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The extreme rays of several cones of complex and real diagonally dominant matrices, and their duals, are identified. Several results on lattices of faces of cones are given. It is then shown that the dual (in the real space of hermitian matrices) of the cone of hermitian diagonally dominant matrices cannot be the image of the cone of positive semidefinite matrices under any nonsingular linear transformation; in particular, it cannot be the image of the cone of positive semidefinite matrices under the Ljapunov transformation $H \mapsto AH + HA^*$ determined by a positive stable matrix A.

1. **Preliminaries.** Let X denote a finite-dimensional real vector space. A *cone* K in X is a nonempty subset of X satisfying

$$\alpha, \beta \geqq 0, x, y \in K \Longrightarrow \alpha x + \beta y \in K .$$

A cone K is *closed* if it is closed in the usual topology of X. If K has interior with respect to this topology, equivalently, if $X = K - K$, then K is *full*. If K satisfies

$$x \in K, -x \in K \Longrightarrow x = 0 ,$$

then is K *pointed*. Associated with a closed cone K is a reflexive and transitive order relation defined by

$$x \leqq y \Longleftrightarrow y - x \in K ,$$

which is a partial order iff K is pointed.

The element $y \in K$ is *extremal* if $x, y - x \in K$ (i.e., $0 \leqq x \leqq y$) $\Rightarrow$ $x = \alpha y$ for some $\alpha \geqq 0$. For each $y \in X$, $\Gamma(y) = \{\alpha y \mid \alpha \geqq 0\}$ is the ray generated by y; if y is extremal in K, then $\Gamma(y)$ is an extreme ray of K. A closed pointed cone K is the convex hull of its extremals (cf. [9], p. 167). (If K is not pointed it has no extremals.)

Given a closed, pointed cone K, let $\mathscr{E}(K)$ denote any minimal generating set of extremals of K: i.e., every element of $\mathscr{E}(K)$ is extremal, and all extremals of K are positive multiples of elements of $\mathscr{E}(K)$; and $\mathscr{E}(K)$ is minimal with respect to these properties.

If we have given an inner product on X, then we may define another cone,

$$K^* = \{x \in X \mid (x, y) \geqq 0 \quad \text{for all} \quad y \in K\} ,$$

* *Dedicated to Alfred Brauer on his eightieth birthday, April* 9, 1974.

the cone dual in X to K, or, briefly, the *dual cone* of K. It is well-known that in determining the elements of K^*, we need only consider inner products with extremals of K, in fact, given any $\mathscr{E}(K)$,

$$K^* = \{x \in X \mid (x, y) \geqq 0 \quad \text{for all} \quad y \in \mathscr{E}(K)\} .$$

We shall often consider cones K which lie in a subspace V of a space X; in this case, K^V will denote the cone dual in V to K.

2. A particular top heavy cone in C_n. Let C_n be the space of complex n-tuple row vectors, with standard basis $E_1, \cdots, E_n$. We shall think of C_n as a real inner product space, with inner product

$$(x, y) = \operatorname{re} \sum_{j=1}^{n} \bar{y}_j x_j .$$

Let

$$T = \left\{ x \in C_n \,\middle|\, x_1 \geqq \sum_{j=2}^{n} |x_j| \right\} ;$$

clearly T is a closed, pointed cone, full in

$$U = \{x \in C_n \mid \operatorname{im} x_1 = 0\} .$$

As a cone in U, T is top heavy (cf. [3]).

The proof of the first lemma depends on the following simple fact: if $a, b \in C$, and either $|a + b| = |a| + |b|$ or $|a - b| = |a| - |b|$, then for some $\varepsilon \in C$, $|\varepsilon| = 1$, $a = |a|\varepsilon$ and $b = |b|\varepsilon$.

LEMMA 1. *For T, U as defined above, we have*

$$\mathscr{E}(T) = \{E_1 + \varepsilon E_j \mid j = 2, \cdots, n; |\varepsilon| = 1\} ,$$
$$T^U = \{x \in U \mid x_1 \geqq |x_j|, j = 2, \cdots, n\} .$$

Then T^U is pointed, and full in U;

$$\mathscr{E}(T^U) = \{E_1 + \sum_{j=2}^{n} \varepsilon_j E_j \mid |\varepsilon_j| = 1, j = 2, \cdots, n\} .$$

Proof. Consider $y = E_1 + \varepsilon E_j$, $j > 1$, $|\varepsilon| = 1$. Suppose $x \in T$, for which $y - x \in T$. Then

$$x_1 \geqq \sum_{k=2}^{n} |x_k|$$
$$1 - x_1 \geqq \sum_{\substack{k=2 \\ k \neq j}}^{n} |x_k| + |\varepsilon - x_j| .$$

Adding,

$$1 \geqq 2\sum_{\substack{k=2\\k\neq j}}^{n} |x_k| + |x_j| + |\varepsilon - x_j| \geqq 2\sum_{\substack{k=2\\k\neq j}}^{n} |x_k| + 1\,,$$

so that $x_k = 0, k = 2, \cdots, n, k \neq j$. It follows that

$$1 - x_1 \geqq |\varepsilon - x_j| \geqq 1 - |x_j|\,,$$

so that $x_1 \leqq |x_j|$; but $x_1 \geqq |x_j|$, so $x_1 = |x_j|$. We now have

$$x_1 = |x_j|, 1 - x_1 = |\varepsilon - x_j|, \quad \text{and} \quad 1 = |\varepsilon|\,.$$

By our previous observation, $x_j = \varepsilon x_1$, hence $x = (x_1)y$. It follows that y is extremal in T.

Conversely, suppose x is extremal in T. If $x = x_1E_1$, then

$$x = \frac{1}{2}x_1(E_1 + E_2) + \frac{1}{2}x_1(E_1 - E_2)\,,$$

which is impossible. Also, clearly,

$$x_1 = \sum_{j=2}^{n} |x_j|\,.$$

Now we may write

$$x = \tilde{\sum_{j=2}^{n}} |x_j|(E_1 + (x_j/|x_j|)E_j)\,,$$

where the summation $\tilde{\sum}$ is taken over indices j for which $x_j \neq 0$. Since x is extremal, there must be exactly one nonzero x_j, $j > 1$, and

$$x = |x_j|(E_1 + (x_j/|x_j|)E_j)\,.$$

We have proved our statement about $\mathscr{E}(T)$.

For $x \in U$, $y = E_1 + \varepsilon E_j$, $j > 1$, and $|\varepsilon| = 1$, we have

$$(x, y) = \mathrm{re}\,(x_1 + \bar{\varepsilon}x_j) = x_1 + \mathrm{re}\,(\bar{\varepsilon}x_j)\,.$$

Clearly, now $x \in T^U$ iff

$$x_1 \geqq |x_j|\,, \qquad\qquad j = 1, \cdots, n\,.$$

(This characterization of T^U is essentially contained in Theorem 2 of [3].)

Consider $y = E_1 + \sum_{j=2}^{n} \varepsilon_j E_j$, where $|\varepsilon_j| = 1$, $j = 1, \cdots, n$. Suppose $x \in T^U$ such that $y - x \in T^U$. For $j > 1$,

$$|x_j| + |\varepsilon_j - x_j| \leqq x_1 + (1 - x_1) = |\varepsilon_j| \leqq |x_j| + |\varepsilon_j - x_j|\,,$$

i.e., $|x_j| + |\varepsilon_j - x_j| = |\varepsilon_j|$. It follows that $x_j = \varepsilon_j x_1$. Since this holds for all $j > 1$, $x = (x_1)y$, and y is extremal in T^U.

Conversely, suppose x is extremal in T^U. We may assume that $x_1 = 1$. If, for some $j > 1$, $|x_j| < 1$, then let $\delta = 1 - |x_j| > 0$. Define

$$y = \frac{1}{2}(x + \delta E_j),\ z = \frac{1}{2}(x - \delta E_j)\ .$$

We have $x = y + z$, $y, z \in T^U$, and y and z are linearly independent, a contradiction. Thus we must have

$$x = E_1 + \sum_{j=2}^{n} \varepsilon_j E_j\ , \qquad |\varepsilon_j| = 1,\ j = 2, \cdots, n\ .$$

We have proved our statement $\mathscr{E}(T^U)$.

We note that for the analogous cone in real n-tuple space $\boldsymbol{R}_n$, with the standard inner product,

$$T_R = \left\{x \in \boldsymbol{R}_n \middle| x_1 \geqq \sum_{j=2}^{n} |x_j|\right\}\ ,$$

we have corresponding characterizations of $\mathscr{E}(T_R)$, T^*, and $\mathscr{E}(T^*)$, with formally identical proofs.

3. Cones of real and complex diagonally dominant matrices. Let $\boldsymbol{C}_{n,n}$ denote the set of $n \times n$ matrices with complex entries; we shall regard $\boldsymbol{C}_{n,n}$ as a real inner product space with inner product $(A, B) = \operatorname{re} \operatorname{tr} B^*A$. Similarly, $\boldsymbol{R}_{n,n}$ will denote the set of $n \times n$ matrices with real entries; $\boldsymbol{R}_{n,n}$ is a real inner product space with inner product $(A, B) = \operatorname{tr} B^*A$.

A matrix $A = [a_{jk}] \in \boldsymbol{C}_{n,n}$ is said to be *diagonally dominant* if

$$|a_{jj}| \geqq \sum_{k \neq j} |a_{jk}|\ ; \qquad j = 1, \cdots, n\ .$$

Neither the set of all diagonally dominant matrices, nor the set of all real diagonally dominant matrices, is a cone. However, in the real case, if we restrict ourselves to diagonally dominant matrices with nonnegative diagonal entries, we obtain a closed, pointed, full cone:

$$D_R = \left\{A \in \boldsymbol{R}_{n,n} \middle| a_{jj} \geqq \sum_{k \neq j} |a_{jk}|,\ j = 1, \cdots, n\right\}\ .$$

In the complex case, there are three closed cones analogous to D_R:

$$D_1 = \left\{A \in \boldsymbol{C}_{n,n} \middle| a_{jj} \geqq \sum_{k \neq j} |a_{jk}|,\ j = 1, \cdots, n\right\}\ ,$$

$$D_2 = \left\{A \in \boldsymbol{C}_{n,n} \middle| \operatorname{re} a_{jj} \geqq \sum_{k \neq j} |a_{jk}|,\ \operatorname{im} a_{jj} \geqq 0,\ j = 1, \cdots, n\right\}\ ,$$

$$D_3 = \left\{A \in \boldsymbol{C}_{n,n} \middle| \operatorname{re} a_{jj} \geqq \sum_{k \neq j} |a_{jk}|,\ j = 1, \cdots, n\right\}\ .$$

Clearly

$$D_1 \subseteqq D_2 \subseteqq D_3 ,$$

and

$$D_R = D_i \cap R_{n,n} , \qquad i = 1, 2, 3 .$$

To discuss further the structure of these cones, observe first that $C_{n,n} = V \oplus W$, where

$$V = \{A \in C_{n,n} \mid \operatorname{im} a_{jj} = 0,\ j = 1, \cdots, n\} ,$$
$$W = \{A \in C_{n,n} \mid \operatorname{re} a_{jj} = a_{jk} = 0,\ j, k = 1, \cdots, n,\ k \neq j\} ,$$

and that (with our real inner product), $W = V^{\perp}$.

Let K_1 be a cone in V, and let K_2 be a cone in W. Then $(K_1 + K_2)^* = K_1^V + K_2^W$. Also, let

$$J = \{A \in W \mid \operatorname{im} a_{jj} \geqq 0,\ j = 1, \cdots, n\} ;$$

J is a closed, pointed cone. As a cone in W, J is full and self-dual, i.e., $J^W = J$.

Now $D_1 \subseteqq V$, and is full in V; $D_2 = D_1 + J$, and $D_3 = D_1 + W$. Clearly D_1 is pointed but not full, D_2 is both pointed and full, and D_3 is full but not pointed. Moreover, $D_1^* = D_1^V + W$ is full but not pointed, $D_2^* = D_1^V + J$ is both full and pointed, and $D_3^* = D_1^V$ is pointed but not full.

We next determine $\mathscr{E}(D_1)$, D_1^V, and $\mathscr{E}(D_1^V)$. We can then easily determine the extremals, the duals, and the extremals of the duals, of all our diagonally dominant cones.

Let E_{jk} denote the $n \times n$ matrix with (j, k)th entry one, and all other entries zero. Define

$$\mathscr{E}_0 = \{iE_{jj} \mid j = 1, \cdots, n\} ,$$
$$\mathscr{E}_1 = \{E_{jj} + \varepsilon E_{jk} \mid |\varepsilon| = 1,\ j, k = 1, \cdots, n,\ k \neq j\} ,$$
$$\mathscr{E}_2 = \left\{E_{jj} + \sum_{\substack{k=1 \\ k\neq j}}^{n} \varepsilon_k E_{jk} \,\middle|\, |\varepsilon_k| = 1,\ j, k = 1, \cdots, n,\ k \neq j\right\} .$$

Clearly $\mathscr{E}(J) = \mathscr{E}_0$.

LEMMA 2. *For D_1 defined above, $\mathscr{E}(D_1) = \mathscr{E}_1$, and*

$$D_1^V = \{A \in V \mid a_{jj} \geqq |a_{jk}| \ \textit{for all}\ j, k = 1, \cdots, n,\ k \neq j\} .$$

Also, D_1^V is a closed, pointed cone, full in V, and $\mathscr{E}(D_1^V) = \mathscr{E}_2$.

Proof. Observe that $D_1 = T_1 + \cdots + T_n$, where

$$T_j = \left\{A \in V \,\middle|\, a_{jj} \geqq \sum_{k \neq j} |a_{jk}|,\ a_{pq} = 0,\ p, q = 1, \cdots, n,\ p \neq j\right\}$$

is contained in $U_j = \{A \in V \mid \operatorname{im} a_{jj} = 0;\ a_{pq} = 0,\ p, q = 1, \cdots, n,\ p \neq j\}$, $j = 1, \cdots, n$; each T_j is essentially the cone T of Lemma 1. Since $V = U_1 \oplus \cdots \oplus U_n$ (in fact, $U_k \subseteqq U_j^{\perp}$ for all $j, k = 1, \cdots, n, k \neq j$), the extremals of D_1 are precisely those matrices which are extremals of some T_j; this proves that $\mathscr{E}_1 = \mathscr{E}(D_1)$. Also,

$$D_1^V = T_1^{U_1} + \cdots + T_n^{U_n},$$

and the extremals of D_1^V are precisely those matrices which are extremals of some $T_j^{U_j}$, proving that $\mathscr{E}_2 = \mathscr{E}(D_1^V)$.

It is now clear that the extremals of D_2 are precisely those matrices which are extremals of D_1 or J (and similarly for D_2^*). Although D_3 is not pointed, and has no extremals, every matrix of D_3 is a nonnegative linear combination of extremals of D_1, J, and $-J$ (and similarly for $D_1{}^*$).

The result for T_R analogous to Lemma 1 can be used to establish the corresponding results for D_R. Note that D_R and D_R^* are polyhedral cones. We summarize these results in Theorem 1.

THEOREM 1. *Let D_R, D_1, D_2, D_3 be defined as above. Then*

$$\mathscr{E}(D_1) = \mathscr{E}_1,\ \mathscr{E}(D_2) = \mathscr{E}_0 \cup \mathscr{E}_1, \quad \textit{and} \quad \mathscr{E}(D_R) = \mathscr{E}_1 \cap \boldsymbol{R}_{n,n}.$$

Also

$$\begin{aligned}
D_1^* &= \{A \in \boldsymbol{C}_{n,n} \mid \operatorname{re} a_{jj} \geqq |a_{jk}| \textit{ for all } j, k = 1, \cdots n, k \neq j\},\\
D_2^* &= \{A \in \boldsymbol{C}_{n,n} \mid \operatorname{re} a_{jj} \geqq |a_{jk}|, \operatorname{im} a_{jj} \geqq 0 \textit{ for all } j, k = 1, \cdots,\\
&\qquad n, k \neq j\},\\
D_3^* &= \{A \in \boldsymbol{C}_{n,n} \mid a_{jj} \geqq |a_{jk}| \textit{ for all } j, k = 1, \cdots, n, k \neq j\},\\
D_R^* &= D_1^* \cap \boldsymbol{R}_{n,n} = D_2^* \cap \boldsymbol{R}_{n,n} = D_3^* \cap \boldsymbol{R}_{n,n},
\end{aligned}$$

and

$$\mathscr{E}(D_2^*) = \mathscr{E}_0 \cup \mathscr{E}_2,\ \mathscr{E}(D_3^*) = \mathscr{E}_2, \ \textit{and}\ \mathscr{E}(D_R^*) = \mathscr{E}_2 \cap \boldsymbol{R}_{n,n}.$$

The characterization of D_R^* given above appeared previously in [5]. Also, the full set D of complex diagonally dominant matrices is the object of study in [4]. In that paper, the authors introduce a set of weakly diagonally dominant matrices which is in some sense dual to D, and which contains our D_1^* (cf. their Theorem 3.5). However, their work is an altogether different spirit from ours, and there is almost no overlap.

4. Cones of hermitian diagonally dominant matrices. One of the outstanding problems of matrix theory is to determine conditions under which a cone K in the real space $\mathscr{H}$ of hermitian matrices in $C_{n,n}$ is the image under a Ljapunov transformation $L_A(H) = AH + HA^*$ of the cone PSD of positive semidefinite hermitian matrices, where $A \in C_{n,n}$ is a (positive) stable matrix (i.e., all eigenvalues of A have positive real parts). One of the necessary conditions is that $K \supseteqq \mathrm{PSD}$ (cf. Loewy [7]).

We wish to study the possibility of cones of diagonally dominant matrices being images of PSD under Ljapunov transformations. Since the cones D_i and D_i^*, $i = 1, 2, 3$, are not contained in $\mathscr{H}$, we shall consider instead

$$D_{\mathscr{H}} = \left\{A \in \mathscr{H} \,\middle|\, a_{jj} \geqq \sum_{\substack{k=1\\k\neq j}}^{n} |a_{jk}|,\ j = 1, \cdots, n\right\},$$

$$_{\mathscr{H}}D = \{A \in \mathscr{H} \,|\, a_{jj} \geqq |a_{jk}|,\ j, k = 1, \cdots, n,\ k \neq j\},$$

and their duals in $\mathscr{H}$. Note that $D_{\mathscr{H}} = D_i \cap \mathscr{H}$ and $_{\mathscr{H}}D = D_i^* \cap \mathscr{H}$, $i = 1, 2, 3$, and that they are both pointed, and full as cones in $\mathscr{H}$.

It is clear that $D_{\mathscr{H}}$ is properly contained in PSD, so that $D_{\mathscr{H}}$ cannot be the image of PSD under a Ljapunov mapping. It follows, however, that $D_{\mathscr{H}}^{\mathscr{H}} \supseteqq \mathrm{PSD}$, so that $D_{\mathscr{H}}^{\mathscr{H}}$ satisfies the necessary condition given above. We will later show that $D_{\mathscr{H}}^{\mathscr{H}}$ cannot be the image of PSD under *any* nonsingular linear transformation. We have that $\mathrm{PSD} \not\subseteq {}_{\mathscr{H}}D$:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

is in PSD, not $_{\mathscr{H}}D$. Finally, since

$$A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

is in $_{\mathscr{H}}D$, but not PSD, $_{\mathscr{H}}D \not\subseteq \mathrm{PSD}$, hence $\mathrm{PSD} \not\subseteq {}_{\mathscr{H}}D^{\mathscr{H}}$. Thus neither $_{\mathscr{H}}D$ nor $_{\mathscr{H}}D^{\mathscr{H}}$ could be the image of PSD under a Ljapunov transformation.

We next determine inequalities defining the matrices of $D_{\mathscr{H}}^{\mathscr{H}}$, and, for the sake of completeness, those defining the matrices of $_{\mathscr{H}}D^{\mathscr{H}}$, and minimal generating sets of extremals for all four cones.

THEOREM 2. *Given $D_{\mathscr{H}}$ and $_{\mathscr{H}}D$ as defined above. Then*

$$\begin{aligned}
\mathscr{E}(D_{\mathscr{H}}) &= \{E_{jj} \mid j = 1, \cdots, n\} \cup \{E_{jj} + \varepsilon E_{jk} + \bar{\varepsilon} E_{kj} + E_{kk} \mid \\
&\qquad |\varepsilon| = 1, j, k = 1, \cdots, n, k \neq j\}, \\
D_{\mathscr{H}}^{\mathscr{H}} &= \{A \in \mathscr{H} \mid a_{jj} \geqq 0, a_{jj} - 2|a_{jk}| + a_{kk} \geqq 0, \\
&\qquad j, k = 1, \cdots, n, j < k\}, \\
\mathscr{E}(D_{\mathscr{H}}^{\mathscr{H}}) &= \left\{2E_{jj} + \sum_{\substack{k=1 \\ k \neq j}}^{n} (\varepsilon_k E_{jk} + \bar{\varepsilon}_k E_{kj}) \right| \\
&\qquad \left. |\varepsilon_k| = 1, j, k = 1, \cdots, n, k \neq j \right\}, \\
\mathscr{E}({}_{\mathscr{H}}D) &= \left\{\sum_{j \in \alpha} E_{jj} + \sum_{\substack{j,k \in \alpha \\ j<k}} (\varepsilon_{jk} E_{jk} + \bar{\varepsilon}_{jk} E_{kj}) \right| \\
&\qquad \left. |\varepsilon_{jk}| = 1, j, k \in \alpha, j < k, \text{ for all } \alpha \subseteqq \{1, \cdots, n\} \right\}, \\
{}_{\mathscr{H}}D^{\mathscr{H}} &= \left\{A \in \mathscr{H} \,\middle|\, \sum_{j \in \alpha} a_{jj} - 2 \sum_{\substack{j,k \in \alpha \\ j<k}} |a_{jk}| \geqq 0 \text{ for all } \alpha \right. \\
&\qquad \left. \subseteqq \{1, \cdots, n\} \right\}, \\
\mathscr{E}({}_{\mathscr{H}}D^{\mathscr{H}}) &= \{2E_{jj} + \varepsilon_k E_{jk} + \bar{\varepsilon}_k E_{kj} \mid |\varepsilon_k| = 1, j, k = 1, \cdots, \\
&\qquad n, k \neq j\}.
\end{aligned}$$

The analogous results hold for cones of diagonally dominant matrices in the real space $\mathscr{S}$ of symmetric matrices in $\boldsymbol{R}_{n,n}$.

Proof of Theorem 2. We first determine $\mathscr{E}(D_{\mathscr{H}})$. Clearly each E_{jj} is extremal in $D_{\mathscr{H}}$; consider a matrix of the form $A = E_{jj} + \varepsilon E_{jk} + \bar{\varepsilon} E_{kj} + E_{kk}$, $j \neq k$, $|\varepsilon| = 1$. If there exists $B \in D_{\mathscr{H}}$ for which $A - B \in D_{\mathscr{H}}$, clearly $b_{pp} = 0$ for $p \notin \{j, k\}$, so that $b_{pq} = 0$ unless $\{p, q\} \subseteqq \{j, k\}$. Suppose $B = b_{jj}E_{jj} + b_{jk}E_{jk} + \bar{b}_{jk}E_{kj} + b_{kk}E_{kk}$; by Lemma 1, $b_{jk} = b_{jj}\varepsilon$, $\bar{b}_{jk} = b_{kk}\bar{\varepsilon}$, implying $b_{jj} = b_{kk}$, and $B = b_{jj}A$. Thus A is extremal in $D_{\mathscr{H}}$.

Conversely, for any $B \in D_{\mathscr{H}}$, we may write

$$\begin{aligned}
B = &\sum_{\substack{j,k=1 \\ j<k}}^{n} (|b_{jk}| E_{jj} + b_{jk} E_{jk} + \bar{b}_{jk} E_{kj} + |b_{jk}| E_{kk}) \\
&+ \sum_{j=1}^{n} \Big(b_{jj} - \sum_{k \neq j} |b_{jk}|\Big) E_{jj}.
\end{aligned}$$

For B to be extremal in $D_{\mathscr{H}}$, exactly one term (between the two summations) must be nonzero, and B has the desired form.

For arbitrary $B \in \mathscr{H}$,

$$\begin{aligned}
&(E_{jj}, B) = b_{jj}, \\
&(E_{jj} + \varepsilon E_{jk} + \bar{\varepsilon} E_{kj} + E_{kk}, B) = b_{jj} + b_{kk} + \mathrm{re}\,(\bar{b}_{jk}\varepsilon + b_{jk}\bar{\varepsilon}).
\end{aligned}$$

Clearly $B \in D_{\mathscr{H}}^{\mathscr{H}}$ iff

$$b_{jj} \geqq 0, b_{jj} + b_{kk} - 2|b_{jk}| \geqq 0, j, k = 1, \cdots, n, k \neq j .$$

We next determine $\mathscr{E}(D_{\mathscr{H}}^{\mathscr{H}})$. Consider a matrix of the form

$$A = 2E_{jj} + \sum_{k \neq j} (\varepsilon_k E_{jk} + \bar{\varepsilon}_k E_{kj}), |\varepsilon_k| = 1, k = 1, \cdots, n, k \neq j .$$

Suppose $B \in D_{\mathscr{H}}^{\mathscr{H}}$ for which $A - B \in D_{\mathscr{H}}^{\mathscr{H}}$. For $p = 1, \cdots, n$, $p \neq j$, $b_{pp} = 0$; and from this, we have $b_{pq} = 0$ if $p \neq j$, and $q \neq j$. Suppose $j \neq q$; then by the argument of Lemma 1, applied to $(2, 2|\varepsilon_q|)$ and $(b_{jj}, 2b_{jq})$, we have $b_{jq} = \varepsilon_q b_{jj}$. Thus $B = b_{jj}A$, and A is extremal in $D_{\mathscr{H}}^{\mathscr{H}}$.

Conversely, suppose A is extremal in $D_{\mathscr{H}}^{\mathscr{H}}$, with at least two positive diagonal entries; withous loss of generality, suppose $a_{11} > 0$. Now define $B \in \mathscr{H}$ by

$$b_{11} = a_{11}, b_{1k} = \begin{cases} a_{1k}, & \text{if} \quad a_{11} \geqq 2|a_{1k}| \\ \dfrac{a_{11}}{2|a_{1k}|} a_{1k}, & \text{if} \quad a_{11} < 2|a_{1k}| \end{cases} ,$$

$$b_{k1} = \bar{b}_{1k}, k = 2, \cdots, n, \quad \text{and} \quad b_{pq} = 0, p, q = 2, \cdots, n .$$

Clearly $b_{jj} \geqq 0$, $j = 1, 2, \cdots, n$, $b_{jj} + b_{kk} - 2|b_{jk}| = 0$, $j, k = 2, \cdots, n$, $k \neq j$, and, by calculation,

$$b_{11} + b_{kk} - 2|b_{1k}| \geqq 0, k = 2, \cdots, n ,$$

i.e., $B \in D_{\mathscr{H}}^{\mathscr{H}}$. Also, if $C = A - B \in \mathscr{H}$, $c_{jj} \geqq 0$, $j = 1, \cdots, n$, $c_{jj} + c_{kk} - 2|c_{jk}| = a_{jj} + a_{kk} - 2|a_{jk}| \geqq 0$, $j, k = 2, \cdots, n$, $k \neq j$, and, by an easy calculation, $c_{11} + c_{kk} - 2|c_{1k}| = a_{kk} - 2|a_{1k} - b_{1k}| \geqq a_{kk} - 2|a_{1k}| \geqq 0$, $k = 2, \cdots, n$, i.e., $C \in D_{\mathscr{H}}^{\mathscr{H}}$. The assumption that A had more than one positive diagonal entry would imply that $A = B + C$, with $B, C \in D_{\mathscr{H}}^{\mathscr{H}}$ and linearly independent, a contradiction. It follows that our extremal A has exactly one positive diagonal entry.

Suppose that $a_{jj} > 0$, $a_{kk} = 0$, $k = 1, \cdots, n$, $k \neq j$. We have $a_{pq} = 0$ unless $p = j$ or $q = j$, i.e.,

$$A = a_{jj}E_{jj} + \sum_{\substack{k=1 \\ j<k}} (a_{jk}E_{jk} + \bar{a}_{jk}E_{kj}) .$$

Using Lemma 1 again, $2|a_{jk}| = a_{jj}$, $k = 1, \cdots, n$, $k \neq j$, and A is a positive multiple of some matrix of our given set of extremals. We have proved that $\mathscr{E}(D_{\mathscr{H}}^{\mathscr{H}})$ has the specified form.

We next determine $\mathscr{E}({}_{\mathscr{H}}D)$. Consider

$$A = \sum_{j \in \alpha} E_{jj} + \sum_{\substack{j,k \in \alpha \\ j<k}} (\varepsilon_{jk}E_{jk} + \bar{\varepsilon}_{jk}E_{kj}) ,$$

where $|\varepsilon_{jk}| = 1$, $j, k \in \alpha$, $j < k$, and $\alpha \subseteqq \{1, 2, \cdots, n\}$. We have $A \in {}_{\mathscr{H}}D$. Let $|\alpha|$ denote the number of elements of α. If $|\alpha| = 1$, A is obviously extremal. Suppose $1 < |\alpha| \leqq n$; without loss of generality, we may assume $|\alpha| = n$, and $\alpha = \{1, 2, \cdots, n\}$. If $B \in {}_{\mathscr{H}}D$ and $A - B \in {}_{\mathscr{H}}D$, then, using Lemma 1,

$$b_{jk} = b_{jj}\varepsilon_{jk}\ , \qquad j, k = 1, \cdots, n, k \neq j\ .$$

Also, $b_{jj} = b_{jk}/\varepsilon_{jk} = \overline{b_{jk}/\varepsilon_{jk}} = b_{kj}/\varepsilon_{kj} = b_{kk}$, $k \neq j$, so that $B = b_{jj}A$. Thus A is extremal in ${}_{\mathscr{H}}D$.

Conversely, suppose A is extremal in ${}_{\mathscr{H}}D$. We first show that all nonzero diagonal entries of A are equal. If not, without loss of generality,

$$a_{11} = a_{22} = \cdots = a_{j-1,j-1} > a_{jj} \geqq \cdots \geqq a_{nn}\ ,$$

where $a_{jj} > 0$. We define B by

$$b_{pq} = \begin{cases} a_{jj} & 1 \leqq p = q < j \\ a_{jj}a_{pq}/a_{11} & 1 \leqq p, q < j, p \neq q \\ a_{pq} & \text{otherwise}\ . \end{cases}$$

Then B is hermitian, and $B \in {}_{\mathscr{H}}D$, for

$$|b_{pq}| = \begin{cases} a_{jj}|a_{pq}|/a_{11} \leqq a_{jj} = b_{pp}\ , & 1 \leqq p, q < j\ , \\ |a_{pq}| \leqq a_{qq} \leqq a_{jj} = b_{pp}\ , & 1 \leqq p < j, q \geqq j\ , \\ |a_{pq}| \leqq a_{pp} = b_{pp}\ , & p \geqq j\ . \end{cases}$$

Let $C = B - A$; first, $c_{pq} = 0$ if $p \geqq j$ or $q \geqq j$. Also, if $1 \leqq p, q < j, p \neq q$,

$$|c_{pq}| = (1 - a_{jj}/a_{11})|a_{pq}| = \frac{|a_{pq}|}{a_{11}}(a_{11} - a_{jj}) \leqq a_{11} - a_{jj} = c_{pp}\ ,$$

so that $C_{\mathscr{H}} \in D$. In this case, B and C are linearly independent, and A is not extremal.

Suppose now that all nonzero diagonal entries of A are equal. Without loss of generality, we may assume all diagonal entries are one. If $|a_{jk}| < 1$ for some $j, k, k \neq j$, then, as in Lemma 1, we could write

$$A = \frac{1}{2}(A + \delta E_{jk} + \bar{\delta} E_{kj}) + \frac{1}{2}(A - \delta E_{jk} - \bar{\delta} E_{kj})\ ,$$

where the summands are in ${}_{\mathscr{H}}D$ and are linearly independent. Thus $|a_{jk}| = 1$ for all j, k. This completes the proof that $\mathscr{E}({}_{\mathscr{H}}D)$ has the desired form.

To determine the inequalities that characterize matrices in ${}_{\mathscr{H}}D^{\mathscr{C}}$, we consider the inner product of an arbitrary $B \in \mathscr{H}$ with an element A of $\mathscr{E}({}_{\mathscr{H}}D)$:

$$(A, B) = \sum_{j \in \alpha} b_{jj} + \sum_{\substack{j,k \in \alpha \\ j<k}} (\varepsilon_{jk} b_{jk} + \bar{\varepsilon}_{jk} \bar{b}_{jk}) ,$$

where $|\varepsilon_{jk}| = 1$, $j, k \in \alpha$, $j < k$, and $\alpha \subseteqq \{1, 2, \cdots, n\}$. Clearly now $B \in {}_{\mathscr{H}}D^{\mathscr{H}}$ iff, for all $\alpha \subseteqq \{1, 2, \cdots, n\}$

$$\sum_{j \in \alpha} b_{jj} - 2 \sum_{\substack{j,k \in \alpha \\ j<k}} |b_{jk}| \geqq 0 .$$

Finally, we must determine $\mathscr{E}({}_{\mathscr{H}}D^{\mathscr{H}})$. Let $A = 2E_{jj} + \varepsilon E_{jk} + \bar{\varepsilon} E_{kj}$ for some $j \neq k$, and some $|\varepsilon| = 1$. Without loss of generality, we can assume $j = 1$. If $B \in {}_{\mathscr{H}}D^{\mathscr{H}}$, such that $A - B \in {}_{\mathscr{H}}D^{\mathscr{H}}$, then clearly $b_{pq} = 0$ unless $p = 1$ or $q = 1$. This implies that for vector $a = 2E_1 + 2\varepsilon E_k \in T$, we have $b = (b_{11}, 2b_{12}, \cdots, 2b_{1n}) \in T$ and $a - b \in T$. By Lemma 1, $b = (b_{11}/2)a$, and $B = (b_{11/2})A$; i.e., A is extremal in ${}_{\mathscr{H}}D^{\mathscr{H}}$.

Before we prove that all extremals of ${}_{\mathscr{H}}D^{\mathscr{H}}$ have the desired form, we give some additional definitions, and a lemma. For $\alpha \subseteqq \{1, \cdots, n\}$ and $A \in \mathscr{H}$, let

$$f_A(\alpha) = \sum_{j \in \alpha} a_{jj} - 2 \sum_{\substack{j,k \in \alpha \\ j<k}} |a_{jk}| ,$$

i.e., $A \in {}_{\mathscr{H}}D^{\mathscr{H}}$ iff $f_A(\alpha) \geqq 0$ for all $\alpha \subseteqq \{1, \cdots, n\}$. When the choice of A is obvious, we will write $f(\alpha)$ for $f_A(\alpha)$.

LEMMA A. *Given α, β, subsets of $\{1, \cdots, n\}$, and $A \in {}_{\mathscr{H}}D^{\mathscr{H}}$, then $f(\alpha \cup \beta) = f(\alpha \cap \beta) = 0$ whenever $f(\alpha) = f(\beta) = 0$,* i.e., $\{\alpha \subseteqq \{1, \cdots, n\} \mid f(\alpha) = 0\}$ *is a sublattice of* $\{1, \cdots, n\}$.

Proof of Lemma A. By hypothesis,

$$\sum_{j \in \alpha} a_{jj} = 2 \sum_{\substack{j,k \in \alpha \\ j<k}} |a_{jk}|, \sum_{j \in \beta} a_{jj} = 2 \sum_{\substack{j,k \in \beta \\ j<k}} |a_{jk}| .$$

Adding, and observing that, whenenver $j, k \in \alpha \cap \beta$, $j < k$, then a_{jk} appears in the right hand side of each equation, we have

$$\sum_{j \in \alpha \cap \beta} a_{jj} + \sum_{j \in \alpha \cup \beta} a_{jj} \leqq 2 \sum_{\substack{j,k \in \alpha \cap \beta \\ j<k}} |a_{jk}| + 2 \sum_{\substack{j,k \in \alpha \cup \beta \\ j<k}} |a_{jk}| .$$

But now, because $A \in {}_{\mathscr{H}}D^{\mathscr{H}}$, we must also have the opposite inequality, so that we have equality, and more; $f(\alpha \cap \beta) = f(\alpha \cup \beta) = 0$.

Proof of Theorem 2 (continued). Suppose A is extremal in ${}_{\mathscr{H}}D^{\mathscr{H}}$. First, we must show that A cannot have more than one nonzero diagonal entry. Suppose A has more than one nonzero diagonal entry; without loss of generality, $a_{11} > 0$. Clearly, in the terminology of Lemma A, if $f(\alpha) > 0$ for all α, $1 \in \alpha$, for sufficiently small $\delta > 0$, $A = \delta E_{11} + (A - \delta E_{11})$, where δE_{11} and $A - \delta E_{11}$ are in ${}_{\mathscr{H}}D^{\mathscr{H}}$ and are linearly independent, a contradiction. If $f(\alpha) = 0$ for at least one $\alpha, 1 \in \alpha$, let μ be the intersection of all α, $1 \in \alpha$ for which $f(\alpha) = 0$. Since $a_{11} > 0$, μ contains at least one index besides 1. For $\delta > 0$, let

$$B_\delta = \delta\Big[2\Big(\sum_{\substack{j \in \mu \\ j \neq 1}} |a_{1j}|\Big)E_{11} + \sum_{\substack{j \in \mu \\ j \neq 1}} (a_{1j}E_{1j} + \overline{a_{1j}}E_{j1})\Big] .$$

Clearly $B_\delta \in {}_{\mathscr{H}}D^{\mathscr{H}}$ for all $\delta > 0$. We shall show that $A - B_\delta \in {}_{\mathscr{H}}D^{\mathscr{H}}$ for sufficiently small $\delta > 0$; it will follow that A, extremal in ${}_{\mathscr{H}}D^{\mathscr{H}}$, cannot have more than one nonzero diagonal entry.

For $\alpha \subseteqq \{1, \cdots, n\}$, $1 \notin \alpha$, $f_{A-B_\delta}(\alpha) = f_A(\alpha) \geqq 0$. For $\alpha \subseteqq \{1, \cdots, n\}$, $1 \in \alpha$, for which $f_A(\alpha) > 0$, clearly $f_{A-B_\delta}(\alpha) > 0$ for sufficiently small $\delta > 0$. Suppose $1 \in \alpha$ and $f_A(\alpha) = 0$. Then, since $\mu \subseteqq \alpha$,

$$\begin{aligned} f_{A-B_\delta}(\alpha) &= \Big(\sum_{j \in \alpha} a_{jj} - 2\delta \sum_{\substack{j \in \mu \\ j \neq 1}} |a_{1j}|\Big) - 2\Big(\sum_{\substack{j,k \in \alpha \\ j<k}} |a_{jk}| - \sum_{\substack{j \in \alpha \\ j \neq 1}} \delta |a_{1j}|\Big) \\ &= 2\delta\Big(\sum_{\substack{j \in \alpha \\ j \neq 1}} |a_{1j}| - \sum_{\substack{j \in \mu \\ j \neq 1}} |a_{1j}|\Big) \geqq 0 . \end{aligned}$$

We have shown that $A - B_\delta \in {}_{\mathscr{H}}D^{\mathscr{H}}$ for sufficiently small $\delta > 0$.

We have now shown that A, extremal in ${}_{\mathscr{H}}D^{\mathscr{H}}$, has one nonzero diagonal entry, say a_{11}. Suppose there were more than one nonzero off-diagonal entry in the first row, one being a_{1j}, $j > 1$. Then for

$$B = \frac{1}{2}(A - a_{1j}E_{1j} - \overline{a_{1j}}E_{j1}),\ C = A - B ,$$

B, $C \in {}_{\mathscr{H}}D^{\mathscr{H}}$, and linearly independent, which is impossible. By Lemma 1, then, A is a positive multiple of $E_{11} - \varepsilon E_{1j} - \bar{\varepsilon} E_{j1}$, where $|\varepsilon| = 1$. This completes the proof of Theorem 2.

5. Faces of cones. In what follows all cones K will be closed and pointed. A *face* of a cone K in a real space X is a subcone F of K satisfying

$$x \in K,\ y - x \in K \quad (\text{i.e., } 0 \leqq x \leqq y),\ y \in F \Longrightarrow x \in F .$$

The set of all faces of K will be denoted by $\mathscr{F}(K)$. If S is a nonempty subset of K, let $\Phi(S)$ denote the smallest face of K containing S. Observe that $S \subseteqq \Phi(S)$, $\Phi(\Phi(S)) = \Phi(S)$, and $\Phi(S) \subseteqq \Phi(T)$ whenever $S \subseteqq T$. We define

$$\begin{aligned} F \vee G &= \Phi(F \cup G) \\ F \wedge G &= F \cap G . \end{aligned}$$

With these definitions of sup and inf, $\mathscr{F}(K)$ is a complete lattice [1].

The results of this section extend some of [1], and will be useful in later sections. If S is a nonempty subset of K, let $\Gamma(S)$ be the smallest subcone of K containing S. Obviously $\Gamma(S) \subseteqq \Phi(S)$.

LEMMA 3. *Let S be a nonempty subset of K. Then* $\Phi(S) = \{x \in K \mid y - x \in K$ *for some* $y \in \Gamma(S)\}$.

REMARK. This is proved in [1] for $S = \{x\}$.

Proof. Let $G = \{x \in K \mid y - x \in K$ for some $y \in \Gamma(S)\}$. We obviously have $\Gamma(S) \subseteqq G$. Also, G is a face of K. To see this, we have first that G is a subcone of K: if $x_1, x_2 \in G$ and $\alpha_1, \alpha_2 > 0$, then there exist $y_1, y_2 \in \Gamma(S)$ such that $y_1 - x_1 \in K$, $y_2 - x_2 \in K$. Now $\alpha_1 x_1 + \alpha_2 x_2 \in K$, $\alpha_1 y_1 + \alpha_2 y_2 \in \Gamma(S)$, and

$$(\alpha_1 y_1 + \alpha_2 y_2) - (\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1(y_1 - x_1) + \alpha_2(y_2 - x_2) \in K ,$$

so that $\alpha_1 x_1 + \alpha_2 x_2 \in G$. To see that G is actually a face of K, consider $x \in G$, $z \in K$, such that $x - z \in K$. There exist $y \in \Gamma(S)$, $y - x \in K$. Now

$$(y - x) + (x - z) = y - z \in K ,$$

so that $z \in G$. Thus G is a face of K, and $G \supseteqq S$, so that $G \supseteqq \Phi(S)$.

To prove the opposite inclusion, let F be any face of K containing S (and also $\Gamma(S)$). Pick $x \in G \subseteqq K$; there exists $y \in \Gamma(S) \subseteqq F$ for which $y - x \in K$. As F is a face, $x \in F$. Thus $G \subseteqq F$, and

$$G \subseteqq \Phi(S) = \bigcap \{F \mid F \text{ a face of } K, F \supseteqq S\} .$$

We remark, as a converse to Lemma 2.9 of [1], that given F, a face of K, then $F = \Phi(x)$ for any x in the relative interior of F (i.e., the interior of F as a subset of the linear span of F).

THEOREM 2. *Let S, T be the nonempty subsets of K. Then*

$$\Phi(S + T) = \Phi(S \cup T) = \Phi(\Phi(S) + \Phi(T)) = \Phi(S) \vee \Phi(T) .$$

REMARK. This extends Proposition 3.2(b) of [1].

Proof. The last equality follows from Proposition 3.2(a) of [1]. We shall complete the proof by showing $\Phi(S + T) \subseteqq \Phi(\Phi(S) + \Phi(T)) \subseteqq \Phi(S \cup T) \subseteqq \Phi(S + T)$. As $S \subseteqq \Phi(S)$ and $T \subseteqq \Phi(T)$, $S + T \subseteqq \Phi(S) + \Phi(T)$; hence $\Phi(S + T) \subseteqq \Phi(\Phi(S) + \Phi(T))$. Also, $\Phi(S) \subseteqq \Phi(S \cup T)$ and $\Phi(T) \subseteqq \Phi(S \cup T)$; since $\Phi(S \cup T)$ is a cone, $\Phi(S) + \Phi(T) \subseteqq \Phi(S \cup T)$, hence $\Phi(\Phi(S) + \Phi(T)) \subseteqq \Phi(S \cup T)$.

Before showing the last inclusion, we first show that $S \subseteqq \Phi(S + T)$. Pick $x \in S$; then for $y \in T$, $x + y \in S + T$ and $(x + y) - x \in K$, implying $x \in \Phi(S + T)$. Now, $S \subseteqq \Phi(S + T)$, and also $T \subseteqq \Phi(S + T)$; thus $S \cup U \subseteqq \Phi(S + T)$ and $\Phi(S \cup T) \subseteqq \Phi(S + T)$.

COROLLARY 1. *Let* $x_1, x_2, \cdots, x_r \in K$. *Then*

$$\begin{aligned}\Phi(x_1 + x_2 + \cdots + x_r) &= \Phi(\{x_1, x_2, \cdots, x_r\}) \\ &= \Phi(x_1) \vee \Phi(x_2) \vee \cdots \vee \Phi(x_r) .\end{aligned}$$

Note that for nonempty subsets S, T of K, $\Phi(S) \vee \Phi(T) = \Phi(S)$ iff $T \subseteqq \Phi(S)$ iff $\Phi(T) \subseteqq \Phi(S)$.

COROLLARY 2. *Suppose* $x_1, x_2, \cdots, x_r$ *satisfy*

$$x_j \notin \Phi(x_1 + \cdots + x_{j-1}),\ j = 2, \cdots, r .$$

Then $\Phi(x_1)$, $\Phi(x_1) \vee \Phi(x_2) = \Phi(x_1 + x_2), \cdots, \Phi(x_1) \vee \Phi(x_2) \vee \cdots \vee \Phi(x_r) = \Phi(x_1 + x_2 + \cdots + x_r)$ *is a strictly increasing sequence of faces of* K.

6. The faces of PSD. A characterization of the faces of PSD has been part of the oral tradition of the subject. Since we shall need this result later, we will state and prove it here. An early version of the proof we shall present was developed informally by Hans Schneider. For brevity, we will denote the lattice $\mathscr{F}$(PSD) of faces of PSD simply by $\mathscr{F}$; the elements of $\mathscr{F}$ are ordered by inclusion.

We will deal with several subsets of $\mathscr{H}$. In each case, we will assume the order induced throughout $\mathscr{H}$ by PSD:

$$A \leqq B \Longleftrightarrow B - A \in \mathrm{PSD} .$$

Also, for $A \in \boldsymbol{C}_{n,n}$, $\mathscr{R}(A)$ and $\mathscr{N}(A)$ denote, respectively, the range and nullspace of A in $\boldsymbol{C}_n$.

Let $\mathscr{P}$ denote the set of projections in $\mathscr{H}$, i.e., $A \in \mathscr{P}$ iff $A \in \mathscr{H}$ and $A^2 = A$. It is well-known (cf. [6], p. 55) that the order on $\mathscr{P}$ induced by PSD is equivalent to

$$A \leqq B \Longleftrightarrow \mathscr{R}(A) \subseteqq \mathscr{R}(B) ,$$

and, since $A, B \in \mathscr{H}$, also to

$$A \leqq B \Longleftrightarrow \mathscr{N}(A) \supseteqq \mathscr{N}(B) .$$

This last formula also provides information on the faces of PSD.

LEMMA 4. *Let* $A, B \in \mathrm{PSD}$. *Then* $B \in \Phi(A)$ *iff* $\mathscr{N}(B) \supseteqq \mathscr{N}(A)$.

Proof. The result is trivial if $A = 0$ or $B = 0$; we shall assume

$A \neq 0$ and $B \neq 0$.

First assume $B \in \Phi(A)$ and choose $\kappa > 0$ such that $\kappa B \leqq A$. If $x \in \mathscr{N}(A)$, then

$$0 \leqq x^*(A - \kappa B)x = -\kappa x^* B x \leqq 0 ,$$

so that $x \in \mathscr{N}(B)$.

Conversely, suppose $\mathscr{N}(B) \supseteqq \mathscr{N}(A)$. Let

$$\Sigma = \{v \in \boldsymbol{C}_n \mid v \in \mathscr{R}(A) = (\mathscr{N}(A))^\perp \quad \text{and} \quad v^*v = 1\} .$$

On the compact set Σ, $v^*Av > 0$, and hence

$$0 \leqq \lambda = \sup\{v^*Bv/v^*Av \mid v \in \Sigma\} < \infty .$$

For $x \in \boldsymbol{C}_n$, we may write $x = u + v$, $u \in \mathscr{N}(A)$, $v \in \mathscr{R}(A)$. Now $x^*Ax = v^*Av$, and $x^*Bx = v^*Bv$, since $u \in \mathscr{N}(A) \subseteqq \mathscr{N}(B)$. As $B \neq 0$, for some $x \in \boldsymbol{C}_n$ we have $x^*Bx = v^*Bv > 0$, so that $\lambda > 0$. Furthermore, for all $x \in \boldsymbol{C}_n$,

$$x^*Ax = v^*Av \geqq \lambda^{-1}v^*Bv = \lambda^{-1}x^*Bx ,$$

so that $\lambda A \geqq B$, i.e., $B \in \Phi(A)$.

COROLLARY 3. *Let $A, B \in$ PSD. Then $\Phi(A) = \Phi(B)$ iff $\mathscr{N}(A) = \mathscr{N}(B)$.*

We shall also need the easily-proved fact that for $A, B \in$ PSD, $\mathscr{N}(A) \cap \mathscr{N}B = \mathscr{N}(A + B)$. Recall that, for $A, B \in \mathscr{P}$, $A \vee B$ and $A \wedge B$ are defined to be the hermitian projections onto, respectively, $\mathscr{R}(A) + \mathscr{R}(B)$ and $\mathscr{R}(A) \cap \mathscr{R}(B)$. We have that

$$\begin{aligned} \mathscr{N}(A \vee B) = (\mathscr{R}(A \vee B))^\perp &= (\mathscr{R}(A) + \mathscr{R}(B))^\perp = \mathscr{R}(A)^\perp \cap \mathscr{R}(B)^\perp \\ &= \mathscr{N}(A) \cap \mathscr{N}(B) = \mathscr{N}(A + B) , \\ \mathscr{N}(A \wedge B) = (\mathscr{R}(A \wedge B))^\perp &= (\mathscr{R}(A) \cap \mathscr{R}(B))^\perp = \mathscr{R}(A)^\perp + \mathscr{R}(B)^\perp \\ &= \mathscr{N}(A) + \mathscr{N}(B) . \end{aligned}$$

In fact, $\mathscr{P}$ is a complete lattice, isomorphic to the lattice of subspaces of $\boldsymbol{C}_n$.

THEOREM 4. *The map on $\mathscr{P}$ given by $A \to \Phi(A)$ is an order-preserving lattice isomorphism of $\mathscr{P}$ onto $\mathscr{F}$.*

Proof. That Φ is $1 - 1$ follows from the corollary to Lemma 4, since for $A, B \in \mathscr{P}$, $A = B$ iff $\mathscr{N}(A) = \mathscr{N}(B)$.

To show that Φ is onto, pick $F \in \mathscr{F}$. Then $F = \Phi(B)$ for some $B \in$ PSD. Let A be the hermitian projection with $\mathscr{N}(A) = \mathscr{N}(B)$; now $\Phi(A) = \Phi(B) = F$. Finally, that Φ and its inverse are order-preserving follows from Lemma 4, since for $A, B \in \mathscr{P}$,

$$\Phi(B) \subseteqq \Phi(A) \Longleftrightarrow B \in \Phi(A) \Longleftrightarrow \mathscr{N}(B) \supseteqq \mathscr{N}(A) \Longleftrightarrow B \leqq A .$$

It follows (cf. [2], p. 24) that $\Phi(A \vee B) = \Phi(A) \vee \Phi(B)$, $\Phi(A \wedge B) = \Phi(A) \wedge \Phi(B)$ for all $A, B \in \mathscr{P}$.

COROLLARY 4. *Let $\mathscr{P}_1$ be the set of hermitian projections of rank* 1. *Then* $\mathscr{E}(\mathrm{PSD}) = \mathscr{P}_1$.

NOTE. This corollary has appeared recently in [8].

REMARK. We are indebted to the referee for pointing out that Corollary 4 has been discovered several times and that a suitable early reference is R. V. Kadison, Isometries of operator algebras, *Ann. of Math.*, 54 (1951), 325–338. In addition, Lemma 4 has appeared in an equivalent form (but with no proof) in O. Taussky, Positive-definite matrices and their role in the study of the characteristic roots of general matrices, *Advances in Math.*, 2 (1968), 175–186.

7. Linear mappings of cones. Let K be a cone in real space X, and τ a linear transformation from X into real space Y. Then τK is a cone in Y, and $x \leqq y$ (i.e., $y - x \in K$) in X implies that $\tau x \leqq \tau y$ (i.e., $\tau y - \tau x \in \tau K$) in τX. If K is closed, so is τK; if K is full, then τK is full in τX.

If τ is one-to-one, then also $\tau x \leqq \tau y$ implies $x \leqq y$. For this case, we give a general result on the lattices of faces of K and τK.

THEOREM 5. *Let K be a cone in X, and $\tau: X \rightarrow Y$ a one-to-one linear transformation. Then if F is a face of K, τF is a face of τK.*

If we define $\tau_\Phi: \mathscr{F}(K) \rightarrow \mathscr{F}(\tau K)$ by $\tau_\Phi(F) = \tau F$, then τ_Φ is an order preserving lattice isomorphism of $\mathscr{F}(K)$ onto $\mathscr{F}(\tau K)$.

Proof. Suppose $F \in \mathscr{F}(K)$. Clearly τF is a subcone of τK. Suppose $0 \leqq u \leqq v$, $v \in \tau F$; then $u = \tau x$ for some $x \in K$, $v = \tau y$ for some $y \in F$, and $0 \leqq x \leqq y$. It follows that $x \in F$, and $u \in \tau F$.

Now τ_Φ is clearly one-to-one and onto; also, both τ_Φ and $(\tau_\Phi)^{-1}$ are order preserving.

We apply our results now to prove that the cone $D^{\mathscr{H}}_{\mathscr{H}}$ in $\mathscr{H}$ is not the image of PSD under any nonsingular linear transformation of $\mathscr{H}$. By Theorem 4, we have that PSD satisfies the Jordan-Dedekind chain condition (cf. [2], p. 5): all maximal chains between $\{0\}$ and K in $\mathscr{F}(K)$ have the same length. In PSD this common length is n. By Theorem 5, any nonsingular linear transformation of $\mathscr{H}$ induces a lattice isomorphism of $\mathscr{F} = \mathscr{F}(\mathrm{PSD})$, and must preserve this chain length. We are done if we exhibit in $\mathscr{F}(D^{\mathscr{H}}_{\mathscr{H}})$ a chain of length greater than n.

Define, for $j = 1, 2, \cdots, n$,

$$A_{2j-1} = 2E_{jj} + \sum_{\substack{k=1\\k\neq j}}^{n} (E_{jk} + E_{kj}),\ A_{2j} = 2E_{jj} - \sum_{\substack{k=1\\k\neq j}}^{n} (E_{jk} + E_{kj}) .$$

We have seen that these matrices are extremal in $D^{\mathscr{H}}_{\mathscr{H}}$. It is easily computed that, for $j = 1, 2, \cdots, n - 1$,

$$\Phi\Big(\sum_{k=1}^{2j} A_k = 4\sum_{k=1}^{j} E_{kk}\Big) = \{C \in D^{\mathscr{H}}_{\mathscr{H}}: c_{pq} = 0,\ p, q = j + 1, \cdots, n\} ,$$

$$\Phi\Big(\sum_{k=1}^{2j-1} A_k\Big) = \Phi\Big(\sum_{k=1}^{2j-2} A_k\Big) + \Gamma(A_{2j-1}) ,$$

and that

$$\Phi\Big(\sum_{k=1}^{2n-1} A_k\Big) = \Phi\Big(\sum_{k=1}^{2n} A_k\Big) = \Phi(I) = D^{\mathscr{H}}_{\mathscr{H}} .$$

Thus the faces

$$\Phi(A_1),\ \Phi(A_1 + A_2) = \Phi(A_1) \vee \Phi(A_2),\ \cdots ,$$
$$\Phi(A_1 + A_2 + \cdots + A_{2n-1}) = \Phi(A_1) \vee \Phi(A_2) \vee \cdots \vee \Phi(A_{2n-1}) = D^{\mathscr{H}}_{\mathscr{H}}$$

form a properly ascending chain of length $2n - 1$.

Our arguments can be seen to apply to the real case: the cone

$$D^{\mathscr{S}}_{\mathscr{S}} = \{A \in \mathscr{S} \mid a_{jj} \geqq 0,\ a_{jj} + a_{kk} \geqq 2|a_{jk}|,\ j, k = 1, \cdots, n,\ k \neq j\}$$

is not the image, under a nonsingular linear transformation of the cone PSD_R of real symmetric positive definite matrices.

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