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THE KRULL INTERSECTION THEOREM

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Let R be a commutative ring, I an ideal in R , and A an R -module. We always have $0 \subseteq 0^s \subseteq I \cap \bigcap_{n=1}^{\infty} I^n A \subseteq \bigcap_{n=1}^{\infty} I^n A$ where S is the multiplicatively closed set $\{1 - i \mid i \in I\}$ and $0^s = 0_s \cap A = \{a \in A \mid \exists s \in S \ni sa = 0\}$. It is of interest to know when some containment can be replaced by equality. The Krull intersection theorem states that for R Noetherian and A finitely generated $I \cap \bigcap_{n=1}^{\infty} I^n A = \bigcap_{n=1}^{\infty} I^n A$. Since $\bigcap_{n=1}^{\infty} I^n A$ is finitely generated, $\bigcap_{n=1}^{\infty} I^n A = 0^s$. Thus if $I \subseteq \text{rad}(R)$, the Jacobson radical of R , or R is a domain and A is torsion-free, we have $\bigcap_{n=1}^{\infty} I^n A = 0$. In this note we show that for a Prüfer domain R and a torsion-free R -module A , $I \cap \bigcap_{n=1}^{\infty} I^n A = \bigcap_{n=1}^{\infty} I^n A$. We also consider the condition (*): $\bigcap_{n=1}^{\infty} I^n = 0$ for every ideal I in the commutative ring R . It is shown that a polynomial ring in any set of indeterminants over a Noetherian domain and the integral closure of a Noetherian domain satisfy (*).

Let R be a ring and A an R -module. If $x \in R$ and $x \notin Z(A)$, the zero divisors of A , then $(x) \bigcap_{n=1}^{\infty} (x)^n A = \bigcap_{n=1}^{\infty} (x)^n A$. Actually we can take I to be invertible and A torsion-free. However, the assumption $x \notin Z(A)$ is essential. For example, let $p \in R$ be neither a unit nor a zero divisor and let $F = Rx \oplus (\sum_{i=1}^{\infty} Ry_i)$ be the free R -module on $\{x, y_1, y_2, \dots\}$. Let $A = F/G$ where $G = (x-py_1, x-p^2y_2, \dots)$; it is not difficult to see that $(p) \bigcap_{n=1}^{\infty} (p)^n A \neq \bigcap_{n=1}^{\infty} (p)^n A$. Using this result, one can show that the following are equivalent: (1) $\dim R = 0$, (2) for every finitely generated (principal) ideal I and every R -module A , $I \bigcap_{n=1}^{\infty} I^n A = \bigcap_{n=1}^{\infty} I^n A$. The first theorem gives another affirmative case.

THEOREM 1. *Let R be a reduced ring and let I be a finitely generated ideal with $\text{rank } I \leq 1$. Then $\bigcap_{n=1}^{\infty} I^n = I \bigcap_{n=1}^{\infty} I^n$. If R is quasi-local or R is a domain, then $\bigcap_{n=1}^{\infty} I^n = 0$.*

Proof. First suppose R is a domain. By localization we can assume $\sqrt{I} = M$ the maximal ideal of R . If $B = \bigcap_{n=1}^{\infty} I \neq 0$, then $\sqrt{B} = M$, so there exists an integer m such that $I^m \subseteq B$. Then $I^m = I^{m+1}$ which implies $I^m = 0$ by Nakayama's lemma. Next suppose R is quasi-local, by passing to R/P where P is a minimal prime we get $\bigcap_{n=1}^{\infty} I^n \subseteq P$. Since R is reduced, we have $\bigcap_{n=1}^{\infty} I^n \subseteq \text{nil}(R) = 0$. The general case now follows by localization.

Another affirmative case is R a Prüfer domain and A a torsion-

free R -module. We first consider the quasi-local case.

LEMMA 1. *Let V be a valuation domain, I an ideal in V , and A a torsion-free V -module. Then $ib \in B = \bigcap_{n=1}^{\infty} I^n A$ where $i \in I$ and $b \in A$ implies $i \in \bigcap_{n=1}^{\infty} I^n$ or $b \in B$. In particular, $B = IB$.*

Proof. Suppose $i \notin \bigcap_{n=1}^{\infty} I^n$, then there exists an integer N such that $i \in I^{N-1} - I^N$. Now $ib \in I^m A$ for $m > N$ implies $ib = j^N j^{m-N} a$ for some $j \in I$ and $a \in A$. Now $i \notin I^N$ gives $j^N = si$ for some $s \in V$. Hence $ib = sij^{m-N} a$ so $b = sj^{m-N} a \in I^{m-N} A$ since A is torsion-free. Therefore $b \in B$.

THEOREM 2. *Let R be a Prüfer domain, I an ideal in R , A a torsion-free R -module, and $B = \bigcap_{n=1}^{\infty} I^n A$. Then $B = IB$.*

Proof. Let $y \in B$ and $J = (IB : y)$; it suffices to show $J = R$. Let M be a fixed maximal ideal; we show that $J \not\subseteq M$. Now $y \in B \subseteq B_M \subseteq \bigcap_{n=1}^{\infty} I_M^n A_M = I_M^2 \bigcap_{n=1}^{\infty} I_M^n A_M$ by Lemma 1, so $y = i^2(b/s)$ where $i \in I$, $b \in A$, $s \in R - M$ and $b/s \in \bigcap_{n=1}^{\infty} I_M^n A_M$. Let N be any maximal ideal of R , then $i^2 b = sy \in B \subseteq \bigcap_{n=1}^{\infty} I_N^n A_N$ so by Lemma 1, $i \in \bigcap_{n=1}^{\infty} I_N^n$ or $ib \in \bigcap_{n=1}^{\infty} I_N^n A$. In either case, $ib \in \bigcap_{n=1}^{\infty} I_N^n A_N$ for every maximal ideal N of R , so $ib \in B$. Therefore, $s \in J - M$.

We remark that for a Prüfer domain, $\bigcap_{n=1}^{\infty} I^n$ need not be a prime ideal, but is always a radical ideal.

Consider the condition (*) on a ring. Local rings and Noetherian domains satisfy this condition. The next two propositions are straight forward and the proofs are omitted.

PROPOSITION 1. *If R satisfies (*), then $Z(R) \subseteq \text{rad}(R)$. Conversely, if R is Noetherian, then $Z(R) \subseteq \text{rad}(R)$ implies (*).*

PROPOSITION 2. *If R satisfies (*), then R_M satisfies (*) for every maximal ideal M . If R_M satisfies (*) for every maximal ideal M , then $\bigcap_{n=1}^{\infty} I^n = I \bigcap_{n=1}^{\infty} I^n$ for every ideal I in R . If $Z(R) \subseteq \text{rad}(R)$, then R satisfies (*).*

The next theorem generalizes the Krull intersection theorem to rings which are locally Noetherian.

THEOREM 3. *Let R be a ring and A an R -module such that $\bigcap_{n=1}^{\infty} P_p^n A_p = 0$ for every $P \in \text{spec}(R)$, then $\bigcap_{n=1}^{\infty} I^n A = 0^s$ for every ideal I in R .*

Proof. Let T be the saturation of $S = \{1 - i \mid i \in I\}$, so $T =$

$R - \bigcup_{P \in \mathcal{S}} P_\alpha$ where $\mathcal{S} = \{P \in \text{spec}(R) \mid P \cap T = \emptyset\}$. Then setting $B = \bigcap_{n=1}^\infty I^n A$ yields $B_P \cong \bigcap_{n=1}^\infty I_P^n A_P = 0$ for every $P \in \mathcal{S}$. Hence $(T^{-1}B)_{T^{-1}P} = 0$ for every $P \in \mathcal{S}$, but the $T^{-1}P \in \mathcal{S}$ are precisely the prime ideals of $T^{-1}R$. Therefore $T^{-1}B = 0$, hence $B_s = 0$ and the result follows.

The next proposition will be used to prove that a polynomial ring in any number of indeterminants over a Noetherian domain satisfies (*).

PROPOSITION 3. *Let R be a Noetherian ring, I an ideal in $R[X]$, and $B = \bigcap_{n=1}^\infty I^n$. Then $B = (B \cap R)R[X]$.*

Proof. First suppose $I \cap R = 0$, we show that $B = 0$. Suppose $0 \neq g(x) \in B$, by the Krull intersection theorem there exists a polynomial $f(x) = a_0x^n + \dots + a_n \in I$ such that $g(x)(1 - f(x)) = 0$. Since $1 - f(x) \in Z(R[X])$, there exists $0 \neq c \in R$ such that $c(1 - f(x)) = 0$. Hence $0 = ca_0 = \dots = ca_{n-1} = c(a_n - 1)$ so $c = ca_n$. But $ca_n = cf(x) \in I \cap R = 0$ so $c = ca_n = 0$, a contradiction. For the general case, let $J = I^m \cap R$, passing to $(R/J)[X]$ yields $B \cong JR[X]$, hence $B \cong \bigcap_{n=1}^\infty (I^n \cap R)[X] = (B \cap R)[X] \subseteq B$.

THEOREM 4. *Let R be a Noetherian domain and $T = R[\{X_\alpha\}]$ a polynomial ring over R in any set $\{X_\alpha\}$ of indeterminants. Then T satisfies (*).*

Proof. We may assume the set of indeterminants is countable and hence index it by the positive integers. By Proposition 2 we may assume that (R, \mathcal{M}) is local and we only need show that $\bigcap_{n=1}^\infty M^n = 0$ where M is a maximal ideal in T with $M \cap R = \mathcal{M}$. Let K be the algebraic closure of $k(\{z_\beta\})$ where $\{z_\beta\}$ is an uncountable set of indeterminants over $k = R/\mathcal{M}$. There exists a local ring (B, N) with $B \supseteq R$ faithfully flat, $N = \mathcal{M}B$ and $B/N = K[1]$. Now $B \supset R$ faithfully flat implies $MB[\{X_i\}] \neq B[\{X_i\}]$ so $MB[\{X_i\}] \subseteq M^*$ a maximal ideal in $B[\{X_i\}]$. It is sufficient to show $\bigcap_{n=1}^\infty M^{*n} = 0$. Since

$$[B[\{X_i\}]/M^*: B/N]$$

is countable and $B/N = K$ is uncountable and algebraically closed, $B[\{X_i\}]/M^* = K$. Thus $M^* = (\mathcal{M}, X_1 - r_1, X_2 - r_2, \dots)$ for suitable $r_i \in B$. Since a given polynomial involves only finitely many indeterminants, it suffices to show $\bigcap_{n=1}^\infty (\mathcal{M}, X_1 - r_1, x_m - r_m)^n = 0$ in $B[X_1, \dots, X_m]$. Since $(\mathcal{M}, X_1 - r_1, \dots, X_m - r_m)^n \cap B[X_1, \dots, X_{m-1}] = (\mathcal{M}, X_1 - r_1, \dots, X_{m-1} - r_{m-1})^n$, the result follows from Proposition 3 and induction.

The last theorem gives another class of rings where (*) holds.

THEOREM 5. *Let R be Noetherian domain and R' its integral closure. Then any ring between R and R' satisfies (*).*

Proof. Let $R \subseteq T \subseteq R'$ be a ring, since $T \subseteq R'$ is integral, any ideal of T is contained in the contraction of an ideal of R' , thus we may assume $T = R'$. It suffices to prove the result for (R, M) a local domain. Now $R \subseteq \hat{R}/N \subseteq \hat{R}/P_1 \oplus \dots \oplus \hat{R}/P_n \subseteq (\hat{R}/P_1)' \oplus \dots \oplus (\hat{R}/P_n)'$ where \hat{R} is the completion of R , $N = P_1 \cap \dots \cap P_n$, and P_1, \dots, P_n are the minimal primes of \hat{R} . Now each \hat{R}/P_i is a complete local domain, so each $(\hat{R}/P_i)'$ is a Noetherian domain and hence satisfies (*). Every maximal ideal \mathcal{M} of R' has the form $\mathcal{M} = M^* \cap R'$ for some maximal ideal M^* of $(\hat{R}/P_1)' \oplus \dots \oplus (\hat{R}/P_n)'$ [2, p. 119]. Hence $M^* = (\hat{R}/P_i)' \oplus \dots \oplus N \oplus \dots \oplus (\hat{R}/P_n)'$ where N is a maximal ideal in $(\hat{R}/P_i)'$ for some i . Then $\bigcap_{n=1}^{\infty} \mathcal{M}^n = \bigcap_{n=1}^{\infty} (M^* \cap R')^n \subseteq (\bigcap_{n=1}^{\infty} M^{*n}) \cap R' = I_i \cap R'$ where $I_i = (\hat{R}/P_i)' \oplus \dots \oplus 0 \oplus \dots \oplus (\hat{R}/P_n)'$. Suppose $I_i \cap R' \neq 0$, then $I_i \cap R \neq 0$ since $R \subseteq R'$ is integral. But $0 \neq a \in I_i \cap R$ implies $a \in P_i \subseteq Z(\hat{R})$, a contradiction.

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Keith Roy Allen, <i>Dendritic compactification</i>	1
Daniel D. Anderson, <i>The Krull intersection theorem</i>	11
George Phillip Barker and David Hilding Carlson, <i>Cones of diagonally dominant matrices</i>	15
David Wilmot Barnette, <i>Generalized combinatorial cells and facet splitting</i>	33
Stefan Bergman, <i>Bounds for distortion in pseudoconformal mappings</i>	47
Nguyễn Phương Các, <i>On bounded solutions of a strongly nonlinear elliptic equation</i>	53
Philip Throop Church and James Timourian, <i>Maps with 0-dimensional critical set</i>	59
G. Coquet and J. C. Dupin, <i>Sur les convexes ubiquitaires</i>	67
Kandiah Dayanithy, <i>On perturbation of differential operators</i>	85
Thomas P. Dence, <i>A Lebesgue decomposition for vector valued additive set functions</i>	91
John Riley Durbin, <i>On locally compact wreath products</i>	99
Allan L. Edelson, <i>The converse to a theorem of Conner and Floyd</i>	109
William Alan Feldman and James Franklin Porter, <i>Compact convergence and the order bidual for $C(X)$</i>	113
Ralph S. Freese, <i>Ideal lattices of lattices</i>	125
R. Gow, <i>Groups whose irreducible character degrees are ordered by divisibility</i>	135
David G. Green, <i>The lattice of congruences on an inverse semigroup</i>	141
John William Green, <i>Completion and semicompletion of Moore spaces</i>	153
David James Hallenbeck, <i>Convex hulls and extreme points of families of starlike and close-to-convex mappings</i>	167
Israel (Yitzchak) Nathan Herstein, <i>On a theorem of Brauer-Cartan-Hua type</i>	177
Virgil Dwight House, Jr., <i>Countable products of generalized countably compact spaces</i>	183
John Sollion Hsia, <i>Spinor norms of local integral rotations. I</i>	199
Hugo Junghenn, <i>Almost periodic compactifications of transformation semigroups</i>	207
Shin'ichi Kinoshita, <i>On elementary ideals of projective planes in the 4-sphere and oriented Θ-curves in the 3-sphere</i>	217
Ronald Fred Levy, <i>Showering spaces</i>	223
Geoffrey Mason, <i>Two theorems on groups of characteristic 2-type</i>	233
Cyril Nasim, <i>An inversion formula for Hankel transform</i>	255
W. P. Novinger, <i>Real parts of uniform algebras on the circle</i>	259
T. Parthasarathy and T. E. S. Raghavan, <i>Equilibria of continuous two-person games</i>	265
John Pfaltzgraff and Ted Joe Suffridge, <i>Close-to-starlike holomorphic functions of several variables</i>	271
Esther Portnoy, <i>Developable surfaces in hyperbolic space</i>	281
Maxwell Alexander Rosenlicht, <i>Differential extension fields of exponential type</i>	289
Keith William Schrader and James Lewis Thornburg, <i>Sufficient conditions for the existence of convergent subsequences</i>	301
Joseph M. Weinstein, <i>Reconstructing colored graphs</i>	307