

# Pacific Journal of Mathematics

**THE KRULL INTERSECTION THEOREM**

DANIEL D. ANDERSON

## THE KRULL INTERSECTION THEOREM

D. D. ANDERSON

Let  $R$  be a commutative ring,  $I$  an ideal in  $R$ , and  $A$  an  $R$ -module. We always have  $0 \subseteq 0^s \subseteq I \bigcap_{n=1}^{\infty} I^n A \subseteq \bigcap_{n=1}^{\infty} I^n A$  where  $S$  is the multiplicatively closed set  $\{1 - i \mid i \in I\}$  and  $0^s = 0_s \cap A = \{a \in A \mid \exists s \in S \exists sa = 0\}$ . It is of interest to know when some containment can be replaced by equality. The Krull intersection theorem states that for  $R$  Noetherian and  $A$  finitely generated  $I \bigcap_{n=1}^{\infty} I^n A = \bigcap_{n=1}^{\infty} I^n A$ . Since  $\bigcap_{n=1}^{\infty} I^n A$  is finitely generated,  $\bigcap_{n=1}^{\infty} I^n A = 0^s$ . Thus if  $I \subseteq \text{rad}(R)$ , the Jacobson radical of  $R$ , or  $R$  is a domain and  $A$  is torsion-free, we have  $\bigcap_{n=1}^{\infty} I^n A = 0$ . In this note we show that for a Prüfer domain  $R$  and a torsion-free  $R$ -module  $A$ ,  $I \bigcap_{i=1}^{\infty} I^n A = \bigcap_{i=1}^{\infty} I^n A$ . We also consider the condition (\*):  $\bigcap_{n=1}^{\infty} I^n = 0$  for every ideal  $I$  in the commutative ring  $R$ . It is shown that a polynomial ring in any set of indeterminants over a Noetherian domain and the integral closure of a Noetherian domain satisfy (\*).

Let  $R$  be a ring and  $A$  an  $R$ -module. If  $x \in R$  and  $x \notin Z(A)$ , the zero divisors of  $A$ , then  $(x) \bigcap_{n=1}^{\infty} (x)^n A = \bigcap_{n=1}^{\infty} (x)^n A$ . Actually we can take  $I$  to be invertible and  $A$  torsion-free. However, the assumption  $x \notin Z(A)$  is essential. For example, let  $p \in R$  be neither a unit nor a zero divisor and let  $F = Rx \oplus (\sum_{i=1}^{\infty} Ry_i)$  be the free  $R$ -module on  $\{x, y_1, y_2, \dots\}$ . Let  $A = F/G$  where  $G = (x-psy_1, x-p^2y_2, \dots)$ ; it is not difficult to see that  $(p) \bigcap_{n=1}^{\infty} (p)^n A \neq \bigcap_{n=1}^{\infty} (p)^n A$ . Using this result, one can show that the following are equivalent: (1)  $\dim R = 0$ , (2) for every finitely generated (principal) ideal  $I$  and every  $R$ -module  $A$ ,  $I \bigcap_{n=1}^{\infty} I^n A = \bigcap_{n=1}^{\infty} I^n A$ . The first theorem gives another affirmative case.

**THEOREM 1.** *Let  $R$  be a reduced ring and let  $I$  be a finitely generated ideal with  $\text{rank } I \leq 1$ . Then  $\bigcap_{n=1}^{\infty} I^n = I \bigcap_{n=1}^{\infty} I^n$ . If  $R$  is quasi-local or  $R$  is a domain, then  $\bigcap_{n=1}^{\infty} I^n = 0$ .*

*Proof.* First suppose  $R$  is a domain. By localization we can assume  $\sqrt{I} = M$  the maximal ideal of  $R$ . If  $B = \bigcap_{n=1}^{\infty} I^n \neq 0$ , then  $\sqrt{B} = M$ , so there exists an integer  $m$  such that  $I^m \subseteq B$ . Then  $I^m = I^{m+1}$  which implies  $I^m = 0$  by Nakayama's lemma. Next suppose  $R$  is quasi-local, by passing to  $R/P$  where  $P$  is a minimal prime we get  $\bigcap_{n=1}^{\infty} I^n \subseteq P$ . Since  $R$  is reduced, we have  $\bigcap_{n=1}^{\infty} I^n \subseteq \text{nil}(R) = 0$ . The general case now follows by localization.

Another affirmative case is  $R$  a Prüfer domain and  $A$  a torsion-

free  $R$ -module. We first consider the quasi-local case.

**LEMMA 1.** *Let  $V$  be a valuation domain,  $I$  an ideal in  $V$ , and  $A$  a torsion-free  $V$ -module. Then  $ib \in B = \bigcap_{n=1}^{\infty} I^n A$  where  $i \in I$  and  $b \in A$  implies  $i \in \bigcap_{n=1}^{\infty} I^n$  or  $b \in B$ . In particular,  $B = IB$ .*

*Proof.* Suppose  $i \notin \bigcap_{n=1}^{\infty} I^n$ , then there exists an integer  $N$  such that  $i \in I^{N-1} - I^N$ . Now  $ib \in I^m A$  for  $m > N$  implies  $ib = j^N j^{m-N} a$  for some  $j \in I$  and  $a \in A$ . Now  $i \notin I^N$  gives  $j^N = si$  for some  $s \in V$ . Hence  $ib = sij^{m-N} a$  so  $b = sj^{m-N} a \in I^{m-N} A$  since  $A$  is torsion-free. Therefore  $b \in B$ .

**THEOREM 2.** *Let  $R$  be a Prüfer domain,  $I$  an ideal in  $R$ ,  $A$  a torsion-free  $R$ -module, and  $B = \bigcap_{n=1}^{\infty} I^n A$ . Then  $B = IB$ .*

*Proof.* Let  $y \in B$  and  $J = (IB : y)$ ; it suffices to show  $J = R$ . Let  $M$  be a fixed maximal ideal; we show that  $J \not\subseteq M$ . Now  $y \in B \subseteq B_M \subseteq \bigcap_{n=1}^{\infty} I_M^n A_M = I_M^2 \bigcap_{n=1}^{\infty} I_M^n A_M$  by Lemma 1, so  $y = i^2(b/s)$  where  $i \in I$ ,  $b \in A$ ,  $s \in R - M$  and  $b/s \in \bigcap_{n=1}^{\infty} I_M^n A_M$ . Let  $N$  be any maximal ideal of  $R$ , then  $i^2 b = sy \in B \subseteq \bigcap_{n=1}^{\infty} I_N^n A_N$  so by Lemma 1,  $i \in \bigcap_{n=1}^{\infty} I_N^n$  or  $ib \in \bigcap_{n=1}^{\infty} I_N^n A$ . In either case,  $ib \in \bigcap_{n=1}^{\infty} I_N^n A_N$  for every maximal ideal  $N$  of  $R$ , so  $ib \in B$ . Therefore,  $s \in J - M$ .

We remark that for a Prüfer domain,  $\bigcap_{n=1}^{\infty} I^n$  need not be a prime ideal, but is always a radical ideal.

Consider the condition (\*) on a ring. Local rings and Noetherian domains satisfy this condition. The next two propositions are straight forward and the proofs are omitted.

**PROPOSITION 1.** *If  $R$  satisfies (\*), then  $Z(R) \subseteq \text{rad}(R)$ . Conversely, if  $R$  is Noetherian, then  $Z(R) \subseteq \text{rad}(R)$  implies (\*).*

**PROPOSITION 2.** *If  $R$  satisfies (\*), then  $R_M$  satisfies (\*) for every maximal ideal  $M$ . If  $R_M$  satisfies (\*) for every maximal ideal  $M$ , then  $\bigcap_{n=1}^{\infty} I^n = I \bigcap_{n=1}^{\infty} I^n$  for every ideal  $I$  in  $R$ . If  $Z(R) \subseteq \text{rad}(R)$ , then  $R$  satisfies (\*).*

The next theorem generalizes the Krull intersection theorem to rings which are locally Noetherian.

**THEOREM 3.** *Let  $R$  be a ring and  $A$  an  $R$ -module such that  $\bigcap_{n=1}^{\infty} P_P^n A_P = 0$  for every  $P \in \text{spec}(R)$ , then  $\bigcap_{n=1}^{\infty} I^n A = 0^s$  for every ideal  $I$  in  $R$ .*

*Proof.* Let  $T$  be the saturation of  $S = \{1 - i \mid i \in I\}$ , so  $T =$

$R - \bigcup_{P_\alpha \in \mathcal{S}} P_\alpha$  where  $\mathcal{S} = \{P \in \text{spec}(R) \mid P \cap T = \emptyset\}$ . Then setting  $B = \bigcap_{n=1}^{\infty} I^n A$  yields  $B_P \subseteq \bigcap_{n=1}^{\infty} I_P^n A_P = 0$  for every  $P \in \mathcal{S}$ . Hence  $(T^{-1}B)_{T^{-1}P} = 0$  for every  $P \in \mathcal{S}$ , but the  $T^{-1}P \in \mathcal{S}$  are precisely the prime ideals of  $T^{-1}R$ . Therefore  $T^{-1}B = 0$ , hence  $B_s = 0$  and the result follows.

The next proposition will be used to prove that a polynomial ring in any number of indeterminants over a Noetherian domain satisfies (\*).

**PROPOSITION 3.** *Let  $R$  be a Noetherian ring,  $I$  an ideal in  $R[X]$ , and  $B = \bigcap_{n=1}^{\infty} I^n$ . Then  $B = (B \cap R)R[X]$ .*

*Proof.* First suppose  $I \cap R = 0$ , we show that  $B = 0$ . Suppose  $0 \neq g(x) \in B$ , by the Krull intersection theorem there exists a polynomial  $f(x) = a_0x^n + \dots + a_n \in I$  such that  $g(x)(1 - f(x)) = 0$ . Since  $1 - f(x) \in Z(R[X])$ , there exists  $0 \neq c \in R$  such that  $c(1 - f(x)) = 0$ . Hence  $0 = ca_0 = \dots = ca_{n-1} = c(a_n - 1)$  so  $c = ca_n$ . But  $ca_n = cf(x) \in I \cap R = 0$  so  $c = ca_n = 0$ , a contradiction. For the general case, let  $J = I^m \cap R$ , passing to  $(R/J)[X]$  yields  $B \subseteq JR[X]$ , hence  $B \subseteq \bigcap_{n=1}^{\infty} (I^n \cap R)[X] = (B \cap R)[X] \subseteq B$ .

**THEOREM 4.** *Let  $R$  be a Noetherian domain and  $T = R[\{X_\alpha\}]$  a polynomial ring over  $R$  in any set  $\{X_\alpha\}$  of indeterminants. Then  $T$  satisfies (\*).*

*Proof.* We may assume the set of indeterminants is countable and hence index it by the positive integers. By Proposition 2 we may assume that  $(R, \mathcal{M})$  is local and we only need show that  $\bigcap_{n=1}^{\infty} M^n = 0$  where  $M$  is a maximal ideal in  $T$  with  $M \cap R = \mathcal{M}$ . Let  $K$  be the algebraic closure of  $k(\{z_\beta\})$  where  $\{z_\beta\}$  is an uncountable set of indeterminants over  $k = R/\mathcal{M}$ . There exists a local ring  $(B, N)$  with  $B \supseteq R$  faithfully flat,  $N = \mathcal{M}B$  and  $B/N = K$  [1]. Now  $B \supset R$  faithfully flat implies  $MB[\{X_i\}] \neq B[\{X_i\}]$  so  $MB[\{X_i\}] \subseteq M^*$  a maximal ideal in  $B[\{X_i\}]$ . It is sufficient to show  $\bigcap_{n=1}^{\infty} M^{*n} = 0$ . Since

$$[B[\{X_i\}]/M^* : B/N]$$

is countable and  $B/N = K$  is uncountable and algebraically closed,  $B[\{X_i\}]/M^* = K$ . Thus  $M^* = (\mathcal{M}, X_1 - r_1, X_2 - r_2, \dots)$  for suitable  $r_i \in B$ . Since a given polynomial involves only finitely many indeterminants, it suffices to show  $\bigcap_{n=1}^{\infty} (\mathcal{M}, X_1 - r_1, x_m - r_m)^n = 0$  in  $B[X_1, \dots, X_m]$ . Since  $(\mathcal{M}, X_1 - r_1, \dots, X_m - r_m)^n \cap B[X_1, \dots, X_{m-1}] = (\mathcal{M}, X_1 - r_1, \dots, X_{m-1} - r_{m-1})^n$ , the result follows from Proposition 3 and induction.

The last theorem gives another class of rings where (\*) holds.

**THEOREM 5.** *Let  $R$  be Noetherian domain and  $R'$  its integral closure. Then any ring between  $R$  and  $R'$  satisfies (\*).*

*Proof.* Let  $R \subseteq T \subseteq R'$  be a ring, since  $T \subseteq R'$  is integral, any ideal of  $T$  is contained in the contraction of an ideal of  $R'$ , thus we may assume  $T = R'$ . It suffices to prove the result for  $(R, M)$  a local domain. Now  $R \subseteq \hat{R}/N \subseteq \hat{R}/P_1 \oplus \dots \oplus \hat{R}/P_n \subseteq (\hat{R}/P_1)' \oplus \dots \oplus (\hat{R}/P_n)'$  where  $\hat{R}$  is the completion of  $R$ ,  $N = P_1 \cap \dots \cap P_n$ , and  $P_1, \dots, P_n$  are the minimal primes of  $\hat{R}$ . Now each  $\hat{R}/P_i$  is a complete local domain, so each  $(\hat{R}/P_i)'$  is a Noetherian domain and hence satisfies (\*). Every maximal ideal  $\mathcal{M}$  of  $R'$  has the form  $\mathcal{M} = M^* \cap R'$  for some maximal ideal  $M^*$  of  $(\hat{R}/P_1)' \oplus \dots \oplus (\hat{R}/P_n)'$  [2, p. 119]. Hence  $M^* = (\hat{R}/P_i)' \oplus \dots \oplus N \oplus \dots \oplus (\hat{R}/P_n)'$  where  $N$  is a maximal ideal in  $(\hat{R}/P_i)'$  for some  $i$ . Then  $\bigcap_{n=1}^{\infty} \mathcal{M}^n = \bigcap_{n=1}^{\infty} (M^* \cap R')^n \subseteq (\bigcap_{n=1}^{\infty} M^{*n}) \cap R' = I_i \cap R'$  where  $I_i = (\hat{R}/P_i)' \oplus \dots \oplus 0 \oplus \dots \oplus (\hat{R}/P_n)'$ . Suppose  $I_i \cap R' \neq 0$ , then  $I_i \cap R \neq 0$  since  $R \subseteq R'$  is integral. But  $0 \neq a \in I_i \cap R$  implies  $a \in P_i \subseteq Z(\hat{R})$ , a contradiction.

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