Pacific Journal of Mathematics

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Vol. 57, No. 1 January 1975

PACIFIC JOURNAL OF MATHEMATICS
Vol. 57, No. 1, 1975

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**I. Introduction. Consider the Dirichlet problem for a bounded domain $G \subset R^n (n \geqq 2)$ having smooth boundary $\partial G$:**

$$(1) \qquad \begin{aligned} &\mathscr{A}u + p(u) = -D_i f_i + f \\ &z|_{\partial G} = 0 , \end{aligned}$$

**where $\mathscr{A}$ is a second order differential operator of Leray-Lions type mapping a real Sobolev space $W_0^{1,q}(G)(1 < q < \infty)$ into its dual; $f, f_i (i = 1, \cdots, n)$ are given functions. We have used the notation $D_i$ for the derivative in the distribution sense $\partial/\partial x_i$ and the convention that if an index is repeated then summation over that index from 1 to $n$ is implied. We shall assume that the real function $p(t)$ is continuous and satisfies the condition**

$$(2) \qquad p(t)t \geqq 0 \qquad \forall t \in R ,$$

**but otherwise $(p)t$ is not subject to any growth condition.**

**In this paper we discuss the existence of a solution of equation (1) in $W_0^{1,q}(G) \cap L^\infty(G)$.**

Many papers appearing recently have studied equations and inequations involving strongly nonlinear elliptic operators of the type (1). For equations we mention among others [1], [2], [7]; in [1] and [2] the existence of a solution in $W_0^{m,q}(G)$ when the operator $\mathscr{A}$ has arbitrary order $2m$ is established under the additional hypothesis:

Given $\varepsilon > 0$, there exists $K_\varepsilon > 0$ such that

$$(3) \qquad p(t)s \leqq \varepsilon p(s)s + K_\varepsilon[1 + p(t)t] \qquad \forall t, s \in R$$

[3], [9] among others deal with strongly nonlinear inequations in $W^{m,q}(G)$.

For an operator $\mathscr{A}$ of second order, [4] proves the existence of a solution in $W_0^{1,q}(G)$ under the sole condition (2).

Finally let us mention that the existence of bounded solution of other strongly nonlinear equations and inequations has been discussed in [8]. However it seems to us that the technique of this paper is different from ours; it consists of multiplying the equation with a nonlinear expression of $u$; it also seems that our method when applied to some concrete cases yields different results in the sense that we only require the functions in the right hand side of (1) to be in $L^r(G)$ for some $r > 1$ and not in $L^\infty(G)$ as in [8].

**II. Main result.** The operator $\mathscr{A}$ is assumed to be of the form

$$\mathscr{A}_u = \frac{\partial}{\partial X_i} a_i(x, u, \nabla u) + a_0(x, u, \nabla u) \tag{4}$$

where $\nabla u = \operatorname{grad} u$ and the functions $a_i$ satisfy the following conditions:

(i) Each $a_i$ $(i = 0, 1, \cdots, n)$ is a function defined on $G \times R \times R^n$ and of Caratheodory type: $a_i(x, \eta, \zeta)$ is measurable in $x$ for fixed $\eta \in R$, $\zeta \in R^n$ and is continuous in $(\eta, \zeta) \in R \times R^n$ for almost all fixed $x \in G$. Moreover there exist a constant $c$, a number $q$, $1 < q < \infty$, a function $k(x) \geqq 0$ a.e. on $G$, $k(x) \in L^{q^*}(G)(1/q + 1/q^* = 1)$, such that

$$|a_i(x, \eta, \zeta)| \leqq c(k(x) + |\eta|^{q-1} + |\zeta|^{q-1}) \tag{5}$$

for $i = 0, 1, \cdots, n$; a.a. $x \in G$ and $\forall (\eta, \zeta) \in R \times R^n$.

(ii) For a.a. $x \in G$,

$$[a_i(x, \eta, \zeta) - a_i(x, \eta, \zeta')](\zeta - \zeta') > 0 \quad \text{if} \quad \zeta \neq \zeta' \tag{6}$$

(iii) For a.a. $x \in G$ and bounded $\eta$,

$$a_i(x, \eta, \zeta)\zeta_i/(|\zeta| + |\zeta|^{q-1}) \longrightarrow \infty \quad \text{as} \quad |\zeta| \longrightarrow \infty \tag{7}$$

Condition (5) implies that the semilinear form

$$\mathscr{A}(u, v) = \int_G [a_i(x, u, \varDelta u)D_i v + a_0(x, u, \nabla u)v]dx$$

is defined for all $u, v \in W_0^{1,q}(G)$ and there is $\mathscr{A}u \in W^{-1,q^*}(G)$ such that ($\langle\cdot,\cdot\rangle$ denotes the pairing between $W_0^{1,q}(G)$ and $W^{-1,q^*}(G)$)

$$\mathscr{A}(u, v) = \langle \mathscr{A}u, v\rangle \quad \forall v \in W_0^{1,q}(G) .$$

It is known that the mapping $\mathscr{A}: W_0^{1,q}(G) \to W^{-1,q^*}(G)$ is continuous and bounded ([6], Chapter 2, Section 2.6). Moreover, under the hypotheses (6) and (7), $\mathscr{A}$ is pseudo-monotone and therefore it is of type $(M)$: *If $u_j \rightharpoonup u$ in $W_0^{1,q}(G)$, $\mathscr{A}u_j \rightharpoonup \chi$ in $W^{-1,q^*}(G)$ and* limsup $\langle \mathscr{A}u_j, u_j - u\rangle \leqq 0$ *then $\mathscr{A}u = \chi$.* (Here and in the sequel "$\rightharpoonup$" and "$\rightarrow$" denote weak and strong convergence respectively.) We prove

THEOREM. *Suppose that the differential operator $\mathscr{A}$ of the form (4) satisfies conditions (5), (6), (7) and the coercivity condition:*

*There exists a constant $\nu > 0$ such that for all $v \in W_0^{1,q}(G)$*

$$\mathscr{A}(v, v) \geqq \nu \|v\|_{W_0^{1,q}(G)} . \tag{8}$$

*Suppose also that the continuous function $p(.)$ satisfies the condition $p(t)t \geqq 0$, $\forall t \in R$. If $f_i \in L^s(G)$ with $s \geqq q^*$, $s > n/(q-1)$, $i = 1, \cdots,$*

*$n$; and $f(x)$ and the function $k(x)$ in* (5) *both belong to $L^r(G)$ with $r \geqq q^*$, $r > n/q$, then the Dirichlet problem* (1) *has a solution $u \in L^\infty(G) \cap W_0^{1,q}(G)$ in the sense that*

$$\mathscr{A}(u, v) + \int_G p(u)v\,dx = \int_G (f_i D_i v + fv)dx \quad \forall v \in W_0^{1,q}(G) .$$

*Proof.* We note that if $q > n$ then by the Sobolev imbedding theorem, any function in $W_0^{1,q}(G)$ is continuous on $G$ and hence bounded. Consequently, in this case it suffices to prove the existence of a solution in $W_0^{1,q}(G)$. This can be done by partially repeating and slightly modifying the proof given below for the case $q \leqq n$. We also note that if $q > n$ then $q^* > n/(q-1)$ so that the theorem holds if $f_i, f, k(x) \in L^{q^*}(G) (i = 1, \cdots, n)$.

So let us suppose that $q \leqq n$. For each positive integer $N$ we denote by $p_N(t)$ the function

$$\begin{aligned} p_N(t) &= p(t) \quad \text{if} \quad |p(t)| \leqq N , \\ &= N \quad \text{if} \quad p(t) > N , \\ &= -N \quad \text{if} \quad p(t) < -N . \end{aligned}$$

The mapping $T_N: u \longrightarrow \mathscr{A}u + p_N(u)$ from $W_0^{1,q}(G)$ into $W^{-1,q^*}(G)$ is of type $(M)$. In fact, consider a sequence $u_j \rightharpoonup u$ in $W_0^{1,q}(G)$ with $T_N u_j \rightharpoonup \chi$ in $W^{-1,q^*}(G)$ and $\limsup_j \langle T_N u_j, u_j - u \rangle \leqq 0$. By the Sobolev imbedding theorem, we can assume without loss of generality that $u_j(x) \longrightarrow u(x)$ for a.a. $x \in G$. Condition (2) on $p(t)$ and Fatou's lemma then give

$$\liminf_j \int_G p_N(u_j)u_j\,dx \geqq \int_G p_N(u)u\,dx .$$

On the other hand, Lebesgue's dominated convergence theorem gives

$$\lim_j \int_G p_N(u_j)v\,dx = \int_G p_N(u)v\,dx \quad \forall v \in W_0^{1,q}(G) .$$

We then deduce that $p_N(u_j) \rightharpoonup p_N(u)$ in $W^{-1,q^*}(G)$, hence $\mathscr{A}u_j \rightharpoonup \chi - p_N(u)$ as $j \longrightarrow \infty$ and

$$\limsup_j \langle \mathscr{A}u_j, u_j - u \rangle \leqq 0 .$$

Since $\mathscr{A}$ has property $(M)$, it follows that $\mathscr{A}u = \chi - p_N(u)$ i.e. $T_N u = \chi$. It is clear that $T_N$ is also bounded and hemicontinuous. The coercivity of $\mathscr{A}$ implies that of $T_N$. Therefore (cf. e.g. [6], Remark 2.1, page 173) there exists $u_N \in W_0^{1,q}(G)$ such that for all $v \in W_0^{1,q}(G)$

$$(9) \qquad \langle \mathscr{A}u_N + p_N(u_N), v \rangle = \langle -D_i f_i + f, v \rangle$$

*We now find a bound for the $L^\infty$-norm of $u_N$.*

Taking $v = u_N$ in (9) and bearing in mind that $p_N(t)t \geqq 0$, we obtain from the coercivity condition (8) that

$$\|u_N\|_{W_0^{1,q}(G)} < C \tag{10}$$

here and in the sequel $C$ denotes various constants independent of $N$. Next we take in (9)

$$v(x) = \max\{u_N(x) - h, 0\}$$

where $h \geqq 1$. If we denote by $A_h$ the set $\{x \mid x \in G, u_N(x) > h\}$ then

$$\begin{aligned}\int_{A_h}&[a_i(x, u_N, \nabla u_N)D_i u_N + a_0(x, u_N, \nabla u_N)(u_N - h)]dx \\ &+ \int_{A_h} p_N(u_N)(u_N - h)dx \\ &= \int_{A_h}[f_i D_i u_N + f\cdot(u_N - h)]dx\,.\end{aligned} \tag{11}$$

On the set $A_h$, $u_N(x) > h \geqq 1$, hence by condition (2), $p(u_N(x)) \geqq 0$. Therefore, taking into account the coercivity condition (8) and condition (5), from (11) we obtain

$$\begin{aligned}\nu\int_{A_h}|\nabla u_N|^q dx \leqq C\int_{A_h}&[f_i D_i u_N + f\cdot(u_N - h) \\ &+ k(x)(u_N - h) + u_N^{q-1}(u_N - h) + (u_N - h)|\nabla u_N|^{q-1}]dx\end{aligned} \tag{12}$$

We now make use of the well known inequalities

$$\begin{aligned}u_N\cdot|\Delta u_N|^{q-1} &\leqq (\nu/4)|\nabla u_N|^q + Cu_N^q \\ |f_i\cdot D_i u_N| &\leqq (\nu/4n)|\nabla u_N|^q + C|f_i|^{q^*} \qquad (i = 1, \cdots, n)\end{aligned}$$

We then deduce from (12) that

$$\int_{A_h}|\nabla u_N|^q dx \leqq C\int_{A_h}\left[1 + \sum_{i=1}^n |f_i|^{q^*} + \{|f| + k(x)\}u_N + u_N^q\right]dx \tag{13}$$

By hypothesis $q > 1$, $f(x), k(x) \in L^r(G)$ with $r > n/q$ and $f_i \in L^s(G)$, hence $|f_i|^{q^*} \in L^{s/q^*}(G)$ with $s/q^* > n/q$. Remembering that on $A_h$, $u_N(x) > h \geqq 1$, we obtain from (13) that

$$\int_{A_h}|\nabla u_N|^q\, dx \leqq \int_{A_h}|u_N|^q \varphi(x)dx \tag{14}$$

where $\varphi(x) \geqq 0$ a.e. on $G$, $\varphi(x) \in L^\beta(G)$ with $\beta > n/q$. From (14) Hölder's inequality gives

$$\int_{A_h}|\nabla u_N|^q\, dx \leqq \left[\int_{A_h}|u_N|^\alpha dx\right]^{q/\alpha}\left[\int_{A_h}\varphi^\beta dx\right]^{1/\beta} \tag{15}$$

with $q/\alpha + 1/\beta = 1$. Therefore

$$\int_{A_h} |\nabla u_N|^q \, dx \leqq C \|\varphi\|_{L^\beta(G)} \left[ \left( \int_{A_h} (u_N - h)^\alpha dx \right)^{q/\alpha} + h^q \operatorname{meas}^{q/\alpha} A_h \right] \tag{16}$$

Since $\beta > n/q$, $\alpha < nq/(n-q)$ and we deduce from (16) and (10) by using Theorem 5.1, Chapter 2 of [5] that $\operatorname{ess}_G \max u_N(x) < C$. Similarly, by taking in (9)

$$v(x) = \max\{-u_N(x) - h, 0\} ,$$

we obtain a bound from below for $u_N(x)$. Thus

$$\|u_N\|_{L^\infty(G)} < C \tag{17}$$

*We now pass to the limit as* $N \to \infty$. Because of (10), (17) and the Sobolev imbedding theorem, we can extract a subsequence of positive integers, still denoted by $\{N\}$ for convenience, such that

$u_N \longrightarrow u$ in $W_0^{1,q}(G)$ ,
$u_N(x) \longrightarrow u(x)$ a.e. on $G$ ,
$u_N$ tends to $u$ in the weak* topology of $L^\infty(G)$,
$p_N(u_N)$ tends to $p(u)$ in the weak* topology of $L^\infty(G)$ ,
$\mathscr{A} u_N \longrightarrow \chi$ in $W^{-1,q^*}(G)$ .

Then by the Lebesgue convergence theorem we have

$$\lim_N \int_G p_N(u_N)(u_N - u)dx = 0 .$$

Therefore taking $v = u_N - u$ in equation (9) and letting $N \to \infty$ we obtain

$$\lim_N \langle \mathscr{A} u_N, u_N - u \rangle = 0 .$$

Since $\mathscr{A}$ is of type $(M)$, it then follows that $\mathscr{A} u = \chi$ i.e. $\mathscr{A} u_N \rightharpoonup \mathscr{A} u$ in $W^{-1,q^*}(G)$. From (9) we deduce

$$\langle \mathscr{A} u, v \rangle + \int_G p(u) v dx = \int_G (f_i D_i v + f v) dx \quad \forall v \in W_0^{1,q}(G)$$

with $u \in L^\infty(G) \cap W_0^{1,q}(G)$.

I wish to thank the referee for a number of helpful suggestions.

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Received October 15, 1974.

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Printed in Japan by Intarnational Academic Printing Co., Ltd., Tokyo, Japan

# Pacific Journal of Mathematics

**Vol. 57, No. 1 January, 1975**