

Pacific Journal of Mathematics

**BOUNDS FOR DISTORTION IN PSEUDOCONFORMAL
MAPPINGS**

STEFAN BERGMAN

BOUNDS FOR DISTORTION IN PSEUDOCONFORMAL MAPPINGS

STEFAN BERGMAN

1. Introduction. When considering a conformal mapping of a domain, say¹ B^2 , of the z -plane, it is useful to introduce a metric which is invariant with respect to conformal transformations. The line element of this metric is given by

$$(1.1) \quad ds_B^2(z) = K_B(z, \bar{z}) |dz|^2, \quad B \equiv B^2,$$

where $K_B(z, \bar{z})$ is the kernel function of B^2 . (In the case of $[|z| < 1]$ the metric (1.1) is identical with the hyperbolic metric introduced by Poincaré.) In addition to the invariant metric one can also introduce scalar invariants, for instance,

$$(1.2) \quad J_B(z) = -\frac{1}{C_B(z)}, \quad C_B(z) = -\frac{2}{K^3} \begin{vmatrix} K & K_{0\bar{1}} \\ K_{10} & K_{1\bar{1}} \end{vmatrix}, \quad K_{10} = \frac{\partial K}{\partial z},$$

$$K_{0\bar{1}} = \frac{\partial K}{\partial \bar{z}}.$$

($C_B(z)$ is the curvature of the metric (1) at the point z .)

Using the kernel function $K_{\mathfrak{B}}(z, \bar{z})$, $z = (z_1, \dots, z_n)$, one can generalize this approach to the theory of PCT's (pseudoconformal transformations), i.e., to the mappings of $2n$ dimensional domains by n analytic functions of n complex variables (with a nonvanishing Jacobian). It is of interest to obtain bounds for the invariant $J_{\mathfrak{B}}(z)$, see (3.1), depending on quantities which are in a simple way connected with the domain, for instance, the maximum and minimum (euclidean) distances between the point z and the boundary of the domain.

In the present paper we shall determine such bounds in the case of pseudoconformal mapping of the domain $\mathfrak{B} = \mathfrak{B}^4$ of the z_1, z_2 -space by pairs

$$(1.3) \quad w_k = f_k(z_1, z_2), \quad k = 1, 2,$$

of analytic functions of two complex variables (with nonvanishing Jacobian). The generalization of our procedure to the case of pseudoconformal mappings of domains \mathfrak{B}^{2n} by n functions of n complex variables, $3 \leq n < \infty$, is immediate and will not be discussed in the following.

2. The minima $\lambda_{\mathfrak{B}}(z)$. To obtain the desired bound we use

¹ The upper index at a set indicates its dimension.

the minimum values $\lambda_{\mathfrak{B}}^{\dots}(z)$ of the integral

$$(2.1) \quad \int_{\mathfrak{B}} |f(\zeta)|^2 d\omega, \quad \zeta = (\zeta_1, \zeta_2),$$

($d\omega$ = the volume element), under some additional conditions for f at the point $z = (z_1, z_2)$.

As indicated in [1, pp. 183 and 198 ff.], many invariant quantities arising in the theory of PCT's can be expressed in terms of the minima $\lambda_{\mathfrak{B}}^{\dots}(z)$. For instance,

$$(2.2) \quad K_{\mathfrak{B}}(z, \bar{z}) = \frac{1}{\lambda_{\mathfrak{B}}^1(z)}, \quad J_{\mathfrak{B}}(z) = \frac{\lambda_{\mathfrak{B}}^{01}(z)\lambda_{\mathfrak{B}}^{001}(z)}{[\lambda_{\mathfrak{B}}^1(z)]^3}.$$

Here $\lambda_{\mathfrak{B}}^{X_{00}}(z)$ is the minimum of (2.1) under the condition $f(z) = X_{00}$, $z \in \mathfrak{B}$, $\lambda_{\mathfrak{B}}^{X_{00}X_{10}}$ is the minimum under the condition $f(z) = X_{00}$, $(\partial f(z)/\partial z_1) = X_{10}$ and $\lambda_{\mathfrak{B}}^{X_{00}X_{10}X_{01}}(z)$ is the minimum under the condition $f(z) = X_{00}$, $(\partial f(z)/\partial z_1) = X_{10}$, $(\partial f(z)/\partial z_2) = X_{01}$. (K is a *relative* invariant, see (25), p. 180, of [1].)

Using (23b), p. 179 of [1], one obtains the representations for the $\lambda_{\mathfrak{B}}^{\dots}(z)$ in terms of the kernel function $K \equiv K_{\mathfrak{B}}$ and their partial derivatives $K_{10\bar{0}} = (\partial K/\partial z_1)$, $K_{01\bar{0}} = (\partial K/\partial z_2)$, $K_{00\bar{10}} = (\partial K/\partial \bar{z}_1)$, $K_{00\bar{01}} = \partial K/\partial \bar{z}_2$. Obviously it holds

LEMMA 2.1. *Suppose that $z \in \mathfrak{B} \subset \mathfrak{G}$, then*

$$(2.3) \quad \lambda_{\mathfrak{B}}^{\dots}(z) \leq \lambda_{\mathfrak{G}}^{\dots}(z).$$

Here it is assumed that the minima $\lambda_{\mathfrak{B}}^{\dots}(z)$ on both sides of (2.3) are taken under the same conditions.

Choosing for \mathfrak{G} a domain for which the kernel function $K_{\mathfrak{G}}$ is a simple expression of the equation of its boundary (e.g., choosing for \mathfrak{G} a sphere or certain Reinhardt circular domains, see [2, p. 21]), we obtain the desired inequality.

Using the above method, we shall derive in the next section an inequality for the invariant $J_{\mathfrak{B}}(z)$.

3. **Derivation of bounds for $J_{\mathfrak{B}}(z)$.** Let \mathfrak{B} be a connected domain of the (four-dimensional) z_1, z_2 -space, $z_k = x_k + iy_k$, $k = 1, 2$. Let

$$(3.1) \quad J_{\mathfrak{B}}(z, \bar{z}) \equiv J_{\mathfrak{B}} = \frac{K}{T_{11}^- T_{22}^- - |T_{12}^-|^2}, \quad T_{m\bar{n}} = \frac{\partial^2 \log K}{\partial z_m \partial \bar{z}_n},$$

denote the invariant respect to PCT's, see (37a), p. 183 of [1]. Here with K is the kernel function of \mathfrak{B} and $T_{m\bar{n}}$ are the coefficients of the line element

$$(3.2) \quad ds_{\mathfrak{B}}^2 = \sum_{m=1}^z \sum_{n=1}^2 T_{m\bar{n}} dz_m d\bar{z}_n$$

of the metric which is invariant with respect to PCT's, see [1, p. 182 ff.].

THEOREM I. *Suppose that r is the maximum distance of the point $z, z \in \mathfrak{B}$, to the boundary $\partial\mathfrak{B}$, and ρ is the corresponding minimum distance. Then*

$$(3.3) \quad H(\rho, r) \leq J_{\mathfrak{B}}(z) \leq H(r, \rho),$$

$$H(\rho, r) = \frac{2r^6[P(\rho)]^9}{9\rho^6[P(r)]^9\pi^2}, \quad P(\rho) = \rho^2 - z_1\bar{z}_1 - z_2\bar{z}_2.$$

Proof. By (97), p. 198 of [1],

$$(3.4) \quad J_{\mathfrak{B}}(z) = \frac{\lambda_{\mathfrak{B}}^{01}(z)\lambda_{\mathfrak{B}}^{001}(z)}{[\lambda_{\mathfrak{B}}^1(z)]^3}$$

and in accordance with (2.3) for $\mathfrak{Y} \subset \mathfrak{B} \subset \mathfrak{X}$ the inequality

$$(3.5) \quad \frac{\lambda_{\mathfrak{Y}}^{01}(z)\lambda_{\mathfrak{Y}}^{001}(z)}{[\lambda_{\mathfrak{Y}}^1(z)]^3} \leq J_{\mathfrak{B}}(z) \leq \frac{\lambda_{\mathfrak{X}}^{01}(z)\lambda_{\mathfrak{X}}^{001}(z)}{[\lambda_{\mathfrak{X}}^1(z)]^3}$$

holds. If r is the maximum distance of the point z from the boundary $\partial\mathfrak{B}$, and ρ is the minimum distance of z from $\partial\mathfrak{B}$, then one can use for \mathfrak{X} the hypersphere $|z_1|^2 + |z_2|^2 < r^2$ and for \mathfrak{Y} the hypersphere $|z_1|^2 + |z_2|^2 < \rho^2$. By (23b)², p.179 of [1] and by (5a), p. 22 of [2] it holds for the hypersphere $|z_1|^2 + |z_2|^2 < r^2$,

$$(3.6) \quad \lambda_{\mathfrak{X}}^{01}(z)\lambda_{\mathfrak{X}}^{001}(z) = \frac{\pi^4[P(r)]^8}{36r^6},$$

$$(3.7) \quad \lambda_{\mathfrak{X}}^1(z) = \frac{1}{K_{\mathfrak{X}}(z, \bar{z})} = \frac{\pi^2[P(r)]^3}{2r^2}.$$

Analogous formulas hold for $\lambda_{\mathfrak{Y}}^{01}(z)\lambda_{\mathfrak{Y}}^{001}(z)$ and $\lambda_{\mathfrak{Y}}^1(z)$. Consequently (3.3) holds.

4. An application of Theorem I. A domain which admits the group

$$(4.1) \quad z_k^* = z_k e^{i\varphi_k}, \quad 0 \leq \varphi_k \leq 2\pi, \quad k = 1, 2,$$

² In the last term of the expression for $\lambda^{x_{00}x_{10}x_{01}}(t)$ of (23b) are misprints, in the denominator $\left| \frac{K}{K_{1000}} \frac{K_{0000}}{K_{1010}} \right|$ should be replaced by $\left| \frac{K}{K_{1000}} \frac{K_{0010}}{K_{1010}} \right|$. In the nominator of the last term of (23b) the last term K_{0101} in the third row should be replaced by K_{0110} . In the denominator the first term K_{0101} of the third row should be replaced by K_{0100} .

of PCT's onto itself (automorphisms) is called a Reinhardt circular domain (see [3], pp. 33-34).

A domain, say \mathfrak{R} , bounded by the hypersurface

$$(4.2) \quad |z_2| = r(|z_1|),$$

where $y_2 = r(x_1)$ is a convex curve, is a Reinhardt circular domain. Its kernel function is

$$(4.3) \quad K_{\mathfrak{R}}(z, \bar{z}) = B_{00} + B_{10}z_1\bar{z}_1 + B_{01}z_2\bar{z}_2 + B_{02}z_1^2\bar{z}_1^2 + B_{11}z_1\bar{z}_1z_2\bar{z}_2 + \dots,$$

$$(4.4) \quad B_{m^p}^{-1} = \int_{\mathfrak{R}} |z_1|^{2m} |z_2|^{2p} d\omega,$$

$d\omega$ volume element (B_{m^p} are the inverse of moments of \mathfrak{R}), see [2], p. 20 ff.

LEMMA. *The kernel function $K_{\mathfrak{R}}$ and its derivatives at the center 0 of \mathfrak{R} equal*

$$(4.5) \quad \begin{aligned} K_{\mathfrak{R}} &\equiv K = B_{00}, \\ K_{10\bar{0}} &\equiv K_{z_1}(0) = 0, \quad K_{101\bar{0}} \equiv \frac{\partial^2 K}{\partial z_1 \partial \bar{z}_1} = B_{10}, \quad K_{01\bar{0}} = 0, \\ K_{01\bar{01}} &= B_{01}, \dots \end{aligned}$$

Therefore

$$(4.6) \quad J_{\mathfrak{R}}(0) = \frac{K}{\begin{vmatrix} K & K_{00\bar{10}} & K_{00\bar{01}} \\ K_{10\bar{00}} & K_{10\bar{10}} & K_{10\bar{01}} \\ K_{01\bar{00}} & K_{01\bar{10}} & K_{01\bar{01}} \end{vmatrix}} = \frac{B_{00}^4}{\begin{vmatrix} B_{00} & 0 & 0 \\ 0 & B_{10} & 0 \\ 0 & 0 & B_{01} \end{vmatrix}} = \frac{B_{00}^3}{B_{10}B_{01}}$$

(see [1], p. 183, (37a)).

THEOREM II. *Let $\mathfrak{B} = B(\mathfrak{R})$ be a pseudoconformal image of a Reinhardt circular domain \mathfrak{R} , and let r and ρ be the maximum and minimum distances from the boundary, respectively, of the image $z^0 = (z_1^0, z_2^0) = B(0)$ of the center 0 of \mathfrak{R} in \mathfrak{B} . Then*

$$(4.7) \quad H(\rho, r) \leq \frac{B_{00}^3}{B_{10}B_{01}} \leq H(r, \rho).$$

Here B_{m^n} are the inverse moments (introduced in (4.4)) of \mathfrak{R} .

Proof. Since $J_{\mathfrak{R}}$ is invariant and \mathfrak{B} is a pseudoconformal image of \mathfrak{R}

$$(4.8) \quad J_{\mathfrak{A}}(0) = J_{\mathfrak{A}}(z^0) = \frac{B_{00}^3}{B_{10}B_{01}} .$$

By Theorem I it follows that for $J_{\mathfrak{A}}(z^0)$ the inequality (4.7) holds.

Similar results as above can be obtained for other interior distinguished points, for instance, for critical points of $J_{\mathfrak{A}}(z, \bar{z})$.

REMARK. One obtains a generalization of Theorem I by assuming that \mathfrak{S} and \mathfrak{A} are domains $|z_1|^{2/m} + |z_2|^2 < \rho^2$ and $|z_1|^{2/M} + |z_2|^2 < r^2$, respectively. The kernel function for the above domains is given in (5), p. 21, of [2].

REFERENCES

1. S. Bergman, *The kernel function and conformal mapping*, Math. Surveys V, Amer. Math. Soc., 2nd ed., 1970.
2. ———, *Sur les fonctions orthogonales de plusieurs variables complexes avec applications à la théorie des fonctions analytiques*, Mémorial des Sciences Mathématiques **106**, 1947.
3. H. Behnke-Thullen, *Theorie der Funktionen mehrerer komplexer Veränderlichen*, 2nd ed., Springer-Verlag, 1970.
4. B. Epstein, *Orthogonal Families of Analytic Functions*, MacMillan Co., New York 1965.

Received August 30, 1974.

STANFORD UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)
University of California
Los Angeles, California 90024

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT
University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM
Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

Pacific Journal of Mathematics

Vol. 57, No. 1

January, 1975

Keith Roy Allen, <i>Dendritic compactification</i>	1
Daniel D. Anderson, <i>The Krull intersection theorem</i>	11
George Phillip Barker and David Hilding Carlson, <i>Cones of diagonally dominant matrices</i>	15
David Wilmot Barnette, <i>Generalized combinatorial cells and facet splitting</i>	33
Stefan Bergman, <i>Bounds for distortion in pseudoconformal mappings</i>	47
Nguyễn Phương Các, <i>On bounded solutions of a strongly nonlinear elliptic equation</i>	53
Philip Throop Church and James Timourian, <i>Maps with 0-dimensional critical set</i>	59
G. Coquet and J. C. Dupin, <i>Sur les convexes ubiquitaires</i>	67
Kandiah Dayanithy, <i>On perturbation of differential operators</i>	85
Thomas P. Dence, <i>A Lebesgue decomposition for vector valued additive set functions</i>	91
John Riley Durbin, <i>On locally compact wreath products</i>	99
Allan L. Edelson, <i>The converse to a theorem of Conner and Floyd</i>	109
William Alan Feldman and James Franklin Porter, <i>Compact convergence and the order bidual for $C(X)$</i>	113
Ralph S. Freese, <i>Ideal lattices of lattices</i>	125
R. Gow, <i>Groups whose irreducible character degrees are ordered by divisibility</i>	135
David G. Green, <i>The lattice of congruences on an inverse semigroup</i>	141
John William Green, <i>Completion and semicompletion of Moore spaces</i>	153
David James Hallenbeck, <i>Convex hulls and extreme points of families of starlike and close-to-convex mappings</i>	167
Israel (Yitzchak) Nathan Herstein, <i>On a theorem of Brauer-Cartan-Hua type</i>	177
Virgil Dwight House, Jr., <i>Countable products of generalized countably compact spaces</i>	183
John Sollion Hsia, <i>Spinor norms of local integral rotations. I</i>	199
Hugo Junghenn, <i>Almost periodic compactifications of transformation semigroups</i>	207
Shin'ichi Kinoshita, <i>On elementary ideals of projective planes in the 4-sphere and oriented Θ-curves in the 3-sphere</i>	217
Ronald Fred Levy, <i>Showering spaces</i>	223
Geoffrey Mason, <i>Two theorems on groups of characteristic 2-type</i>	233
Cyril Nasim, <i>An inversion formula for Hankel transform</i>	255
W. P. Novinger, <i>Real parts of uniform algebras on the circle</i>	259
T. Parthasarathy and T. E. S. Raghavan, <i>Equilibria of continuous two-person games</i>	265
John Pfaltzgraff and Ted Joe Suffridge, <i>Close-to-starlike holomorphic functions of several variables</i>	271
Esther Portnoy, <i>Developable surfaces in hyperbolic space</i>	281
Maxwell Alexander Rosenlicht, <i>Differential extension fields of exponential type</i>	289
Keith William Schrader and James Lewis Thornburg, <i>Sufficient conditions for the existence of convergent subsequences</i>	301
Joseph M. Weinstein, <i>Reconstructing colored graphs</i>	307