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The theory of spectral operators, when applied to eigenfunction expansions, covers the unconditionally convergent case. However, by perturbing certain spectral differential operators, J. Schwartz has obtained differential operators which are not spectral but whose eigenfunctions span the whole space. In this paper we show how new norms can be constructed so as to make these perturbed differential operators spectral. This we achieve by showing that such operators have an underlying generalized spectral measure (as defined by V. È. Ljance) and that every generalized spectral measure is essentially a *C*-spectral measure.

It may be shown that the topology defined by the new norm is finer than the outer spectral topology introduced by Ljance. Thus the theory of C-spectral measures both generalizes the concept as well as sharpens some of the results of the theory of generalized spectral measures. In this connection, it will be noted that the norm constructed by Smart is the inner spectral norm of the theory of C-spectral measures.

We denote by $\bar{X}$ a Banach space over the field C of all complex numbers; the norm of $\bar{X}$ is denoted by $||\cdot||$. The set of all bounded linear operators on $\bar{X}$ into itself is denoted $B(\bar{X})$. For any vector space X, we denote by $L(X)$ the set of all linear transformations on X into itself. The adjoint space of $\bar{X}$ is denoted $\bar{X}^*$. If $T \in B(\bar{X})$, we denote by $N(T)$ the kernel of T and by T^* the adjoint of T. B_0 will denote the σ-algebra of all Borel subsets of the complex plane Λ.

1. Spectral measures and their generalizations.

DEFINITION 1 (Ljance). A subring A of B_0 is said to be admissible if and only if it contains along with each of its members every Borel subset of that member.

If an admissible subring is also an algebra then it equals the whole of B_0, which is obviously an admissible subring of itself.

DEFINITION 2. Let X be a dense linear subspace of $\bar{X}$ and A an admissible subring of B_0. Let $E: A \to L(X)$ be a mapping such that

D(i) $E(\delta)E(\sigma) = E(\delta \cap \sigma)$ $(\delta, \sigma \in A)$;

D(ii) For each $x \in X$, the mapping $E(\cdot)x: A \to \bar{X}$ is countably additive in the norm-topology of $\bar{X}$.

We define $p: X \to [0, +\infty)$ as follows:

$$p(x) = \inf \left\{ \sum_{i=1}^{n} \|x_i\| : x = \sum_{i=1}^{n} E(\sigma_i)x_i, \quad \text{where} \right.$$

$$\left. \sigma_i \in B_0;\, x_i \in X \ (i = 1, \cdots, n) \right\} \qquad (x \in X) .$$

It is immediate that p is a semi-norm on X. Also it is easily seen that if $A = B_0$ and $E(\Lambda) = I$, the identity operator on X, then $p(x) \leqq \|x\|$ $(x \in X)$.

DEFINITION 3. Suppose that E is as above. We say that E is a generalized spectral measure if and only if

L(i) $X = \bar{X}$;

L(ii) Range $E \subset B(\bar{X})$;

L(iii) $\bigcap \{N[E(\Delta)]: \Delta \in A\} = \{0\} \subset \bar{X}$; and

L(iv) $\bigcap \{N[E^*(\Delta)]: \Delta \in A\} = \{0\} \subset \bar{X}^*$.

E is said to be a Γ-spectral measure based on X if and only if

J(i) $A = B_0$;

J(ii) $E(\Lambda) = I$, the identity operator on X; and

J(iii) p is a norm on X, compatible with $\|\cdot\|$, that is: If $\{x_n\}$ is a Cauchy-sequence in $\|\cdot\|$ and converges to zero in p, then it also converges to zero in $\|\cdot\|$.

REMARKS. The definition of generalized spectral measures is due to Ljance [3]; however, it will be noted that he confines himself to the Hilbert space case and that we have taken the natural generalization of his concept to Banach spaces. The concept of Γ-spectral measures was introduced in [1]; and C-spectral measures are particular kind of Γ-spectral measures obtained either by restricting or by extending suitably the base space of the latter.

LEMMA 1. *Suppose that E is a generalized spectral measure in $\bar{X}$ and having for its domain the admissible ring A, and let $X =$* sp $\{\bigcup [E(\Delta)\bar{X}]: \Delta \in A\}$ $(=\tilde{X}$, in the notations of Ljance). *For each $\sigma \in B_0$, we define the mapping $\tilde{E}(\sigma): X \to X$ as follows: If $x \in E(\Delta)\bar{X}$, for some $\Delta \in A$, then $\tilde{E}(\sigma)x = E(\Delta \cap \sigma)x$.*

The mapping $\tilde{E}: B_0 \to L(X)$ is well-defined, $\tilde{E}(\Delta) = E(\Delta)|X$ $(\Delta \in A)$ and $\tilde{E}$ satisfies the conditions of Definition 2. Moreover, if the semi-norm p is defined as there we have

(i) $p\{E(\Delta)x\} \leqq p(x)$ $(x \in X; \Delta \in A)$;

(ii) *For each $\Delta \in A$, there exists a finite positive $K(\Delta)$ such that*

$$p(x) \leqq \|x\| \leqq K(\Delta)p(x) \quad (x \in E(\Delta)\bar{X}) .$$

Proof. Let $\sigma \in B_0$ and $x \in X$. If $\Delta_1, \Delta_2 \in A$ such that $x \in E(\Delta_1)\bar{X}$ and $x \in E(\Delta_2)\bar{X}$, then $x \in E(\Delta)\bar{X}$, where $\Delta = \Delta_1 \cap \Delta_2$. Thus $\tilde{E}(\sigma)x = E(\Delta \cap \sigma)x$. But $\Delta \cap \sigma = (\Delta_1 \cap \sigma) \cap \Delta_2$, so that

$$E(\Delta \cap \sigma)x = E(\Delta_1 \cap \sigma)E(\Delta_2)x = E(\Delta_1 \cap \sigma)x$$

and similarly $E(\Delta \cap \sigma)x = E(\Delta_2 \cap \sigma)x$. This proves that $\tilde{E}(\sigma): X \rightarrow X$ is well-defined. Clearly it is linear and, for each $\Delta \in A$, we have $\tilde{E}(\Delta) = E(\Delta) | X$. Further it follows from condition L(iv) of Definition 3 that X is a dense linear subspace of $\bar{X}$. Thus $\tilde{E}$ satisfies the conditions of Definition 2.

Inequality (i) follows immediately from the definition of p. To prove inequality (ii), assume that $\Delta \in A$. Then $E(\Delta)$ is a bounded projector on $\bar{X}$, so that $E(\Delta)\bar{X}$ is a Banach subspace of $\bar{X}$; further, $E(\Delta)\bar{X} \subset X$, so that the function $\tilde{E}(\cdot)z$ is countably additive in the norm-topology for each $z \in E(\Delta)\bar{X}$. Hence it follows by Corollaries IV.10.2 and II.3.21 of [2] that

$$\|\tilde{E}(\sigma)z\| \leqq K_1(\Delta)\|z\| \quad (z \in E(\Delta)\bar{X}) ,$$

for some finite positive $K_1(\Delta)$. Now let $x \in E(\Delta)\tilde{X}$ and $x = \sum_{i=1}^{n} \tilde{E}(\sigma_i)x_i$, where $\sigma_i \in B_0$ and $x_i \in X$ $(i = 1, \cdots, n)$. Then

$$x = E(\Delta)x = \sum_{i=1}^{n} \tilde{E}(\sigma_i)E(\Delta)x_i ,$$

so that

$$\begin{aligned} \|x\| &\leqq K_1(\Delta) \sum_{i=1}^{n} \|E(\Delta)x_i\| \\ &\leqq K_1(\Delta)\|E(\Delta)\| \sum_{i=1}^{n} \|x_i\| . \end{aligned}$$

Hence

$$\|x\| \leqq K(\Delta)p(x) ,$$

where $K(\Delta) = K_1(\Delta)\|E(\Delta)\|$. Finally it will be observed that $\Lambda \in B_0$ and $x = \tilde{E}(\Lambda)x$, so that $p(x) \leqq \|x\|$. This completes the proof of the lemma.

THEOREM 1. *Every generalized spectral measure induces* (as in the above lemma) *a Γ-spectral measure.*

Proof. Suppose that E is a generalized spectral measure in $\bar{X}$, having A for its domain. Let X and $\tilde{E}$ be defined as in the previous lemma. It has already been noted that $\tilde{E}$ satisfies the conditions of Definition 2. So it only remains to verify condition J(iii). To this end assume that $\{x_n\}$ is a Cauchy-sequence in X and that $p(x_n) \rightarrow 0$.

Let x be the limit of the sequence $\{x_n\}$ and let $\Delta \in A$. Then, since $E(\Delta)$ is a bounded linear operator on $\bar{X}$, it follows that $\tilde{E}(\Delta)x_n \rightarrow E(\Delta)x$; also $p\{\tilde{E}(\Delta)x_n\} \leqq p(x_n)$. Thus $||\tilde{E}(\Delta)x_n|| \leqq K(\Delta)p(x_n)$, where $K(\Delta)$ is as in the previous lemma, so that $E(\Delta)x = 0$. Since $\Delta \in A$ is arbitrary, it follows from L(iii) that $x = 0$. This completes the proof of the theorem. Now $\tilde{E}$ may be made into a C-spectral measure [1, Theorem 2.12 (iv)].

It will be noted that the domain A of a generalized spectral measure is not universally fixed; neither is the base X of a Γ-spectral measure. Thus the injectivity of the relation between generalized spectral measures and the induced Γ-spectral measure may be destroyed by trivial extensions. The next proposition asserts that this is the worst that could happen.

PROPOSITION 1. *If the Γ-spectral measures induced by two generalized spectral measures have a common extension, then so do the generalized spectral measures themselves.*

Proof. Let $E_1: A_1 \rightarrow B(\bar{X})$ and $E_2: A_2 \rightarrow B(\bar{X})$ be two generalized spectral measures and let the Γ-spectral measures induced by them be $\tilde{E}_1: B_0 \rightarrow L(X_1)$ and $\tilde{E}_2: B_0 \rightarrow L(X_2)$. Suppose that $E: B_0 \rightarrow L(X)$ is a Γ-spectral measure extending both $\tilde{E}_1$ and $\tilde{E}_2$, that is $X_i \subset X$ and $E(\Delta)|X_i = \tilde{E}_i(\Delta)$ $(\Delta \in B_0;\ i = 1, 2)$. Let A be the set of all Borel sets Δ such that (i) $||E(\Delta)|| < +\infty$ and (ii) $\overline{E(\Delta)X} \subset X$. Since E is a Γ-spectral measure, it follows that $E(\sigma)$ is closable for each $\sigma \in B_0$. This proves that if $\Delta \in A$, $\delta \in B_0$ and $\delta \subset \Delta$, then $\delta \in A$. Now it is easily seen that A is an admissible subring of B_0 and that if, for each $\Delta \in A$, $\bar{E}(\Delta)$ denotes the continuous extension of $E(\Delta)$ to $\bar{X}$, then $\bar{E}: A \rightarrow B(\bar{X})$ is a generalized spectral measure. Further $A_i \subset A$ and $E_i(\Delta) = \bar{E}(\Delta)$ $(\Delta \in A_i;\ i = 1, 2)$. Hence $\bar{E}$ extends both E_1 and E_2, and this completes the proof of proposition.

2. Examples. In this section we shall give some examples of generalized spectral measures. In particular, a class of differential operators studied by Jack Schwartz [4] (see also [2, § XIX.5]) do in fact have, associated with them, eigenfunction expansions which are unconditionally convergent in a suitable norm.

THEOREM 2. *Suppose that T is a discrete operator in $\bar{X}$ and that $\{\lambda_n: n = 1, 2, \cdots\}$ is an enumeration of $\sigma(T)$. For each $n = 1, 2, \cdots$, let $E(\lambda_n)$ denote the spectral projector associated with λ_n. Further suppose that*

(i) $\overline{\text{sp}}\,(T) = \overline{\text{sp}}\,\{\bigcup\,[E(\lambda_n)\bar{X}: n = 1, 2, \cdots]\} = \bar{X}$; *and*

(ii) $C_\infty(T) = \{x \in \bar{X}: E(\lambda_n)x = 0\ (n = 1, 2, \cdots)\} = \{0\}$.

Then E induces a C-spectral measure in the following manner: For each $\Delta \in B_0$, we define $E(\Delta): X \to X$ as $E(\Delta)x = \sum_{\lambda \in \Delta} E(\lambda)x$ $(x \in X)$, where $X = \text{sp}\{\bigcup [E(\lambda_n)\bar{X}: n = 1, 2, \cdots]\}$.

Moreover, if all but finite number of spectral points are simple poles of the resolvent function $R(\cdot; T)$, then T is C-spectral.

All we need to observe in proving this theorem is that A, the set of all bounded Borel subsets of Λ, is an admissible subring of B_0 and that $E: A \to B(\bar{X})$ is a generalized spectral measure.

COROLLARY 1. *Let the situation be as in the previous theorem. Then there exists a norm* $|\cdot|$ *on* $\bar{X}$ *such that each element* $x \in \bar{X}$ *has spectral expansion of the form*

$$x = \sum_{n=1}^{\infty} E(\lambda_n)x ,$$

where the series converges unconditionally in the $|\cdot|$*-norm topology.*

We have only to take $|\cdot|$ to be the outer spectral norm associated with E; and observe that, for each $x \in \bar{X}$, the mapping $E(\cdot)x: B_0 \to \bar{X}$ is countably additive in the $|\cdot|$-norm topology [1, Theorem 2.10].

Examples of operators satisfying the conditions of the theorem are due to Jack Schwartz [2, Theorem XIX.5.8] and Browder [2, Theorem XIV.6.28]. The next theorem is related to the result above.

THEOREM 3. *Let* $\{\xi_n, \xi_n^*: n = 0, 1, 2, \cdots\}$ *be an* $\bar{X}$*-complete biorthogonal system such that* $\{\xi_n^*\}$ *is total in* $\bar{X}^*$. *Then there exists a norm on* $\bar{X}$ *for which the expansion*

$$x = \sum_{n=0}^{\infty} \langle x, \xi_n^* \rangle \xi_n$$

is unconditionally convergent for each element x *of* $\bar{X}$.

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