

# Pacific Journal of Mathematics

**ON BOUNDED SOLUTIONS OF A STRONGLY NONLINEAR  
ELLIPTIC EQUATION**

NGUYỄN PHUONG CÁC

## ON BOUNDED SOLUTIONS OF A STRONGLY NONLINEAR ELLIPTIC EQUATION

NGUYEN PHUONG CẮC

**I. Introduction.** Consider the Dirichlet problem for a bounded domain  $G \subset R^n (n \geq 2)$  having smooth boundary  $\partial G$ :

$$(1) \quad \begin{aligned} \mathcal{A}u + p(u) &= -D_i f_i + f \\ z|_{\partial G} &= 0, \end{aligned}$$

where  $\mathcal{A}$  is a second order differential operator of Leray-Lions type mapping a real Sobolev space  $W_0^{1,q}(G) (1 < q < \infty)$  into its dual;  $f, f_i (i = 1, \dots, n)$  are given functions. We have used the notation  $D_i$  for the derivative in the distribution sense  $\partial/\partial x_i$  and the convention that if an index is repeated then summation over that index from 1 to  $n$  is implied. We shall assume that the real function  $p(t)$  is continuous and satisfies the condition

$$(2) \quad p(t)t \geq 0 \quad \forall t \in R,$$

but otherwise  $(p)t$  is not subject to any growth condition.

In this paper we discuss the existence of a solution of equation (1) in  $W_0^{1,q}(G) \cap L^\infty(G)$ .

Many papers appearing recently have studied equations and inequations involving strongly nonlinear elliptic operators of the type (1). For equations we mention among others [1], [2], [7]; in [1] and [2] the existence of a solution in  $W_0^{m,q}(G)$  when the operator  $\mathcal{A}$  has arbitrary order  $2m$  is established under the additional hypothesis:

Given  $\varepsilon > 0$ , there exists  $K_\varepsilon > 0$  such that

$$(3) \quad p(t)s \leq \varepsilon p(s)s + K_\varepsilon [1 + p(t)t] \quad \forall t, s \in R$$

[3], [9] among others deal with strongly nonlinear inequations in  $W^{m,q}(G)$ .

For an operator  $\mathcal{A}$  of second order, [4] proves the existence of a solution in  $W_0^{1,q}(G)$  under the sole condition (2).

Finally let us mention that the existence of bounded solution of other strongly nonlinear equations and inequations has been discussed in [8]. However it seems to us that the technique of this paper is different from ours; it consists of multiplying the equation with a nonlinear expression of  $u$ ; it also seems that our method when applied to some concrete cases yields different results in the sense that we only require the functions in the right hand side of (1) to be in  $L^r(G)$  for some  $r > 1$  and not in  $L^\infty(G)$  as in [8].

II. Main result. The operator  $\mathcal{A}$  is assumed to be of the form

$$(4) \quad \mathcal{A}_u = \frac{\partial}{\partial X_i} a_i(x, u, \nabla u) + a_0(x, u, \nabla u)$$

where  $\nabla u = \text{grad } u$  and the functions  $a_i$  satisfy the following conditions:

(i) Each  $a_i$  ( $i = 0, 1, \dots, n$ ) is a function defined on  $G \times R \times R^n$  and of Caratheodory type:  $a_i(x, \eta, \zeta)$  is measurable in  $x$  for fixed  $\eta \in R, \zeta \in R^n$  and is continuous in  $(\eta, \zeta) \in R \times R^n$  for almost all fixed  $x \in G$ . Moreover there exist a constant  $c$ , a number  $q, 1 < q < \infty$ , a function  $k(x) \geq 0$  a.e. on  $G, k(x) \in L^{q^*}(G) (1/q + 1/q^* = 1)$ , such that

$$(5) \quad |a_i(x, \eta, \zeta)| \leq c(k(x) + |\eta|^{q-1} + |\zeta|^{q-1})$$

for  $i = 0, 1, \dots, n$ ; a.a.  $x \in G$  and  $\forall (\eta, \zeta) \in R \times R^n$ .

(ii) For a.a.  $x \in G$ ,

$$(6) \quad [a_i(x, \eta, \zeta) - a_i(x, \eta, \zeta')](\zeta - \zeta') > 0 \quad \text{if } \zeta \neq \zeta'$$

(iii) For a.a.  $x \in G$  and bounded  $\eta$ ,

$$(7) \quad a_i(x, \eta, \zeta) \zeta_i / (|\zeta| + |\zeta|^{q-1}) \longrightarrow \infty \quad \text{as } |\zeta| \longrightarrow \infty$$

Condition (5) implies that the semilinear form

$$\mathcal{A}(u, v) = \int_G [a_i(x, u, \Delta u) D_i v + a_0(x, u, \nabla u) v] dx$$

is defined for all  $u, v \in W_0^{1,q}(G)$  and there is  $\mathcal{A}u \in W^{-1,q^*}(G)$  such that  $\langle \cdot, \cdot \rangle$  denotes the pairing between  $W_0^{1,q}(G)$  and  $W^{-1,q^*}(G)$

$$\mathcal{A}(u, v) = \langle \mathcal{A}u, v \rangle \quad \forall v \in W_0^{1,q}(G).$$

It is known that the mapping  $\mathcal{A}: W_0^{1,q}(G) \rightarrow W^{-1,q^*}(G)$  is continuous and bounded ([6], Chapter 2, Section 2.6). Moreover, under the hypotheses (6) and (7),  $\mathcal{A}$  is pseudo-monotone and therefore it is of type (M): *If  $u_j \rightharpoonup u$  in  $W_0^{1,q}(G)$ ,  $\mathcal{A}u_j \rightharpoonup \chi$  in  $W^{-1,q^*}(G)$  and  $\limsup \langle \mathcal{A}u_j, u_j - u \rangle \leq 0$  then  $\mathcal{A}u = \chi$ . (Here and in the sequel “ $\rightharpoonup$ ” and “ $\rightarrow$ ” denote weak and strong convergence respectively.) We prove*

**THEOREM.** *Suppose that the differential operator  $\mathcal{A}$  of the form (4) satisfies conditions (5), (6), (7) and the coercivity condition:*

*There exists a constant  $\nu > 0$  such that for all  $v \in W_0^{1,q}(G)$*

$$(8) \quad \mathcal{A}(v, v) \geq \nu \|v\|_{W_0^{1,q}(G)}.$$

*Suppose also that the continuous function  $p(\cdot)$  satisfies the condition  $p(t) \geq 0, \forall t \in R$ . If  $f_i \in L^s(G)$  with  $s \geq q^*, s > n/(q-1), i = 1, \dots,$*

$n$ ; and  $f(x)$  and the function  $k(x)$  in (5) both belong to  $L^r(G)$  with  $r \geq q^*$ ,  $r > n/q$ , then the Dirichlet problem (1) has a solution  $u \in L^\infty(G) \cap W_0^{1,q}(G)$  in the sense that

$$\mathcal{A}(u, v) + \int_G p(u)v dx = \int_G (f_i D_i v + f v) dx \quad \forall v \in W_0^{1,q}(G).$$

*Proof.* We note that if  $q > n$  then by the Sobolev imbedding theorem, any function in  $W_0^{1,q}(G)$  is continuous on  $G$  and hence bounded. Consequently, in this case it suffices to prove the existence of a solution in  $W_0^{1,q}(G)$ . This can be done by partially repeating and slightly modifying the proof given below for the case  $q \leq n$ . We also note that if  $q > n$  then  $q^* > n/(q - 1)$  so that the theorem holds if  $f_i, f, k(x) \in L^{q^*}(G) (i = 1, \dots, n)$ .

So let us suppose that  $q \leq n$ . For each positive integer  $N$  we denote by  $p_N(t)$  the function

$$\begin{aligned} p_N(t) &= p(t) & \text{if } |p(t)| \leq N, \\ &= N & \text{if } p(t) > N, \\ &= -N & \text{if } p(t) < -N. \end{aligned}$$

The mapping  $T_N: u \rightarrow \mathcal{A}u + p_N(u)$  from  $W_0^{1,q}(G)$  into  $W^{-1,q^*}(G)$  is of type (M). In fact, consider a sequence  $u_j \rightarrow u$  in  $W_0^{1,q}(G)$  with  $T_N u_j \rightarrow \chi$  in  $W^{-1,q^*}(G)$  and  $\limsup_j \langle T_N u_j, u_j - u \rangle \leq 0$ . By the Sobolev imbedding theorem, we can assume without loss of generality that  $u_j(x) \rightarrow u(x)$  for a.a.  $x \in G$ . Condition (2) on  $p(t)$  and Fatou's lemma then give

$$\liminf_j \int_G p_N(u_j) u_j dx \geq \int_G p_N(u) u dx.$$

On the other hand, Lebesgue's dominated convergence theorem gives

$$\lim_j \int_G p_N(u_j) v dx = \int_G p_N(u) v dx \quad \forall v \in W_0^{1,q}(G).$$

We then deduce that  $p_N(u_j) \rightarrow p_N(u)$  in  $W^{-1,q^*}(G)$ , hence  $\mathcal{A}u_j \rightarrow \chi - p_N(u)$  as  $j \rightarrow \infty$  and

$$\limsup_j \langle \mathcal{A}u_j, u_j - u \rangle \leq 0.$$

Since  $\mathcal{A}$  has property (M), it follows that  $\mathcal{A}u = \chi - p_N(u)$  i.e.  $T_N u = \chi$ . It is clear that  $T_N$  is also bounded and hemicontinuous. The coercivity of  $\mathcal{A}$  implies that of  $T_N$ . Therefore (cf. e.g. [6], Remark 2.1, page 173) there exists  $u_N \in W_0^{1,q}(G)$  such that for all  $v \in W_0^{1,q}(G)$

$$(9) \quad \langle \mathcal{A}u_N + p_N(u_N), v \rangle = \langle -D_i f_i + f, v \rangle$$

We now find a bound for the  $L^\infty$ -norm of  $u_N$ .

Taking  $v = u_N$  in (9) and bearing in mind that  $p_N(t)t \geq 0$ , we obtain from the coercivity condition (8) that

$$(10) \quad \|u_N\|_{W_0^{1,q}(G)} < C$$

here and in the sequel  $C$  denotes various constants independent of  $N$ . Next we take in (9)

$$v(x) = \max \{u_N(x) - h, 0\}$$

where  $h \geq 1$ . If we denote by  $A_h$  the set  $\{x | x \in G, u_N(x) > h\}$  then

$$(11) \quad \begin{aligned} & \int_{A_h} [\alpha_i(x, u_N, \nabla u_N) D_i u_N + \alpha_0(x, u_N, \nabla u_N)(u_N - h)] dx \\ & + \int_{A_h} p_N(u_N)(u_N - h) dx \\ & = \int_{A_h} [f_i D_i u_N + f \cdot (u_N - h)] dx. \end{aligned}$$

On the set  $A_h$ ,  $u_N(x) > h \geq 1$ , hence by condition (2),  $p(u_N(x)) \geq 0$ . Therefore, taking into account the coercivity condition (8) and condition (5), from (11) we obtain

$$(12) \quad \begin{aligned} \nu \int_{A_h} |\nabla u_N|^q dx & \leq C \int_{A_h} [f_i D_i u_N + f \cdot (u_N - h) \\ & + k(x)(u_N - h) + u_N^{q-1}(u_N - h) + (u_N - h) |\nabla u_N|^{q-1}] dx \end{aligned}$$

We now make use of the well known inequalities

$$\begin{aligned} u_N \cdot |\Delta u_N|^{q-1} & \leq (\nu/4) |\nabla u_N|^q + C u_N^q \\ |f_i \cdot D_i u_N| & \leq (\nu/4n) |\nabla u_N|^q + C |f_i|^{q^*} \quad (i = 1, \dots, n) \end{aligned}$$

We then deduce from (12) that

$$(13) \quad \int_{A_h} |\nabla u_N|^q dx \leq C \int_{A_h} \left[ 1 + \sum_{i=1}^n |f_i|^{q^*} + \{|f| + k(x)\} u_N + u_N^q \right] dx$$

By hypothesis  $q > 1$ ,  $f(x), k(x) \in L^r(G)$  with  $r > n/q$  and  $f_i \in L^s(G)$ , hence  $|f_i|^{q^*} \in L^{s/q^*}(G)$  with  $s/q^* > n/q$ . Remembering that on  $A_h$ ,  $u_N(x) > h \geq 1$ , we obtain from (13) that

$$(14) \quad \int_{A_h} |\nabla u_N|^q dx \leq \int_{A_h} |u_N|^q \varphi(x) dx$$

where  $\varphi(x) \geq 0$  a.e. on  $G$ ,  $\varphi(x) \in L^\beta(G)$  with  $\beta > n/q$ . From (14) Hölder's inequality gives

$$(15) \quad \int_{A_h} |\nabla u_N|^q dx \leq \left[ \int_{A_h} |u_N|^\alpha dx \right]^{q/\alpha} \left[ \int_{A_h} \varphi^\beta dx \right]^{1/\beta}$$

with  $q/\alpha + 1/\beta = 1$ . Therefore

$$(16) \quad \int_{A_h} |\nabla u_N|^q dx \leq C \|\varphi\|_{L^\beta(G)} \left[ \left( \int_{A_h} (u_N - h)^\alpha dx \right)^{q/\alpha} + h^q \text{meas}^{q/\alpha} A_h \right]$$

Since  $\beta > n/q$ ,  $\alpha < nq/(n - q)$  and we deduce from (16) and (10) by using Theorem 5.1, Chapter 2 of [5] that  $\text{ess}_G \max u_N(x) < C$ . Similarly, by taking in (9)

$$v(x) = \max \{-u_N(x) - h, 0\},$$

we obtain a bound from below for  $u_N(x)$ . Thus

$$(17) \quad \|u_N\|_{L^\infty(G)} < C$$

We now pass to the limit as  $N \rightarrow \infty$ . Because of (10), (17) and the Sobolev imbedding theorem, we can extract a subsequence of positive integers, still denoted by  $\{N\}$  for convenience, such that

$$\begin{aligned} u_N &\longrightarrow u \text{ in } W_0^{1,q}(G), \\ u_N(x) &\longrightarrow u(x) \text{ a.e. on } G, \\ u_N &\text{ tends to } u \text{ in the weak* topology of } L^\infty(G), \\ p_N(u_N) &\text{ tends to } p(u) \text{ in the weak* topology of } L^\infty(G), \\ \mathcal{A}u_N &\longrightarrow \chi \text{ in } W^{-1,q^*}(G). \end{aligned}$$

Then by the Lebesgue convergence theorem we have

$$\lim_N \int_G p_N(u_N)(u_N - u) dx = 0.$$

Therefore taking  $v = u_N - u$  in equation (9) and letting  $N \rightarrow \infty$  we obtain

$$\lim_N \langle \mathcal{A}u_N, u_N - u \rangle = 0.$$

Since  $\mathcal{A}$  is of type (M), it then follows that  $\mathcal{A}u = \chi$  i.e.  $\mathcal{A}u_N \rightarrow \mathcal{A}u$  in  $W^{-1,q^*}(G)$ . From (9) we deduce

$$\langle \mathcal{A}u, v \rangle + \int_G p(u)v dx = \int_G (f_i D_i v + f v) dx \quad \forall v \in W_0^{1,q}(G)$$

with  $u \in L^\infty(G) \cap W_0^{1,q}(G)$ .

I wish to thank the referee for a number of helpful suggestions.

#### REFERENCES

1. F. E. Browder, *Existence theory for boundary value problem for quasilinear elliptic systems with strongly nonlinear lower order terms*, Proc. Sympos. Pure Math., **23** (1973), 269-286.

2. P. Hess, *On linear mappings of monotone type with respect to two Banach spaces*, J. Math. Pures et Appl., **52** (1973), 13-26.
3. ———, *Variational inequalities for strongly nonlinear elliptic operators*, J. Math. Pures et Appl., **52** (1973), 285-298.
4. ———, *A strongly nonlinear elliptic boundary value problem*, J. Math. Anal. and Appl., **43** (1973), 241-249.
5. O. A. Ladyzenskaya and N. N. Ural'tseva, *Linear and quasi-linear elliptic equations*, Translated from the Russian. Academic Press, New York 1968.
6. J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Gauthier-Villars, Paris 1969.
7. B. A. Tôn, *Pseudo-monotone operators in Banach spaces and nonlinear elliptic equations*, Math. Z., **121** (1971), 243-252.
8. ———, *On strongly nonlinear elliptic variational inequalities*, Pacific J. Math., **48** (1973), 279-291.
9. Nguyen P. Các, *On strongly nonlinear variational inequalities* J. Math. Pures et Appl.

Received October 15, 1974.

UNIVERSITY OF IOWA

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

RICHARD ARENS (Managing Editor)  
University of California  
Los Angeles, California 90024

J. DUGUNDJI  
Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

R. A. BEAUMONT  
University of Washington  
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM  
Stanford University  
Stanford, California 94305

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA  
STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
NAVAL WEAPONS CENTER



# Pacific Journal of Mathematics

Vol. 57, No. 1

January, 1975

Keith Roy Allen, <i>Dendritic compactification</i> . . . . .	1
Daniel D. Anderson, <i>The Krull intersection theorem</i> . . . . .	11
George Phillip Barker and David Hilding Carlson, <i>Cones of diagonally dominant matrices</i> . . . . .	15
David Wilmot Barnette, <i>Generalized combinatorial cells and facet splitting</i> . . . . .	33
Stefan Bergman, <i>Bounds for distortion in pseudoconformal mappings</i> . . . . .	47
Nguyễn Phương Các, <i>On bounded solutions of a strongly nonlinear elliptic equation</i> . . . . .	53
Philip Throop Church and James Timourian, <i>Maps with 0-dimensional critical set</i> . . . . .	59
G. Coquet and J. C. Dupin, <i>Sur les convexes ubiquitaires</i> . . . . .	67
Kandiah Dayanithy, <i>On perturbation of differential operators</i> . . . . .	85
Thomas P. Dence, <i>A Lebesgue decomposition for vector valued additive set functions</i> . . . . .	91
John Riley Durbin, <i>On locally compact wreath products</i> . . . . .	99
Allan L. Edelson, <i>The converse to a theorem of Conner and Floyd</i> . . . . .	109
William Alan Feldman and James Franklin Porter, <i>Compact convergence and the order bidual for <math>C(X)</math></i> . . . . .	113
Ralph S. Freese, <i>Ideal lattices of lattices</i> . . . . .	125
R. Gow, <i>Groups whose irreducible character degrees are ordered by divisibility</i> . . . . .	135
David G. Green, <i>The lattice of congruences on an inverse semigroup</i> . . . . .	141
John William Green, <i>Completion and semicompletion of Moore spaces</i> . . . . .	153
David James Hallenbeck, <i>Convex hulls and extreme points of families of starlike and close-to-convex mappings</i> . . . . .	167
Israel (Yitzchak) Nathan Herstein, <i>On a theorem of Brauer-Cartan-Hua type</i> . . . . .	177
Virgil Dwight House, Jr., <i>Countable products of generalized countably compact spaces</i> . . . . .	183
John Sollion Hsia, <i>Spinor norms of local integral rotations. I</i> . . . . .	199
Hugo Junghenn, <i>Almost periodic compactifications of transformation semigroups</i> . . . . .	207
Shin'ichi Kinoshita, <i>On elementary ideals of projective planes in the 4-sphere and oriented <math>\Theta</math>-curves in the 3-sphere</i> . . . . .	217
Ronald Fred Levy, <i>Showering spaces</i> . . . . .	223
Geoffrey Mason, <i>Two theorems on groups of characteristic 2-type</i> . . . . .	233
Cyril Nasim, <i>An inversion formula for Hankel transform</i> . . . . .	255
W. P. Novinger, <i>Real parts of uniform algebras on the circle</i> . . . . .	259
T. Parthasarathy and T. E. S. Raghavan, <i>Equilibria of continuous two-person games</i> . . . . .	265
John Pfaltzgraff and Ted Joe Suffridge, <i>Close-to-starlike holomorphic functions of several variables</i> . . . . .	271
Esther Portnoy, <i>Developable surfaces in hyperbolic space</i> . . . . .	281
Maxwell Alexander Rosenlicht, <i>Differential extension fields of exponential type</i> . . . . .	289
Keith William Schrader and James Lewis Thornburg, <i>Sufficient conditions for the existence of convergent subsequences</i> . . . . .	301
Joseph M. Weinstein, <i>Reconstructing colored graphs</i> . . . . .	307