrow X$ such that $X^{Z_p} = K$. Suppose $H_i(X; Z) = 0$ for $0 < i < m$. Assume $H_m(X; Z)$ contains a finite subgroup A of order prime to p such that A is a trivial left R -submodule of $H_m(X)$. Then there exists a connected, simply connected, finite CW complex Y containing X as a subcomplex and possessing a cellular Z_p action extending α such that*

- (1) $Y^{Z_p} = K$
- (2) $H_i(Y; Z) = 0$ for $0 < i < m$.
- (3) $H_m(Y; Z) = H_m(X; Z)/A$ as an R -module.
- (4) *The inclusion induces an isomorphism of $H_i(X; Z)$ onto $H_i(Y; Z)$ for $i > m$.*

Proof. This is essentially the proof of Theorem 1.1 in [2]. We note that it suffices by induction to assume $A = Z_q$ where q is prime to p . Obtain, by the Hurewicz theorem, a map $k: S^m \rightarrow X$ which realizes a generator of $Z_q \subset H_m(X; Z)$. We shall attach p cells of dimension $(m + 1)$ to X along the maps $k, \alpha k, \alpha^2 k, \dots, \alpha^{p-1} k: S^m \rightarrow X$. Call the resulting CW complex Y_1 ; clearly we obtain a cellular Z_p action on Y_1 by extending α to permute the points in the added cells. Then

$H_i(Y_1; Z) = H_i(X; Z)$ for $i \neq m, m + 1$, and the long exact sequence of the pair (Y_1, X) yields the exact sequence of R modules

$$0 \rightarrow H_{m+1}(X) \rightarrow H_{m+1}(Y_1) \rightarrow R \xrightarrow{\epsilon} H_m(X) \rightarrow H_m(Y_1) \rightarrow 0.$$

Since Z_q is a trivial R module, the map denoted ϵ may be identified with the augmentation map from R onto Z_q . It follows that $H_m(Y_1) = H_m(X)/Z_q$ and

$$0 \rightarrow H_{m+1}(X) \rightarrow H_{m+1}(Y_1) \rightarrow \text{Ker } \epsilon \rightarrow 0 \quad \text{is exact.}$$

By Lemma 3, $\text{Ker } \epsilon \cong R$ and hence is projective. Thus $H_{m+1}(Y_1) = H_{m+1}(X) \oplus R$. The Hurewicz map $h: \pi_{m+1}(Y_1) \rightarrow H_{m+1}(Y_1)$ is surjective (see Hu [1; p. 167] or G. W. Whitehead [3]). Hence we may represent the element $e \in R \subset H_{m+1}(Y_1)$ by a map $j: S^{m+1} \rightarrow Y_1$. As before, attach p cells of dimension $(m + 2)$ to Y_1 along the maps $j, \alpha j, \alpha^2 j, \dots, \alpha^{p-1} j$ to obtain a CW complex Y ; we may extend the map α over Y . Then $H_i(Y) = H_i(Y_1)$ for $i \neq m + 2, m + 1$, and

$$0 \rightarrow H_{m+2}(Y_1) \rightarrow H_{m+2}(Y) \rightarrow R \rightarrow H_{m+1}(Y_1) \rightarrow H_{m+1}(Y) \rightarrow 0$$

is exact. By construction the map of R into $H_{m+1}(Y_1)$ is an isomorphism onto the summand isomorphic to R . Hence

$$H_{m+2}(Y) = H_{m+2}(Y_1) = H_{m+2}(X), \quad H_{m+1}(Y) = H_{m+1}(X).$$

The complex Y satisfies the conclusions of the lemma.

LEMMA B. *Suppose X is a connected, simply connected, finite CW complex with a cellular Z_p action given by $\alpha: X \rightarrow X$ such that $X^{Z_p} = K$. Suppose $H_i(X) = 0$ if $0 < i < m$. Assume $H_m(X)$, $H_{m+1}(X)$, and $H_{m+2}(X)$ all are trivial as R modules, and that $H_m(X) = Z$, $H_{m+1}(X; Z_p) = 0$, $H_{m+2}(X; Z_p) = 0$. Then there exists a connected, simply connected, finite CW complex Y which contains X as a subcomplex and possesses a cellular Z_p action extending α such that*

- (1) $Y^{Z_p} = K$
- (2) $H_i(Y; Z) = 0$ for $0 < i \leq m + 1$
- (3) $H_{m+2}(Y; Z) = Z$ as a trivial R module
- (4) The inclusion induces isomorphisms from $H_i(X; Z)$ onto $H_i(Y; Z)$ for $i > m + 2$.

Proof. Obtain by the Hurewicz theorem a map $k: S^m \rightarrow X$ which represents the generator of $H_m(X)$. Attach p cells of dimension

$(m + 1)$ along the maps $k, \alpha k, \dots, \alpha^{p-1}k$ to obtain a CW complex Y_1 ; and extend the map α over Y_1 via the obvious permutation of points on the added cells. Then $H_i(Y_1) = H_i(X)$ for $i \neq m, m + 1$; and

$$0 \rightarrow H_{m+1}(X) \rightarrow H_{m+1}(Y_1) \rightarrow R \xrightarrow{\epsilon} H_m(X) \rightarrow H_m(Y_1) \rightarrow 0$$

is exact. By construction, ϵ may be regarded as the augmentation map from R onto Z . Hence $H_m(Y_1) = 0$ and

$$0 \rightarrow H_{m+1}(X) \rightarrow H_{m+1}(Y_1) \rightarrow \text{Ker } \epsilon \rightarrow 0$$

is exact. Since $H_{m+1}(X; Z_p) = 0$ and $H_{m+1}(X)$ is a trivial R module, by Lemma A we may obtain a complex $Y_2 \supset Y_1$ with an action extending α so $H_i(Y_2) = 0$ for $0 < i < m + 1$, $H_{m+1}(Y_2) = \text{Ker } \epsilon$, and $H_i(Y_2) = H_i(Y_1) = H_i(X)$ by the inclusion map for $i > m + 1$. Let λ be the homomorphism of Lemma 2. By the Hurewicz theorem we represent $\lambda(e) \in H_{m+1}(Y_2)$ by a map $j: S^{m+1} \rightarrow Y_2$. Adjoin cells to Y_2 along the maps $j, \alpha j, \dots, \alpha^{p-1}j$ to obtain a complex Y_3 with action α . Then $H_i(Y_3) = H_i(Y_2)$ for $i \neq m + 2, m + 1$, and

$$0 \rightarrow H_{m+2}(Y_2) \rightarrow H_{m+2}(Y_3) \rightarrow R \xrightarrow{\lambda} H_{m+1}(Y_2) \rightarrow H_{m+1}(Y_3) \rightarrow 0$$

is exact. Since $\text{Image } \lambda = \text{Ker } \epsilon$, $H_{m+1}(Y_3) = 0$ and

$$0 \rightarrow H_{m+2}(Y_2) \rightarrow H_{m+2}(Y_3) \rightarrow \text{Ker } \lambda \rightarrow 0$$

is exact. Since $H_{m+2}(Y_2) = H_{m+2}(X)$ is a trivial R module and $H_{m+2}(X; Z_p) = 0$, we may apply Lemma A to the subgroup $H_{m+2}(Y_2) \subset H_{m+2}(Y_3)$ to obtain a complex $Y \supset Y_3$ so $H_i(Y) = 0$ for $i < m + 2$ and $H_{m+2}(Y) = \text{Ker } \lambda = Z$. This Y satisfies the conclusions of the lemma.

LEMMA C. *Suppose X is a connected, simply connected, finite CW complex with a cellular Z_p action given by $\alpha: X \rightarrow X$ such that $X^{Z_p} = K$. Suppose $H_i(X) = 0$ if $0 < i < m$. Assume $H_m(X)$, $H_{m+1}(X)$, and $H_{m+2}(X)$ are all trivial as R modules and $H_m(X) = Z_p^s$ for some $s \geq 1$; and both $H_{m+1}(X)$ and $H_{m+2}(X)$ are finite groups of order prime to p . Then $H_{m+2}(X; Z_p) = 0$. Then there exists a connected, simply connected, finite CW complex Y containing X and with a cellular Z_p action extending α such that*

- (1) $Y^{Z_p} = K$
- (2) $H_i(Y; Z) = 0$ for $0 < i \leq m + 1$

- (3) $H_{m+2}(Y; Z) = Z_p^s$ as a trivial R module
 (4) The inclusion induces isomorphisms from $H_i(X)$ onto $H_i(Y)$ for $i > m + 2$.

Proof. Obtain by the Hurewicz theorem a map $k: S^m \rightarrow X$ representing a generator for $Z_p^s = H_m(X)$. Attach p cells of dimension $(m + 1)$ along the maps $k, \alpha k, \dots, \alpha^{p-1}k$ to obtain a complex Y_1 with action α extending the previous α . Then $H_i(Y_1) = H_i(X)$ for $i \neq m, m + 1$, and

$$0 \rightarrow H_{m+1}(X) \rightarrow H_{m+1}(Y_1) \rightarrow R \xrightarrow{\epsilon} H_m(X) \rightarrow H_m(Y_1) \rightarrow 0$$

is exact. By construction, the map ϵ may be identified with the augmentation map of Lemma 1. Then $H_m(Y_1) = 0$ and

$$0 \rightarrow H_{m+1}(X) \rightarrow H_{m+1}(Y_1) \rightarrow \text{Ker } \epsilon \rightarrow 0.$$

Apply Lemma A to the subgroup $H_{m+1}(X)$ of $H_{m+1}(Y_1)$ to obtain a complex $Y_2 \supset Y_1$ so that $H_i(Y_2) = 0$ for $0 < i < m + 1$; $H_{m+1}(Y_2) = \text{Ker } \epsilon$; $H_i(Y_2) = H_i(X)$ for $i > m + 1$.

Let $\mu: R \oplus R \rightarrow \text{Ker } \epsilon$ be the homomorphism in Lemma 1. By the Hurewicz theorem we may represent $\mu(e, 0)$ by a map $j: S^{m+1} \rightarrow Y_2$ and we may represent $\mu(0, e)$ by a map $l: S^{m+1} \rightarrow Y_2$. Adjoin p cells of dimension $(m + 2)$ via $j, \alpha j, \dots, \alpha^{p-1}j$ and also p cells via $l, \alpha l, \dots, \alpha^{p-1}l$; call the resulting complex Y_3 and extend α over Y_3 in the obvious fashion. Then $H_i(Y_3) = H_i(Y_2)$ for $i \neq m + 1, m + 2$; and

$$0 \rightarrow H_{m+2}(Y_2) \rightarrow H_{m+2}(Y_3) \rightarrow R \oplus R \xrightarrow{\mu} H_{m+1}(Y_2) \rightarrow H_{m+1}(Y_3) \rightarrow 0$$

is exact. Since $\text{Image } \mu = \text{Ker } \epsilon$, $H_{m+1}(Y_3) = 0$; and

$$0 \rightarrow H_{m+2}(Y_2) \rightarrow H_{m+2}(Y_3) \rightarrow \text{Ker } \mu \rightarrow 0$$

is exact. Apply Lemma A to the complex Y_3 and the subgroup $H_{m+2}(Y_2) \subset H_{m+2}(Y_3)$; this is possible since $H_{m+2}(X; Z_p) = 0$ and $H_{m+2}(X)$ is a trivial R module. We obtain a complex Y_4 so $H_i(Y_4) = 0$ for $0 < i < m + 1$, $H_{m+2}(Y_4) = \text{Ker } \mu$, $H_i(Y_4) = H_i(X)$ for $i > m + 2$. Let λ be the homomorphism of Lemma 1, and represent $\lambda(e)$ by a map $r: S^{m+2} \rightarrow Y_4$. Attach p cells of dimension $(m + 3)$ to Y_4 along $r, \alpha r, \dots, \alpha^{p-1}r$ to obtain a complex Y . Then $H_i(Y) = H_i(Y_4)$ for $i \neq m + 2, m + 3$ and

$$0 \rightarrow H_{m+3}(Y_4) \rightarrow H_{m+3}(Y) \rightarrow R \xrightarrow{\lambda} H_{m+2}(Y_4) \rightarrow H_{m+2}(Y) \rightarrow 0$$

is exact. By Lemma 1, λ is monic, so $H_{m+3}(Y_4) = H_{m+3}(Y)$; and $H_{m+2}(Y) = \text{Coker } \lambda = Z_p^s$. The complex Y satisfies the conclusions of the lemma.

3. Proof of Theorem I. We must first deal with $\pi_1(K)$. Assume that $n > 1$. Choose a finite generating set b_1, \dots, b_q for $\pi_1(K)$ by Van Kampen's theorem. We may kill b_1 by adjunction of 2-cells along $b_1, \alpha b_1, \dots, \alpha^{p-1} b_1$, yielding a CW complex W . Since the image of b_1 in $H_1(K; Z)$ has order prime to p , we find $H_2(W; Z) = H_2(K; Z) \oplus R$, and we may proceed as in Lemma A to remove the R summand by adjunction of 3-cells. Deal similarly with b_2, b_3, \dots, b_q . In this manner we obtain a simply-connected finite CW complex X_1 with cellular action $\alpha: X_1 \rightarrow X_1$ of period p so $H_i(X_1; Z) = H_i(K; Z)$ for $i > 1$, and $X_1^{Z_p} = K$. Apply Lemma A to the group $H_2(X_1; Z)$. Continuing inductively in this manner, we obtain a simply-connected, finite CW complex X_n such that $H_i(X_n) = H_i(K)$ for $i > n$; $H_i(X_n) = 0$ for $i < n$; $H_n(X_n) = Z$ if Case (3) of Theorem I is pertinent; and $H_n(X_n) = Z_p^s$ if Case (4) of Theorem I is pertinent. Now we apply repeatedly Lemma B for Case (3) and Lemma C for Case (4). After finitely many steps, the process terminates since $H_i(K) = 0$ for sufficiently large i .

If $n = 1$, we modify the above proof slightly. We first kill $H_1(K; Z)$ except for the summand Z or Z_p^s by Lemma A. Call the resulting complex W_1 , and choose a finite generating set b_1, \dots, b_q for $\pi_1(W_1)$. We may assume that the image of b_1 in $H_1(W_1)$ is a generator of $H_1(W_1) = Z$ or Z_p^s . If the image of b_j is represented by $m_j \in Z$ for $j = 2, \dots, q$, then $b_j b_1^{-m_j}$ has image 0 in $H_1(W_1)$, and the elements $b_1, b_2 b_1^{-m_2}, \dots, b_q b_1^{-m_q}$ generate $\pi_1(W_1)$. Kill $b_2 b_1^{-m_2}, \dots, b_q b_1^{-m_q}$ as in the case where $n > 1$; we obtain a complex W_2 for which $\pi_1(W_2) = Z$ or Z_p^s , and $H_i(W_2; Z) = H_i(K; Z)$ for $i \geq 2$. Apply Lemma B or C to W_2 . The remainder of the proof follows as for the case $n > 1$.

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