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Let $f: M^n \rightarrow N^p$ be C^n with $n - p = 0$ or 1 , let $p \geq 2$, and let $R_{p-1}(f)$ be the critical set of f . If $\dim(R_{p-1}(f)) \leq 0$, then (1.1) at each $x \in M^n$, f is locally topologically equivalent to one of the following maps:

- (a) the projection map $\rho: R^n \rightarrow R^p$,
- (b) $\sigma: C \rightarrow C$ defined by $\sigma(z) = z^d$ ($d = 2, 3, \dots$), where C is the complex plane, or
- (c) $\tau: C \times C \rightarrow C \times R$ defined by $\tau(z, w) = (2z \cdot \bar{w}, |w|^2 - |z|^2)$, where \bar{w} is the complex conjugate of w .

Under the additional hypothesis that $\dim(f(R_{p-1}(f))) \leq p-2$ this result was proved in an earlier paper of the authors. They show here that $\dim(R_{p-1}(f)) \leq 0$ implies something like $\dim(f(R_{p-1}(f))) \leq p-2$.

For general background material, the reader is referred to that earlier paper [5]. The branch set B_f [5, p. 616, (1.5)] is the set of points at which f fails to be locally topologically equivalent to ρ . A map $g: J^{n-m} \times R^m \rightarrow L^{p-m} \times R^m$ is called a *layer map* if for each $t \in R^m$, $g(J^{n-m} \times \{t\}) \subset L^{p-m} \times \{t\}$.

1.2. Outline of the proof. We suppose that f is not an open map, and from some technical differential lemmas of §3 obtain in (3.4) by restriction and change of coordinates a layer map satisfying the hypotheses of (2.1). By that lemma $\dim(B_f) = p-1$, so that $\dim(R_{p-1}(f)) = p-1$, contradicting the hypothesis of (1.1). Thus f is open, and from the local structure for open maps given in [7] we conclude in (4.1) that $\dim(f(B_f)) \leq p-2$. This is (essentially) the additional hypothesis assumed in [5], and our conclusion results. A global structure theorem is also given (4.5).

2. A topological lemma. In order to read the proof of (2.1) the reader will need to have at hand the definition and certain properties of spoke sets [7, (2.1), (2.2), (2.3)].

LEMMA 2.1. Let $f: D^2 \times R^{p-1} \rightarrow R \times R^{p-1}$ be a layer map with $B_f \neq \emptyset$, $f(\partial D^2 \times \{t\})$ a single point not in $f(B_f)$, and $\dim(B_f \cap (D^2 \times \{t\})) = \dim(f(B_f \cap (D^2 \times \{t\}))) \leq 0$ for each $t \in R^{p-1}$. Then $\dim B_f = p-1$.

Proof. The last hypothesis implies that $\dim f(B_f) \leq p-1$ [9, p.

44, Theorem IV 3], so that $\dim B_f \leq p - 1$ [9, p. 91, Theorem VI 7]. If $p = 1$ and $B_f = \emptyset$, then f is open and a contradiction results from [7, (3.1)(b) or (d)]. Thus, for $p = 1$ $\dim B_f = 0$, i.e., $p - 1$. Hence we may suppose that $p \geq 2$, and will prove that $\dim B_f \geq p - 1$.

Let $I = [0, 1]$, let $I^{p-1} \subset I^p$ be $\{(x_1, x_2, \dots, x_p): x_p = 0\}$, let $r = 0, 1, \dots, p - 1$, and, for $a \in I^{p-1}$, let $\Gamma_{a,r} = \{x \in I^{p-1}: x_i = a_i \text{ for } i \geq r + 1\}$. For

$$X \subset \Gamma_{a,r} \text{ and } \alpha > 0,$$

let $X(r, \alpha) = \{x \in I^{p-1}: (x_1, \dots, x_r, a_{r+1}, \dots, a_{p-1}) \in X \text{ and } |x_i - a_i| < \alpha \text{ for } i \geq r + 1\}$. Thus $\Gamma_{a,r}(r, \alpha) = \{x \in I^{p-1}: |x_i - a_i| < \alpha \text{ for } i \geq r + 1\}$.

Consider statement S_r : (1) for every $\varepsilon > 0$ and $a \in I^{p-1}$, there are a triangulation \mathfrak{X} of the r -cell $\Gamma_{a,r}$ and $\alpha > 0$, and (2) for every closed r -simplex σ of \mathfrak{X} , there are spoke sets $L_{j,\sigma} (j = 0, 1, \dots, q(\sigma))$ satisfying conclusions (i)–(vi) of [7, (2.1) and (2.2)] with W replaced by $\text{Cl}[\sigma(r, \alpha)]$ and $E = B_f \cap (D^2 \times I^{p-1})$. Moreover, (3) let σ and τ be closed r -simplices of \mathfrak{X} , and let $D^2 \times \text{Cl}[(\sigma \cap \tau)(r, \alpha)]$ be denoted by T . Then, for any $L_{i,\sigma}$ and $L_{j,\tau}$, one of the following statements is true: $L_{i,\sigma} \cap T = L_{j,\tau} \cap T$, $L_{i,\sigma} \cap T \subset (L_{j,\tau} - \Omega_{j,\tau}) \cap T$, $L_{j,\tau} \cap T \subset (L_{j,\sigma} - \Omega_{j,\sigma}) \cap T$, or $L_{i,\sigma} \cap (L_{j,\tau} - \Omega_{j,\tau}) \cap T = \emptyset$.

Since $\Gamma_{a,0} = \{a\}$ and $\{a\}$ is the only 0-simplex of T , statement S_0 follows immediately from [7, (2.2)]. We will suppose that S_r is true ($r < p - 1$) and deduce S_{r+1} .

Let $\varepsilon > 0$ and $a \in I^{p-1}$ be given. For $[u, v] \subset \mathbb{R}$ and $\eta > 0$, let

$$\Psi(u, v, \eta) = \{x \in I^{p-1}: u < x_{r+1} < v \text{ and } |x_i - a_i| < \eta \text{ for } i > r + 1\}.$$

If $c \in \Gamma_{a,r+1}$, then $\Gamma_{c,r} \subset \Gamma_{a,r+1}$ and $\Gamma_{c,r}(r, \eta) = \Psi(c_{r+1} - \eta, c_{r+1} + \eta, \eta)$. For $c \in \Gamma_{a,r+1}$, let $\alpha(c) > 0$, $\mathfrak{X}(c)$, and $\{L_{c,j,\sigma}\}$ be as given by S_r for ε (and a replaced by c). There are $c(i) (i = 1, 2, \dots, m)$ such that $\{\Gamma_{c(i),r}(r, \alpha(c(i)))\}$ covers $\Gamma_{a,r+1}$. We may suppose that $\{c_{r+1}(i)\}$ are in increasing order and the cover is minimal. If the open interval $(c_{r+1}(i) - \alpha(c(i)), c_{r+1}(i) + \alpha(c(i)))$ is denoted by A_i , then $0 \in A_1 - \bigcup_{i \neq 1} A_i$, $1 \in A_m - \bigcup_{i \neq m} A_i$, and $A_i \cap A_j \neq \emptyset$ if and only if $j = i - 1, i$, or $i + 1$. Choose $b(i) \in \Gamma_{a,r+1}$, $0 < b_{r+1}(i) < 1$, and $\gamma > 0$ so that the intervals $F_i = [b_{r+1}(i) - \gamma, b_{r+1}(i) + \gamma]$ are mutually disjoint and $F_i \subset A_i \cap A_{i+1} \cap (0, 1) (i = 1, 2, \dots, m - 1)$.

Let $\Omega = \bigcup_{i,j,\sigma} \Omega_{c(i),j,\sigma}$. Since $B_f \cap \Omega = \emptyset$ (by S_r (2) (iii) and (iv)), there is a δ with $0 < \delta < \min(\varepsilon, d(B_f, \Omega))$ (d is distance). Let $\alpha(b(i)) > 0$, $\mathfrak{X}(b(i))$, and $\{L_{b(i),j,\sigma}\}$ be as given by S_r for ε replaced by δ and a replaced by $b(i) (i = 1, 2, \dots, m - 1)$; let $\beta = \min\{\alpha(b(i)), \alpha(c(i)), \gamma\}$. By S_r (2) (vi) each $\dim L_{b(i),j} < \delta < d(B_f, \Omega)$ and by S_r (2) (iv) $B_f \cap L_{b(i),j} \neq \emptyset$; thus (*) if

$$(D^2 \times \Gamma_{a,r+1}(r, \beta)) \cap L_{b(i),j,\sigma} \cap L_{c(h),k,\tau} \neq \emptyset,$$

then $(D^2 \times \Gamma_{a,r}(r, \beta)) \cap L_{b(i),j,\sigma} \subset (D^2 \times \Gamma_{a,r}(r, \beta)) \cap (L_{c(h),k,\tau} - \Omega_{c(h),k,\tau})$.

Let $d(i) (i = 1, 2, \dots, 2m)$ be the numbers $0, 1, b_{r+1}(i) - \beta$, and $b_{r+1}(i) + \beta (i = 1, \dots, m - 1)$ in increasing order. Then $\Psi(d(2i - 1), d(2i), \beta)$ (resp., $\Psi(d(2i), d(2i + 1), \beta)$) is contained in $\Gamma_{c(i),r}(r, \alpha(c(i)))$ (resp., $\Gamma_{b(i),r}(r, \alpha(b(i)))$).

For each closed r -simplex σ of $\mathfrak{X}(c(i))$ (resp., $\mathfrak{X}(b(i))$), let $\Sigma \subset \Gamma_{a,r+1}$ be the closed $(r + 1)$ -cell defined by $x \in \Sigma$ if and only if $(x_1, \dots, x_r, a_{r+1}, \dots, a_{p-1}) \in \sigma$, $d(2i - 1) \leq x_{r+1} \leq d(2i)$ (resp., $d(2i) \leq x_{r+1} \leq d(2i + 1)$), and $x_i = a_i$ for $i > r + 1$. There is a triangulation \mathfrak{X} of $\Gamma_{a,r+1}$ such that each such Σ is a subpolyhedron [13, Chapter 1, p. 5]. For each closed $(r + 1)$ -simplex ρ of \mathfrak{X} , there is an r -simplex σ of $\mathfrak{X}(c(i))$ or $\mathfrak{X}(b(i))$ with $\rho \subset \Sigma$. Define $L_{j,\rho} = L_{j,\sigma} \cap (D^2 \times \rho(r + 1, \beta))$ ($j = 1, 2, \dots, q(\rho) = q(\sigma)$). It follows that S_{r+1} is satisfied for ε and a , with $\beta > 0$, \mathfrak{X} , and $\{L_{j,\rho}\}$ (conclusion (3) follows from (*) and S_r (3)).

Thus S_{p-1} is true for (say) 0 and any $\varepsilon > 0$; note that $\Gamma_{0,p-1} = I^{p-1}$ itself, and α does not arise in this case.

Let $e = 1, 2, \dots$. Let \mathfrak{X}_e be the triangulation of I^{p-1} and let $\{L_{j,\sigma,e}\}$ be as given in S_{p-1} for $\varepsilon = 1/e$, let $L_e = \bigcup_{j,\sigma} L_{j,\sigma,e}$, and let $\Omega_e = \bigcup_{j,\sigma} \Omega_{j,\sigma,e}$. Each \mathfrak{X}_e is rectilinear in I^{p-1} , so we may suppose that each \mathfrak{X}_{e+1} is a subdivision of \mathfrak{X}_e .

Define an equivalence relation \sim on L_e by: for every $a \in I^{p-1}$, σ , and j , and for every $u, v \in L_{j,\sigma,e} \cap (D^2 \times \{a\})$, $u \sim v$. Let Y_e be the resulting identification space, and let $\omega_e: L_e \rightarrow Y_e$ be the identification map. Let $L_e \cap (D^2 \times \partial I^{p-1})$ be denoted by G_e , and $\omega_e(G_e)$ by ∂Y_e . Then $\omega_e: (L_e, G_e) \rightarrow (Y_e, \partial Y_e)$ is a homotopy equivalence, Y_e is a $(p - 1)$ -dimensional finite polyhedron, viewed as a cell complex [13, Chapter 1, p. 5], its closed $(p - 1)$ -cells are $\omega_e(L_{j,\sigma,e})$, their interiors $\omega_e(L_{j,\sigma,e} \cap (D^2 \times \text{int } \sigma)) = \gamma_{j,\sigma,e}$ are mutually disjoint for distinct pairs (j, σ) .

With the index ξ of [7, (2.1)] $\sum_{j,\sigma} \xi(L_{j,\sigma,e}) \cdot \gamma_{j,\sigma,e}$ is a $(p - 1)$ -chain β_e of $(Y_e, \partial Y_e)$. From the index formula [7, (2.3)] and from (2) (v) and (3) in S_{p-1} (note that $\text{Cl}[(\sigma \cap \tau)(\alpha)]$ is merely $\sigma \cap \tau$ in this case), it follows that β_e is a cycle of $(Y_e, \partial Y_e)$. Since $\xi(D^2 \times \{s\}) = 1$, it follows again from the index formula that $\sum_j \xi(L_{j,\sigma}) = 1$ for each σ , so that $\beta_e \neq 0$. Since $\dim Y_e = p - 1$, β_e defines a nonzero element of $H_{p-1}(Y_e, \partial Y_e; Z) \approx H_{p-1}(L_e, G_e, Z)$ (Z the ring of integers). Let $\eta_e = \omega_e^{-1}(\{\beta_e\}) \in H_{p-1}(L_e, G_e; Z)$.

Since $\Omega_e \cap B_f = \emptyset$ (by S_{p-1} (2) (iv)), there exists $\delta(e)$ with $0 < \delta(e) < d(\Omega_e, B_f)$ ($e = 1, 2, \dots$), and there is a subsequence $\{e(k)\}$ such that $e(1) = 1$ and $1/e(k + 1) < \min \{\delta(e(i)): i \leq k\}$ ($k = 1, 2, \dots$). For every $L_{j,\sigma,e(k+1)}$, there are a unique $\tau \in T_{e(k)}$ with $\sigma \subset \tau$ and $x \in B_f \cap L_{j,\sigma,e(k+1)}$ by S_{p-1} (2) (iv). For a unique i , $x \in L_{i,\tau,e(k)}$ by S_{p-2} (2) (iv) and (V), and from the size of $1/e(k + 1)$ and S_{p-1} (2) (vi), (†) $L_{j,\sigma,e(k+1)} \subset$

$L_{i,\tau,\varepsilon(k)}$. Let $\lambda_{k+1}: (L_{e(k+1)}, G_{e(k+1)}) \rightarrow (L_{e(k)}, G_{e(k)})$ be inclusion. From (†) and the index formula [7, (2.3)] it follows that $\lambda_{k+1}^*(\eta_{e(k+1)}) = \eta_{e(k)} (\neq 0)$. Thus the inverse limit of $\{\eta_{e(k)}\}$ is nonzero, so that the Čech homology group $H_{p-1}(\bigcap_e L_e, \bigcap_e G_e; Z) \neq 0$ by the Continuity Theorem. Hence

$$\dim \left(\bigcap_e L_e \right) \geq p - 1$$

[9, p. 152, Theorem VIII 4], and since $\bigcap_e L_e \subset B_f(S_{p-1}(2))$ (iv) and (vi), $\dim B_f \geq p - 1$.

3. Differential lemmas. The following two lemmas are generalizations of lemmas that have been used repeatedly, and these generalizations will also be used elsewhere.

LEMMA 3.1. *Let $f: M^n \rightarrow N^p$ be C^m , let K^q be a C^m q -manifold ($m = 1, 2, \dots$; or $m = \infty$; or $m = \omega$; $q = 0, 1, \dots, p - 1$), let ρ be a C^m diffeomorphism of a region in N^p onto $K^q \times R^{p-q}$, and let Ω be a nonempty compact subset of $f^{-1}(\rho^{-1}(K^q \times \{0\}))$. If $f|_{\Omega}$ is transverse regular on $\rho^{-1}(K^q \times \{0\})$, then there are $\varepsilon > 0$, a $C^m(n - p + q)$ -manifold L , and a C^m diffeomorphism σ of $L \times S(0, \varepsilon)$ onto a neighborhood of Ω in M^n such that $\rho \circ f \circ \sigma$ is a layer map.*

This is proved in [6, (4.1)] and is a generalization of [8, p. 80, (3.5)] and [3, p. 376, (2.7)]. The condition that “ $f|_{\Omega}$ is transverse regular” means that f is transverse regular at x for each $x \in \Omega$.

LEMMA 3.2. *Let $q = 1, 2, \dots$, let $f: M^n \rightarrow N^p$ be a C^r map with $\max(n - q + 1, 1) \leq r \leq \infty$, let $\Omega \subset M^n$ be compact, and let $Y \subset N^p$ be closed, with $\dim Y \geq q$. Then for some m ($m = 0, 1, \dots, p - q$) there is a C^r embedding λ of $S^m \times R^{p-m}$ in N^p such that $f|_{\Omega}$ is transverse regular on $\lambda(S^m \times \{t\})$ and $\lambda(S^m \times \{t\}) \cap Y \neq \emptyset$ for each $t \in R^{p-m}$.*

If Ω is omitted, “ $f|_{\Omega}$ is transverse regular” is replaced by “ f is transverse regular”, and f is assumed proper, this is [8, p. 80, (3.7)]. The proof is an immediate generalization of that proof. (Although we do not need it in this paper, the same comments apply to [8, p. 82, (3.8)], except that J need not be compact.)

DEFINITION 3.3. Let K^n and L^p be C^r -manifolds with nonempty boundary, and let $f: K^n \rightarrow L^p$ be a C^r ($r \geq 1$) proper map with $f^{-1}(\partial L^p) = \partial K^n$ and $f(R_{p-1}(f)) \subset \text{int } L^p$. Let $D(K^n)$ and $D(L^p)$ be the doubles K^n and L^p , respectively [10, p. 52, (5.10) and p. 62, (6.3)]. We now define a C^r map $g: D(K^n) \rightarrow D(L^p)$, called a *double of f* , such that the restriction of g to each half is C^r equivalent to f [5, p. 616,

(1.3)].

Let $K_i = K \times i$, let $L_i = L \times i$, and let $f_i: K_i \rightarrow L_i$ be defined by $f_i(x, i) = (f(x), i)$ ($i = 0, 1$). Let $J_0 = [0, 1)$ and $J_1 = (-1, 0]$. There is an open neighborhood U of ∂L in L disjoint from $f(R_{p-1}(f))$ and C^r diffeomorphisms $\psi_i: U_i = U \times i \rightarrow \partial L_i \times J_i$ [10, p. 51, (5.9)]. Let $\alpha_i: f_i^{-1}(U_i) \rightarrow U_i$ and $\beta_i: \partial K_i \rightarrow \partial L_i$ be the restrictions of f_i .

There exist manifolds $V_i = V_i^n$ with $\partial V_i = \emptyset$ and $f^{-1}(U_i) \subset V_i$ and $W_i = W_i^p$ with $\partial W_i = \emptyset$ and $U_i \subset W_i$, and a C^r extension $\gamma_i: V_i \rightarrow W_i$ of α_i . By restricting γ_i we may suppose that it is proper. Now γ_i is the projection map of a C^r bundle (e.g. from (3.1) with K a single point), so that α_i and β_i are also. Thus there are diffeomorphisms $\phi_i: f_i^{-1}(U_i) \rightarrow \partial K_i \times J_i$ such that $\psi_i \circ \alpha_i = (\beta_i \times \iota) \circ \phi_i$ (where ι is the identity map on J_i) [11, p. 53, (11.4)].

We may define the (C^r structures on the) doubles $D(K^n)$ and $D(L^p)$ using the maps ϕ_i and ψ_i (identify $(x, 0)$ in ∂K_0 with $(y, 1)$ in ∂K_1 if $\phi_0(x, 0)$ and $\phi_1(y, 1)$ have the same first coordinate), and let $\lambda_i: K_i \rightarrow D(K^n)$ and $\mu_i: L_i \rightarrow D(L^p)$ be the natural (C^r) embeddings. Define g by $g(x) = f_i(x)$ for $x \in K_i$. Clearly g is C^r except possibly on ∂K . If $U' = U_0 \cup U_1$ and $\psi: U' \rightarrow \partial L \times (-1, 1)$ and $\phi: g^{-1}(U') \rightarrow \partial K \times (-1, 1)$ are defined by the ψ_i and ϕ_i , respectively, then $\psi \circ g|g^{-1}(U') = (\beta \times \iota) \circ \phi$ (where ι is the identity map on $(-1, 1)$ and $\beta = \beta_1 = \beta_2$), so that g is C^r everywhere.

LEMMA 3.4. *Let $f: M^n \rightarrow N^p$ be a C^n map with $n - p = 0$ or 1 , $\dim B_f \leq p - 2$, and $\dim(B_f \cap f^{-1}(y)) \leq 0$ for each $y \in N^p$. Then f is open.*

Proof. In case $n = p$, f is light and the conclusion is given by [2, p. 94, (2.3)], so we may suppose that $n = p + 1$. Suppose that f is not open. Let E_f be the set of points at which f fails to be open, and let $x \in E_f$. According to [5, p. 622, (2.6)] there is a connected (not necessarily compact) manifold $K^{p+1} \subset M^{p+1}$ with boundary such that $x \in \text{int } K^{p+1} (= K^{p+1} - \partial K^{p+1})$ and the closure \bar{K}^{p+1} of K^{p+1} in M^{p+1} is compact; there is an open p -cell $D^p \subset N^p$ with $f(K^{p+1}) \subset D^p$; and the restriction map $g: K^{p+1} \rightarrow D^p$ is proper with $B_g \cap \partial K^{p+1} = \emptyset$. Let $\psi = g|_{\text{int } K^{p+1}}$, and let $\Omega \subset \text{int } K^{p+1}$ be the compact set E_ψ . Since f is not open, $\dim \psi(E_\psi) \geq p - 1$ [5, p. 623, (3.4)], and by (3.2) there is a C^{p+1} embedding $\lambda: S^m \times R^{p-m} \rightarrow D^p$ such that $\psi|_\Omega$ is transverse regular on $\lambda(S^m \times \{t\})$ and $\lambda(S^m \times \{t\}) \cap \psi(E_\psi) \neq \emptyset$ for each $t \in R^{p-m}$ and $m = 0$ or 1 . From (3.1) $m \neq 0$ and, for some $\varepsilon > 0$, the restriction of ψ to some neighborhood of E_ψ is C^{p+1} equivalent to the C^{p+1} layer map $\alpha: Q^2 \times R^{p-1} \rightarrow S^1 \times R^{p-1}$ with $E_\alpha \cap (Q^2 \times \{t\}) \neq \emptyset$ for every $t \in R^{p-1}$.

Since $B_\alpha \subset R_{p-1}(\alpha)$ (the Rank Theorem [5, p. 617, (1.6)], $\dim(\alpha(B_\alpha)) \cap$

$(S^1 \times \{t\}) \cong 0$ for each $t \in R^{p-1}$ (by Sard's theorem); and since

$$\dim(B_\alpha \cap \alpha^{-1}(u, t)) \leq 0$$

for each $(u, t) \in S^1 \times R^{p-1}$ by hypothesis, $\dim(B_\alpha \cap (Q^2 \times \{t\})) \leq 0$ [9, p. 91, Theorem VI 7].

Let $(q, s) \in E_\alpha \subset B_\alpha$ (we may suppose that $s = 0$), and let $T \subset Q^2 \times R^{p-1}$ be a closed $(p+1)$ -cell neighborhood of $(q, 0)$. Since $\{(q, 0)\}$ is the component of $\alpha^{-1}(\alpha(q, 0))$ containing $(q, 0)$ [5, p. 622, (3.2)], there is an interval $I \subset S^1$ with $\alpha_0(q) \in \text{int } I$ and $\delta > 0$ such that the component F of $\alpha^{-1}(I \times S(s, \delta))$ is contained in $\text{int } T$. We may suppose that the endpoints of I are regular values of α_0 , and thus, for δ sufficiently small, of α_t for every $t \in S(0, \delta)$. Thus F is an n -manifold with boundary, and each $F_t = F \cap (Q^2 \times \{t\})$ is compact. Let G be the double of F , and let $\beta: G \rightarrow S^1 \times S(0, \delta)$ be the double of the proper map $\alpha|_F: F \rightarrow I \times S(0, \delta)$ (3.3).

Choose an open 2-cell U with $q \in U$ and $U \times \{0\} \subset \text{int } F_0 \subset G_0$, and choose η , $0 < \eta < \delta$, with $U \times S(0, \eta) \subset \text{int } F \subset G$. There exists ξ , $0 < \xi < \eta$, and an interval $J \subset \text{int } I \subset S^1$ such that $\beta_0(q) \in \text{int } J$, the component X of $\beta^{-1}(J \times S(0, \xi))$ containing $(q, 0)$ is contained in $U \times S(0, \xi)$, and the end points of J are regular values of β_t for each $t \in S(0, \xi)$. Thus $X \cap (U \times \{0\})$, call it A^2 , is a 2-disk with holes, and $\alpha_0(\partial A^2) \subset \partial J$.

We now apply [1, p. 196, (3.4)] to β , $K_0 = S^1 \times \{0\}$, $\Gamma_1 = J \times \{0\}$, $K_1 = \partial \Gamma_1$, and ρ the identity map. There exists ζ , $0 < \zeta < \xi$, and a C^{p+1} (layer) diffeomorphism ω of $\beta^{-1}(S^1 \times \{0\}) \times S(0, \zeta)$ onto $\beta^{-1}(S^1 \times S(0, \zeta))$ with $\omega(A^2 \times S(0, \zeta)) = X$. Let D be the closed 2-cell with $A^2 \subset D \subset U$ and $\partial D \subset \partial A^2$, and let $\gamma: D \times S(0, \zeta) \rightarrow \text{int } I \times S(0, \zeta)$ be the restriction of $\beta \circ \omega$. Now $(0, q) \in E_\gamma \subset B_\gamma$ and by (2.1) $\dim B_\gamma = p - 1$, so that $\dim B_f \geq p - 1$, and a contradiction results.

4. Conclusions.

PROPOSITION 4.1. *Let $f: M^{p+1} \rightarrow N^p$ be C^{p+1} with $B_f \neq \emptyset$, $\dim B_f \leq p - 2$, and $\dim(f^{-1}(y) \cap B_f) \leq 0$ for each $y \in N^p$. Then $\dim B_f = p - 3$ and there is a closed set $Y \subset B_f$ such that $\dim Y < p - 3$ and, for every $x \in B_f - Y$, f at x is locally topologically equivalent to*

$$\tau \times \text{id}: R^4 \times R^{p-3} \longrightarrow R^3 \times R^{p-3}.$$

According to the Rank Theorem [5, p. 617, (1.6)] $B_f \subset R_{p-1}(f)$ and the following corollary results.

COROLLARY 4.2. *Let $f: M^{p+1} \rightarrow N^p$ be C^{p+1} with critical set $R_{p-1}(f)$, let $\dim R_{p-1}(f) \leq p - 2$, and let $\dim(f^{-1}(y) \cap R_{p-1}(f)) \leq 0$ for each $y \in N^p$. Then there is a closed set $Y \subset M^{p+1}$ such that $\dim Y < p - 3$*

and, for each $x \in M^{p+1} - Y$, f at x is locally topologically equivalent to either the projection map $\rho: R^{p+1} \rightarrow R^p$ or to

$$\tau \times \text{id}: R^4 \times R^{p-3} \longrightarrow R^3 \times R^{p-3} .$$

Proof of (4.1). By (3.4) f is open, and $p \geq 2$ since $B_f \neq \emptyset$ and $\dim B_f \leq p - 2$. According to [7, (4.1) and (1.1)], if $f: M^{p+1} \rightarrow N^p$ is a C^3 open map with $\dim(B_f \cap f^{-1}(y)) \leq 0$ for each $y \in N^p$, then there is a closed set $X \subset M^{p+1}$ such that $\dim f(X) \leq p - 2$ and, for every $x \in M^{p+1} - X$, there is a natural number $d(x)$ with f at x locally topologically equivalent to the map

$$\phi_{d(x)}: C \times R^{p-1} \longrightarrow R \times R^{p-1}$$

defined by $\phi_{d(x)}(z, t) = (\mathcal{R}(z^{d(x)}), t)$ ($\mathcal{R}(z^{d(x)})$ is the real part of the complex number).

Since $\dim B_f \leq p - 2$ by hypothesis, $B_f \subset X$, so that $\dim f(B_f) \leq p - 2$. Thus f satisfies the hypothesis of [5, p. 626, (4.7)]. (For $n = p + 1$ that proposition is identical with the present one except that the hypothesis $\dim B_f \leq p - 2$ is replaced by $\dim f(B_f) \leq p - 2$.)

COROLLARY 4.3. *If $f: M^{p+1} \rightarrow N^p$ is a C^{p+1} map with $\dim B_f = 0$ and $p \geq 2$, then $p = 3$ and at each $x \in B_f$, f is locally topologically equivalent to τ .*

4.4. Proof of (1.1). From the Rank Theorem [5, p. 617, (1.6)] $B_f \subset R_{p-1}(f)$, and the conclusion for $n - p = 1$ results from (4.3). For $n = p \geq 3$ $\dim(R_{p-1}(f)) \leq 0$ implies $B_f = \emptyset$ [2, p. 94, (2.2)]; for $n = p = 2$, f is light open [2, p. 94, (2.3)], and so has the desired structure (e.g. by [2, p. 90, (1.10)]).

Let G be a compact, connected Lie group, and let M be a closed, connected, oriented G -manifold with orbit space a manifold. The action is called *almost free* if it is free except for the fixed point set F , and F is discrete nonempty set. In [4] Church and Lamotke classified such actions globally, up to equivariant homeomorphism (they also treated the smooth case): invariants are the oriented homeomorphism type of the orbit space and the number (which is even) of fixed points. This classification gives significance to the following corollary of (1.1), a global classification of maps with 0-dimensional critical set.

COROLLARY 4.5. *Let M^{p+1} and N^p be closed, connected, oriented manifolds, and let $f: M^{p+1} \rightarrow N^p$ be a C^{p+1} map with critical set $R_{p-1}(f)$ of dimension at most 0. Then there is a unique factorization $f = h \circ g$, where $g: M^{p+1} \rightarrow K^p$ is the orbit map of a topological S^1 free or almost free action on M^{p+1} (and thus is classified by [4]),*

and $h: K^p \rightarrow N^p$ is an r -to-1 covering map ($r = 1, 2, \dots$).

Proof. By (1.1) either the branch set $B_f = \emptyset$, or $p = 3$ and at each point of B_f f is locally topologically equivalent to τ , i.e., to the cone map of the Hopf fibration $\psi: S^3 \rightarrow S^2$ [5, p. 618, (1.10)]. According to [12, p. 64, (2.5)] there is a natural number k such that $f^{-1}(y)$ has exactly k components for each $y \in N^p - f(B_f)$, and at most k components for each $y \in f(B_f)$. From the local structure, $f^{-1}(y)$ has exactly k components for every $y \in N^p$, and thus according to [12, p. 63, (2.1)] there is a (unique) factorization $f = h \circ g$, where $g: M^{p+1} \rightarrow K^p$ is a C^{p+1} monotone map and $h: K^p \rightarrow N^p$ is an r -to-1 covering map.

In case $B_f = \emptyset$, $B_g = \emptyset$ also, so that g is a bundle map [5, p. 618, (1.9)] with fiber S^1 . The structure group can be reduced to $S^1 = SO(2)$ [12, pp. 64-65], and thus g is the orbit map of a free S^1 action. In case $B_f \neq \emptyset$, the map $\alpha: M^{p+1} - B_g \rightarrow K^p - g(B_g)$ defined by restriction of g is also a free S^1 action; since B_g is discrete, g itself is the orbit map of an almost free action.

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