BOUNDS FOR DISTORTION IN PSEUDOCONFORMAL MAPPINGS

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1. Introduction. When considering a conformal mapping of a domain, say $B^3$, of the $z$-plane, it is useful to introduce a metric which is invariant with respect to conformal transformations. The line element of this metric is given by

$$ds^2_B(z) = K_B(z, \bar{z})|dz|^2, \quad B \equiv B^3,$$

where $K_B(z, \bar{z})$ is the kernel function of $B^3$. (In the case of $|z| < 1$ the metric (1.1) is identical with the hyperbolic metric introduced by Poincaré.) In addition to the invariant metric one can also introduce scalar invariants, for instance,

$$J_B(z) = -\frac{1}{C_B(z)}, \quad C_B(z) = -\frac{2}{K^2} \left[ \frac{K}{K_{10}} \frac{K_{01}}{K_{11}} \right], \quad K_{10} = \frac{\partial K}{\partial z},$$

$$K_{01} = \frac{\partial K}{\partial \bar{z}}.$$

($C_B(z)$ is the curvature of the metric (1) at the point $z$.)

Using the kernel function $K_B(z, \bar{z}), z = (z_1, \cdots, z_n)$, one can generalize this approach to the theory of PCT's (pseudoconformal transformations), i.e., to the mappings of $2n$ dimensional domains by $n$ analytic functions of $n$ complex variables (with a nonvanishing Jacobian). It is of interest to obtain bounds for the invariant $J_B(z)$, see (3.1), depending on quantities which are in a simple way connected with the domain, for instance, the maximum and minimum (euclidean) distances between the point $z$ and the boundary of the domain.

In the present paper we shall determine such bounds in the case of pseudoconformal mapping of the domain $\mathcal{B} = \mathcal{B}^4$ of the $z_1, z_2$-space by pairs

$$w_k = f_k(z_1, z_2), \quad k = 1, 2,$$

of analytic functions of two complex variables (with nonvanishing Jacobian). The generalization of our procedure to the case of pseudoconformal mappings of domains $\mathcal{B}^{2n}$ by $n$ functions of $n$ complex variables, $3 \leq n < \infty$, is immediate and will not be discussed in the following.

2. The minima $\lambda_{j\omega}(z)$. To obtain the desired bound we use

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1 The upper index at a set indicates its dimension.
the minimum values \( \lambda_{\nu}(z) \) of the integral

\[
(2.1) \quad \int_{\mathcal{D}} |f(\zeta)|^2 d\omega, \quad \zeta = (\zeta_1, \zeta_2),
\]

\( (d\omega = \text{the volume element}) \), under some additional conditions for \( f \) at
the point \( z = (z_1, z_2) \).

As indicated in [1, pp. 183 and 198 ff.], many invariant quantities arising in the theory of PCT’s can be expressed in terms of the minima \( \lambda_{\nu}(z) \). For instance,

\[
(2.2) \quad K_\mathcal{D}(z, \bar{z}) = \frac{1}{\lambda_{\mathcal{D}}(z)}, \quad J_\mathcal{D}(z) = \frac{\lambda_{\mathcal{D}}^0(z)\lambda_{\mathcal{D}}^{00}(z)}{[\lambda_{\mathcal{D}}(z)]^3}.
\]

Here \( \lambda_{\mathcal{D}}^0(z) \) is the minimum of (2.1) under the condition \( f(z) = X_{00}, \)
\( z \in \mathbb{D} \), \( \lambda_{\mathcal{D}}^{00}(z) \) is the minimum under the condition \( f(z) = X_{00}, \) \( \frac{\partial f(z)}{\partial \zeta_1} = X_{10} \) and \( \lambda_{\mathcal{D}}^{X_{00}X_{00}}(z) \) is the minimum under the condition \( f(z) = X_{00}, \)
\( \frac{\partial f(z)}{\partial \zeta_1} = X_{10}, \) \( \frac{\partial f(z)}{\partial \zeta_2} = X_{01}. \) \( (K \text{ is a relative invariant, see (25), p. 180, of [1].}) \)

Using (23b), p. 179 of [1], one obtains the representations for the \( \lambda_{\nu}(z) \) in terms of the kernel function \( K = K_\mathcal{D} \) and their partial derivatives \( K_{1000} = (\partial K/\partial z_1), K_{0100} = (\partial K/\partial z_1), K_{0010} = (\partial K/\partial z_2), K_{0001} = \delta K/\delta \bar{z}_2. \) Obviously it holds

**Lemma 2.1.** Suppose that \( z \in \mathbb{D} \subset \mathbb{D} \), then

\[
(2.3) \quad \lambda_{\nu}(z) \leq \lambda_{\mathcal{D}}(z).
\]

Here it is assumed that the minima \( \lambda_{\nu}(z) \) on both sides of (2.3) are taken under the same conditions.

Choosing for \( \mathbb{D} \) a domain for which the kernel function \( K_\mathcal{D} \) is a simple expression of
the equation of its boundary (e.g., choosing for \( \mathbb{D} \) a sphere or certain Reinhardt circular domains, see [2, p. 21]), we obtain the desired inequality.

Using the above method, we shall derive in the next section an inequality for the invariant \( J_\mathcal{D}(z) \).

3. Derivation of bounds for \( J_\mathcal{D}(z) \). Let \( \mathbb{D} \) be a connected domain of the (four-dimensional) \( z, z_\bar{z} \)-space, \( z_k = x_k + iy_k, k = 1, 2. \) Let

\[
(3.1) \quad J_\mathcal{D}(z, \bar{z}) = J_\mathcal{D} = \frac{K}{T_{11}T_{22} - |T_{12}|^2}, \quad T_{mn} = \frac{\partial^2 \log K}{\partial z_m \partial \bar{z}_n},
\]

denote the invariant respect to PCT’s, see (37a), p. 183 of [1]. Here with \( K \) is the kernel function of \( \mathbb{D} \) and \( T_{mn} \) are the coefficients of the line element
of the metric which is invariant with respect to PCT's, see [1, p. 182 ff.].

**Theorem I.** Suppose that \( r \) is the maximum distance of the point \( z, z \in \mathbb{B} \), to the boundary \( \partial \mathbb{B} \), and \( \rho \) is the corresponding minimum distance. Then

\[
H(\rho, r) \leq J_\delta(z) \leq H(\rho, \rho),
\]

where

\[
H(\rho, r) = \frac{2r^{|P(\rho)|^2}}{9\rho^{|P(r)|^2} \pi^2} , \quad P(\rho) = \rho^2 - z_1\bar{z}_1 - z_2\bar{z}_2.
\]

**Proof.** By (97), p. 198 of [1],

\[
J_\delta(z) = \frac{\lambda_\delta^0(z)\lambda_\delta^0(z)}{[\lambda_\delta^1(z)]^3}
\]

and in accordance with (2.3) for \( \mathfrak{B} \subset \mathbb{B} \subset \mathfrak{M} \) the inequality

\[
\frac{\lambda_\delta^0(z)\lambda_\delta^0(z)}{[\lambda_\delta^1(z)]^3} \leq J_\delta(z) \leq \frac{\lambda_\delta^0(z)\lambda_\delta^0(z)}{[\lambda_\delta^3(z)]^3}
\]

holds. If \( r \) is the maximum distance of the point \( z \) from the boundary \( \partial \mathbb{B} \), and \( \rho \) is the minimum distance of \( z \) from \( \partial \mathbb{B} \), then one can use for \( \mathfrak{M} \) the hypersphere \( |z_1|^2 + |z_2|^2 < r^2 \) and for \( \mathfrak{B} \) the hypersphere \( |z_1|^2 + |z_2|^2 < \rho^2 \). By (23b), p. 179 of [1] and by (5a), p. 22 of [2] it holds for the hypersphere \( |z_1|^2 + |z_2|^2 < r^2 \),

\[
\lambda_\delta^0(z)\lambda_\delta^0(z) = \frac{\pi^{|P(\rho)|^2}}{36r^6},
\]

\[
\lambda_\delta^1(z) = \frac{1}{K_\delta(z, \bar{z})} = \frac{\pi^{|P(\rho)|^2}}{2r^2}.
\]

Analogous formulas hold for \( \lambda_\delta^0(z)\lambda_\delta^0(z) \) and \( \lambda_\delta^3(z) \). Consequently (3.3) holds.

4. An application of Theorem I. A domain which admits the group

\[
z_k^\delta = z_k e^{i\varphi_k}, \quad 0 \leq \varphi_k \leq 2\pi, \quad k = 1, 2,
\]

\[\text{In the last term of the expression for } K_{00}^{10} K_{01}^{10} (t) \text{ of (23b) are misprints, in the denominator } K_{1000}^{0005} \text{ should be replaced by } K_{1000}^{0010} K_{0010}^{0010}. \] In the nominator of the last term of (23b) the last term \( K_{0101} \) in the third row should be replaced by \( K_{0110} \). In the denominator the first term \( K_{0101} \) of the third row should be replaced by \( K_{0105} \).
of PCT's onto itself (automorphisms) is called a Reinhardt circular domain (see [3], pp. 33-34).

A domain, say $\mathcal{R}$, bounded by the hypersurface

\begin{equation}
|z_2| = r(|z_2|),
\end{equation}

where $y_2 = r(x_2)$ is a convex curve, is a Reinhardt circular domain. Its kernel function is

\begin{equation}
K_\mathcal{R}(z, \bar{z}) = B_{00} + B_{10}z_1\bar{z}_1 + B_{01}z_2\bar{z}_2 + B_{02}z_1^2\bar{z}_2^2 + B_{11}z_1z_2\bar{z}_1\bar{z}_2 + \cdots,
\end{equation}

\begin{equation}
B_{m,p}^{-1} = \int_\mathcal{R} |z_1|^{2m} |z_2|^{2p} d\omega,
\end{equation}

d$\omega$ volume element ($B_{m,p}$ are the inverse of moments of $\mathcal{R}$), see [2], p. 20 ff.

**Lemma.** The kernel function $K_\mathcal{R}$ and its derivatives at the center $0$ of $\mathcal{R}$ equal

\begin{equation}
K_\mathcal{R} = K = B_{00},
\end{equation}

\begin{equation}
K_{0000} = K_{11}(0) = 0, \quad K_{1010} = \frac{\partial^2 K}{\partial z_1 \partial \bar{z}_1} = B_{10}, \quad K_{0100} = 0,
\end{equation}

\begin{equation}
K_{0110} = B_{01}, \cdots.
\end{equation}

Therefore

\begin{equation}
J_\mathcal{R}(0) = \begin{pmatrix} K & K_{0010} & K_{0001} \\ K_{1000} & K_{1010} & K_{1001} \\ K_{0100} & K_{0110} & K_{0101} \end{pmatrix} = \begin{pmatrix} B_{00}^{-1} & 0 & 0 \\ B_{00} & 0 & B_{10} \\ 0 & B_{10} & 0 \end{pmatrix} = \begin{pmatrix} B_{00}^{-1} \\ B_{00} \end{pmatrix}.
\end{equation}

(see [1], p. 183, (37a)).

**Theorem II.** Let $\mathcal{B} = B(\mathcal{R})$ be a pseudoconformal image of a Reinhardt circular domain $\mathcal{R}$, and let $r$ and $\rho$ be the maximum and minimum distances from the boundary, respectively, of the image $\mathcal{B}(0)$ of the center $0$ of $\mathcal{R}$ in $\mathcal{B}$. Then

\begin{equation}
H(\rho, r) \leq \frac{B_{00}^3}{B_{10}B_{01}} \leq H(r, \rho).
\end{equation}

Here $B_{m,n}$ are the inverse moments (introduced in (4.4)) of $\mathcal{R}$.

**Proof.** Since $J_\mathcal{R}$ is invariant and $\mathcal{B}$ is a pseudoconformal image of $\mathcal{R}$.
(4.8) \[ J_\alpha(0) = J_\alpha(z^0) = \frac{B_{00}}{B_{10}B_{01}}. \]

By Theorem I it follows that for \( J_\alpha(z^0) \) the inequality (4.7) holds.

Similar results as above can be obtained for other interior distinguished points, for instance, for critical points of \( J_\alpha(z, \bar{z}) \).

**REMARK.** One obtains a generalization of Theorem I by assuming that \( \mathcal{F} \) and \( \mathcal{W} \) are domains \( |z_1|^{2/m} + |z_2|^2 < \rho^2 \) and \( |z_1|^{2/m} + |z_2|^2 < r^2 \), respectively. The kernel function for the above domains is given in (5), p. 21, of [2].

**REFERENCES**


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**Stanford University**
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