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**THE CONVERSE TO A THEOREM OF CONNER AND FLOYD**

ALLAN L. EDELSON

## THE CONVERSE TO A THEOREM OF CONNER AND FLOYD

A. L. EDELSON

**If  $W^{2n}$  is a manifold with almost complex structure  $J: \tau(W) \rightarrow \tau(W)$  on its tangent bundle, then a conjugation on  $W$  is a smooth involution  $T: W^{2n} \rightarrow W^{2n}$  whose differential anti-commutes with  $J$ , i.e.,  $T_*J = -JT_*$ . Examples of such actions are those induced by complex conjugation of coordinates in  $P^n(C)$  and  $H_{m,n}(C)$  having fixed point sets  $P^n(R)$  and  $H_{m,n}(R)$  respectively.**

Conner and Floyd have proved that the fixed point set of a conjugation is always an  $n$ -dimensional submanifold if it is nonempty. Furthermore, they show that if  $F^n$  denotes the fixed point set of the conjugation  $T: W^{2n} \rightarrow W^{2n}$  and  $[\ ]_2$  denotes the nonoriented cobordism class, then  $[W^{2n}]_2 = [F^n \times F^n]_2$ . In this article we prove that every closed  $n$ -manifold is the fixed point set of a conjugation on a closed  $2n$ -dimensional almost complex manifold.

The technique of the proof involves modification of the authors previous work on the case of stable almost complex structures, that is a conjugation of an almost complex structure on the stable tangent bundle  $\tau(W^{2n}) \oplus \theta^k$ ,  $k > 0$ . The proof consists of showing that if for every  $n > 0$  the sphere  $S^n$  is fixed point set of a conjugation, then every closed  $n$ -manifold is also. This proof involves a suggestion made by R. Stong. Next we describe an almost complex manifold  $W^{2n}$  having conjugation fixing  $S^n$ . We use generalized equivariant surgery, and rely heavily on the fact that a regular neighborhood of the fixed point set is diffeomorphic to the tangent disc bundle. Note that every manifold is fixed point of a conjugation on an open manifold; namely, the bundle involution on its tangent disc bundle.

**THEOREM.** *Let  $M^n$  be a smooth closed  $n$ -manifold. Then there exists a smooth closed almost complex manifold  $W^{2n}$  with conjugation  $T: W^{2n} \rightarrow W^{2n}$  having fixed point set  $M^n$ .*

*Proof.* It follows from [5] that the nonoriented cobordism ring can be generated by the manifolds  $P^{2n}(R)$  and  $H_{m,n}(R)$ , where the latter is the hypersurface in  $P^m(R) \times P^n(R)$  defined by  $\sum_{i=0}^{\min(m,n)} x_i y_i = 0$ . Complex conjugation of coordinates defines conjugations on the corresponding complex manifolds  $P^{2n}(C)$  and  $H_{m,n}(C)$ , so it follows that the generators of  $\eta_*$  are fixed point sets of conjugations. It then follows from [3] that if  $M^n$  is any manifold, there is an almost com-

plex manifold  $V^{2n}$  with conjugation  $S: V^{2n} \rightarrow V^{2n}$  having fixed point set  $F^n$ , and such that  $M^n$  can be obtained from  $F^n$  by a sequence of surgeries. We will show that any such modification of  $F^n$  can be extended to an equivariant modification of  $V^{2n}$ , which preserves the almost complex structure and conjugation.

We now make the assumption that for every  $n > 0$  there is a closed almost complex manifold  $W^{2n}$  with conjugation  $T: W^{2n} \rightarrow W^{2n}$  having fixed point set  $S^n$ .

LEMMA. *If  $F^n$  is the fixed point set of the conjugation  $S: V^{2n} \rightarrow V^{2n}$ , then any manifold obtained from  $F^n$  by surgery on an imbedded sphere is also the fixed point set of a conjugation on some almost complex manifold.*

*Proof.* Let  $f_0: S^p \rightarrow F^n$ ,  $0 \leq p < n$  be an imbedding with trivial normal bundle. Then  $f_0$  extends to an imbedding  $f: S^p \times D^{n-p} \rightarrow F^n$ . The restriction to  $f(S^p \times D^{n-p})$  of the tangent bundle  $\tau(F^n)$  is trivial and again by [1: 24. 2] the almost complex structure on  $V^{2n}$  defines an isomorphism  $\tau(F^n) \xrightarrow{\cong} \nu(F^n)$  where  $\nu(F^n)$  denotes the normal bundle of  $F^n$  in  $V^{2n}$ . By this isomorphism we can extend  $f$  to an imbedding  $F: S^p \times D^{n-p} \times D^n \rightarrow V^{2n}$ , equivariant with respect to the involution given by  $-1$  in the factor  $D^n$ . This follows since at a fixed point of the involution  $S$ , the representation is multiplication by  $-1$  in  $\nu(F^n)$ . Similarly if  $T: W^{2n} \rightarrow W^{2n}$  is a conjugation with fixed point set  $S^n$ , let  $G: D^{p+1} \times S^{n-p-1} \times D^n \rightarrow W^{2n}$  be the equivariant imbedding induced by the standard inclusion  $g: S^{n-p-1} \rightarrow S^n$ . There is a diffeomorphism

$$h: F(S^p \times (D^{n-p} - \{0\}) \times D^n) \longrightarrow G((D^{p+1} - \{0\}) \times S^{n-p-1} \times D^n)$$

given by  $h(F(u, tv, w)) = G(tu, v, w)$  for  $0 < t \leq 1$ . It is clear that  $h$  is equivariant. The almost complex structures define isomorphisms between the tangent and normal bundles to the fixed point sets, so it follows that the differential  $h_*$  preserves the almost complex structure.

Now let  $M^{2n}$  be the manifold obtained from  $V^{2n} - F(S^p \times \{0\} \times D^n)$  and  $W^{2n} - G(\{0\} \times S^{n-p-1} \times D^n)$  by identifying the submanifolds  $F(S^p \times (D^{n-p} - \{0\}) \times D^n)$  and  $G((D^{p+1} - \{0\}) \times S^{n-p-1} \times D^n)$  using the diffeomorphism  $h$ . Then  $M^{2n}$  has an almost complex structure and conjugation induced by  $T$  and  $S$ . The fixed point set of this conjugation is obtained from  $F^n - F(S^p \times (D^{n-p} - \{0\}) \times \{0\})$  and  $S^n - G((D^{p+1} - \{0\}) \times S^{n-p-1} \times \{0\})$  by identifying the appropriate submanifolds using the restriction of  $h$ . This is precisely the manifold obtained from  $F^n$  by surgery on the imbedded sphere  $f_0(S^p)$ .

We will now construct for each  $S^n$ , an almost complex manifold  $W^{2n}$  with conjugation  $T: W^{2n} \rightarrow W^{2n}$  having  $S^n$  as fixed point set.

Let  $D(S^n)$  denote the tangent disc bundle to  $S^n$  and  $\tau_1(S^n)$  its boundary, the unit tangent bundle. Then  $D(S^n)$  can be described as the submanifold of  $S^{2n+1}$  consisting of vectors  $\{(x, y) \in R^{n+1} \times R^{n+1}\}$  satisfying the conditions  $x \cdot x + y \cdot y = 2$ ,  $x \cdot y = 0$ ,  $0 < x \cdot x \leq 1$ . We take the sphere of radius 2 for convenience. Identifying  $(x, y)$  with the complex vector  $z = x + iy$  in  $C^{n+1}$ , the unit tangent bundle  $\tau_1(S^n)$  is described by the equation  $\sum_0^n Z_i^2 = 0$ . Define involutions  $T_j: D(S^n) \rightarrow D(S^n)$  for  $j = 1, 2$ , by  $T_1(x, y) = (x, -y)$  and  $T_2(x, y) = (-x, y)$ . Then  $T_1$  corresponds to multiplication by  $-1$  in the fibers of  $D(S^n)$  and so has fixed point set equal to  $S^n$ .  $T_2$  reduces to the antipodal involution on  $S^n$  and has no fixed points.

We will now describe almost complex structures  $J_1$  and  $J_2$  on  $D(S^n)$  with respect to which  $T_1$  and  $T_2$  are conjugations. At a point  $(x, y) \in D(S^n)$  the tangent space  $\tau_{(x,y)}(D(S^n))$  consists of all vectors  $(u, v) \in R^{n+1} \times R^{n+1}$  satisfying the equations

- (1)  $x \cdot u + y \cdot v = 0$
- (2)  $y \cdot u + x \cdot v = 0$ .

Define

$$J_1 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} |x| \left( -v + \frac{v \cdot x}{|x|^2} x \right) - \frac{y \cdot u}{|x|^3} x \\ \frac{v \cdot y}{|x|} x + \frac{u}{|x|} - \frac{x \cdot u}{|x|^3} x \end{pmatrix}$$

$$J_2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{u \cdot x}{|y|} y + \frac{v}{|y|} - \frac{y \cdot v}{|y|^3} y \\ |y| \left( -u + \frac{u \cdot y}{|y|^2} y \right) - \frac{x \cdot v}{|y|^3} y \end{pmatrix}.$$

It can be verified that  $J_1^2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -u \\ -v \end{pmatrix}$  and  $J_2^2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -u \\ -v \end{pmatrix}$  so that these formulae describe almost complex structures at the point  $(x, y)$ . The maps  $T_1$  and  $T_2$  extend to  $R^{n+1} \times R^{n+1}$  so their differentials are given by  $T_{1*} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ -v \end{pmatrix}$  and  $T_{2*} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -u \\ v \end{pmatrix}$ . Again it can be verified that  $T_{1*} \circ J_{1(x,y)} = -J_{1(x,-y)} \circ T_{1*}$  and  $T_{2*} \circ J_{2(x,y)} = -J_{2(-x,y)} \circ T_{2*}$ , so that the involutions  $T_1$  and  $T_2$  are in fact conjugate linear. Now define a diffeomorphism  $h: \tau_1(S^n) \rightarrow \tau_1(S^n)$  by  $h(x, y) = (y, x)$ . Form a closed manifold  $W^{2n}$  from two copies of  $D(S^n)$  by identifying them along  $t_1(S^n)$  using  $h$ . Then  $W^{2n}$  can be made into a smooth manifold and since  $h \circ T_1(x, y) = h(x, -y) = (-y, x)$ ,  $T_2 h(x, y) = T_2(y, x) = (-y, x)$ , it follows that  $W^{2n}$  can be given an involution  $T: W^{2n} \rightarrow W^{2n}$  given by  $T_1$  on the first copy of  $D(S^n)$  and by  $T_2$  on the second. It is clear that the fixed point set of  $T$  equals the fixed point set of  $T_1$ , which is  $S^n$ . It remains to show that  $W^{2n}$  is an almost complex manifold.

There are almost complex structures defined on each copy of  $D(S^n)$  so  $W^{2n}$  is almost complex provided the identification map  $h$  has differential which commutes with  $J_1$  and  $J_2$ . We note that there is a commutative diagram.

$$\begin{array}{ccc} t_{(x,y)}D(S^n) & \xrightarrow{h_*} & t_{(y,x)}D(S^n) \\ \downarrow J_1 & & \downarrow J_2 \\ t_{(x,y)}D(S^n) & \longrightarrow & t_{(y,x)}D(S^n) \end{array}$$

This follows since

$$h_* \circ J_{1(x,y)} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{r \cdot y}{|x|} x + \frac{u}{|x|} - \frac{x \cdot u}{|x|^3} x \\ |x| \left( -r + \frac{r \cdot x}{|x|^2} x \right) - \frac{y \cdot y}{|x|^3} x \end{pmatrix} = J_{2(y,x)} \circ h_* \begin{pmatrix} u \\ v \end{pmatrix}.$$

Then  $W^{2n}$  is an almost complex manifold and  $T$  is a conjugation which completes the proof.

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