BOUND FOR DISTORTION IN PSEUDOCONFORMAL MAPPINGS

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1. Introduction. When considering a conformal mapping of a domain, say \( B^2 \), of the \( z \)-plane, it is useful to introduce a metric which is invariant with respect to conformal transformations. The line element of this metric is given by

\[
\text{d}s^2(z) = K_B(z, \bar{z})|\text{d}z|^2, \quad B \equiv B^2,
\]

where \( K_B(z, \bar{z}) \) is the kernel function of \( B^2 \). (In the case of \(|z| < 1\) the metric (1.1) is identical with the hyperbolic metric introduced by Poincaré.) In addition to the invariant metric one can also introduce scalar invariants, for instance,

\[
J_B(z) = -\frac{1}{C_B(z)}, \quad C_B(z) = -\frac{2}{K^2} \begin{vmatrix} K & K_{\bar{z}} \\ K_{\bar{z}} & K \end{vmatrix}, \quad K_{\bar{z}} = \frac{\partial K}{\partial \bar{z}}.
\]

\((C_B(z)\) is the curvature of the metric (1) at the point \( z \).)

Using the kernel function \( K_B(z, \bar{z}), z = (z_1, \ldots, z_n) \), one can generalize this approach to the theory of PCT’s (pseudoconformal transformations), i.e., to the mappings of \( 2n \) dimensional domains by \( n \) analytic functions of \( n \) complex variables (with a nonvanishing Jacobian). It is of interest to obtain bounds for the invariant \( J_B(z) \), see (3.1), depending on quantities which are in a simple way connected with the domain, for instance, the maximum and minimum (euclidean) distances between the point \( z \) and the boundary of the domain.

In the present paper we shall determine such bounds in the case of pseudoconformal mapping of the domain \( \mathcal{B} = \mathbb{B}^4 \) of the \( z_i, z_{\bar{z}} \)-space by pairs

\[
w_k = f_k(z_i, z_{\bar{z}}), \quad k = 1, 2,
\]

of analytic functions of two complex variables (with nonvanishing Jacobian). The generalization of our procedure to the case of pseudoconformal mappings of domains \( \mathcal{B}^m \) by \( n \) functions of \( n \) complex variables, \( 3 \leq n < \infty \), is immediate and will not be discussed in the following.

2. The minima \( \lambda_{\mathcal{B}}(z) \). To obtain the desired bound we use

\(1\) The upper index at a set indicates its dimension.
the minimum values \( \lambda_{\alpha}''(z) \) of the integral

\[
(2.1) \quad \int_\mathcal{B} |f(\zeta)|^2 d\omega, \, \zeta = (\zeta_1, \zeta_2),
\]

\( (d\omega = \text{the volume element}), \) under some additional conditions for \( f \) at the point \( z = (z_1, z_2) \).

As indicated in [1, pp. 183 and 198 ff.], many invariant quantities arising in the theory of PCT's can be expressed in terms of the minima \( \lambda_{\alpha}''(z) \). For instance,

\[
(2.2) \quad K_\alpha(z, \bar{z}) = \frac{1}{\lambda_\alpha(z)}, \quad J_\alpha(z) = \frac{\lambda_\alpha^{01}(z)\lambda_\alpha^{00}(z)}{[\lambda_\alpha^{11}(z)]^3}.
\]

Here \( \lambda_\alpha^{00}(z) \) is the minimum of (2.1) under the condition \( f(z) = X_{00}, \) \( z \in \mathcal{B}, \lambda_\alpha^{X_{00}X_{10}} \) is the minimum under the condition \( f(z) = X_{10}, (\partial f(z)/\partial z_1) = X_{10} \) and \( \lambda_\alpha^{X_{00}X_{10}X_{01}}(z) \) is the minimum under the condition \( f(z) = X_{00}, (\partial f(z)/\partial z_1) = X_{10}, (\partial f(z)/\partial z_2) = X_{01}. \) (\( K \) is a relative invariant, see (25), p. 180, of [1].)

Using (23b), p. 179 of [1], one obtains the representations for the \( \lambda_{\alpha}''(z) \) in terms of the kernel function \( K \equiv K_\alpha \) and their partial derivatives \( K_{000} = (\partial K/\partial z_1), K_{010} = (\partial K/\partial z_2), K_{001} = (\partial K/\partial z_1), K_{000} = (\partial K/\partial z_2). \) Obviously it holds

**Lemma 2.1.** Suppose that \( z \in \mathcal{B} \subset \omega, \) then

\[
(2.3) \quad \lambda_{\alpha}''(z) \leq \lambda_{\alpha}''(z).
\]

Here it is assumed that the minima \( \lambda_{\alpha}''(z) \) on both sides of (2.3) are taken under the same conditions.

Choosing for \( \omega \) a domain for which the kernel function \( K_\alpha \) is a simple expression of the equation of its boundary (e.g., choosing for \( \omega \) a sphere or certain Reinhardt circular domains, see [2, p. 21]), we obtain the desired inequality.

Using the above method, we shall derive in the next section an inequality for the invariant \( J_\alpha(z) \).

3. Derivation of bounds for \( J_\alpha(z) \). Let \( \mathcal{B} \) be a connected domain of the (four-dimensional) \( z_1, z_2 \)-space, \( z_k = x_k + iy_k, k = 1, 2. \) Let

\[
(3.1) \quad J_\alpha(z, \bar{z}) = J_\alpha = \frac{K}{T_{11} T_{22} - |T_{12}|^2}, \quad T_{mn} = \frac{\partial^2 \log K}{\partial z_m \partial \bar{z}_n},
\]

denote the invariant respect to PCT's, see (37a), p. 183 of [1]. Here with \( K \) is the kernel function of \( \mathcal{B} \) and \( T_{mn} \) are the coefficients of the line element
of the metric which is invariant with respect to PCT's, see [1, p. 182 ff.].

**Theorem I.** Suppose that $r$ is the maximum distance of the point $z, z \in \mathcal{B}$, to the boundary $\partial \mathcal{B}$, and $\rho$ is the corresponding minimum distance. Then

(3.3) \[
H(\rho, r) \leq J_\varepsilon(z) \leq H(r, \rho),
\]

Proof. By (97), p. 198 of [1],

(3.4) \[
J_\varepsilon(z) = \frac{\lambda_3^{01}(z) \lambda_3^{01}(\varepsilon)}{[\lambda_3^{01}(z)]^3}
\]

and in accordance with (2.3) for $\mathfrak{A} \subset \mathcal{B} \subset \mathfrak{A}$ the inequality

(3.5) \[
\lambda_3^{01}(z) \lambda_3^{01}(\varepsilon) \leq J_\varepsilon(z) \leq \frac{\lambda_3^{01}(z) \lambda_3^{01}(\varepsilon)}{[\lambda_3^{01}(z)]^3}
\]

holds. If $r$ is the maximum distance of the point $z$ from the boundary $\partial \mathcal{B}$, and $\rho$ is the minimum distance of $z$ from $\partial \mathcal{B}$, then one can use for $\mathfrak{A}$ the hypersphere $|z_1|^2 + |z_2|^2 < r^2$ and for $\mathfrak{A}$ the hypersphere $|z_1|^2 + |z_2|^2 < \rho^2$. By (23b), p. 179 of [1] and by (5a), p. 22 of [2] it holds for the hypersphere $|z_1|^2 + |z_2|^2 < r^2$,

(3.6) \[
\lambda_3^{01}(z) \lambda_3^{01}(\varepsilon) = \frac{\pi^4[P(r)]^8}{36r^8},
\]

(3.7) \[
\lambda_3^{01}(z) = \frac{1}{K_\varepsilon(z, \varepsilon)} = \frac{\pi^4[P(r)]^8}{2r^8}.
\]

Analogous formulas hold for $\lambda_3^{01}(z) \lambda_3^{01}(\varepsilon)$ and $\lambda_3^{01}(z)$. Consequently (3.3) holds.

4. An application of Theorem I. A domain which admits the group

(4.1) \[
z_k^* = z_\varepsilon e^{i\varphi_k}, \quad 0 \leq \varphi_k \leq 2\pi, \quad k = 1, 2,
\]

In the last term of the expression for $\lambda^{2001}_{10} X_{001}$ of (23b) are misprints, in the denominator $\frac{K_{1000}}{K_{1010}}$ should be replaced by $\frac{K_{1000}}{K_{1010}}$. In the nominator of the last term of (23b) the last term $K_{1010}$ in the third row should be replaced by $K_{0110}$. In the denominator the first term $K_{1010}$ of the third row should be replaced by $K_{0110}$. 

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\[ ^5 \text{In the last term of the expression for } \lambda^{2001}_{10} X_{001}(t) \text{ of (23b) are misprints, in the denominator } \frac{K_{1000}}{K_{1010}} \text{ should be replaced by } \frac{K_{1000}}{K_{1010}}. \text{ In the nominator of the last term of (23b) the last term } K_{1010} \text{ in the third row should be replaced by } K_{0110}. \text{ In the denominator the first term } K_{1010} \text{ of the third row should be replaced by } K_{0110}. \]
of PCT's onto itself (automorphisms) is called a Reinhardt circular
domain (see [3], pp. 33–34).

A domain, say \( \mathcal{R} \), bounded by the hypersurface

\[
|z_2| = r(|z_1|),
\]

where \( y_z = r(x_z) \) is a convex curve, is a Reinhardt circular domain. Its kernel function is

\[
K_n(z, \bar{z}) = B_{00} + B_{10}z_1 \bar{z}_1 + B_{01}z_2 \bar{z}_2 + B_{20}z_1^2 \bar{z}_1^2 + B_{11}z_1z_2 \bar{z}_1 \bar{z}_2 + \cdots,
\]

where \( y_2 = r(x_2) \) is a convex curve, is a Reinhardt circular domain.

Its kernel function is

\[
K_n(z, \bar{z}) = B_{00} + B_{10}z_1 \bar{z}_1 + B_{01}z_2 \bar{z}_2 + B_{20}z_1^2 \bar{z}_1^2 + B_{11}z_1z_2 \bar{z}_1 \bar{z}_2 + \cdots,
\]

where \( y_z = r(x_z) \) is a convex curve, is a Reinhardt circular domain.

Its kernel function is

\[
B_{m,p}^{-1} = \int_{\mathcal{R}} |z_1|^{2m} |z_2|^{2p} d\omega,
\]

\( d\omega \) volume element \( (B_{m,p} \) are the inverse of moments of \( \mathcal{R} \)), see [2], p. 20 ff.

**Lemma.** The kernel function \( K_n \) and its derivatives at the center 0 of \( \mathcal{R} \) equal

\[
K_n \equiv K = B_{00},
\]

\[
K_{1000} \equiv K_{z_1}(0) = 0, \quad K_{0100} = \frac{\partial^2 K}{\partial z_2 \partial \bar{z}_1} = B_{10}, \quad K_{0010} = 0,
\]

\[
K_{0001} = B_{01}, \quad \ldots.
\]

Therefore

\[
J_n(0) = \begin{vmatrix} K & K_{0010} & K_{0001} \\ K_{1000} & K_{1010} & K_{0101} \\ K_{0100} & K_{0110} & K_{0011} \end{vmatrix} = \begin{vmatrix} B_{00}^* & 0 & 0 \\ 0 & B_{10} & 0 \\ 0 & 0 & B_{01} \end{vmatrix} = \frac{B_{00}^*}{B_{10}B_{01}}
\]

(see [1], p. 183, (37a)).

**Theorem II.** Let \( \mathcal{B} = B(\mathcal{R}) \) be a pseudoconformal image of a
Reinhardt circular domain \( \mathcal{R} \), and let \( r \) and \( \rho \) be the maximum and
minimum distances from the boundary, respectively, of the image
\( z^* = (z_1^*, z_2^*) = B(0) \) of the center 0 of \( \mathcal{R} \) in \( \mathcal{B} \). Then

\[
H(\rho, r) = \frac{B_{10}^*}{B_{10}B_{01}} \leq H(r, \rho).
\]

*Proof.* Since \( J_n \) is invariant and \( \mathcal{B} \) is a pseudoconformal image
of \( \mathcal{R} \)
By Theorem I it follows that for $J_s(z^0)$ the inequality (4.7) holds. Similar results as above can be obtained for other interior distinguished points, for instance, for critical points of $J_s(z, \overline{z})$.

**Remark.** One obtains a generalization of Theorem I by assuming that $\mathcal{G}$ and $\mathbb{H}$ are domains $|z_1|^2/m + |z_2|^2 < \rho^2$ and $|z_1|^2/m + |z_2|^2 < r^4$, respectively. The kernel function for the above domains is given in (5), p. 21, of [2].

**References**


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