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William Alan Feldman and James Franklin Porter

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COMPACT CONVERGENCE AND THE ORDER BIDUAL FOR $C(X)$

W. A. FELDMAN AND J. F. PORTER

An order-theoretic characterization of the topology of compact convergence on the lattice $C(X)$ of all continuous real-valued functions on X is provided for a realcompact space X, analogous to the order unit characterization for compact X. The approach is to generalize the concept of an order unit to permit consideration of locally convex topologies. The characterization is then achieved by viewing $C(X)$ as a subspace of its order bidual. In addition, the bidual is employed to provide an order-theoretic description of the continuous convergence structure on $C(X)$.

Semiorder-units in a vector lattice and the locally convex topology they generate are introduced in § 1, generalizing the concept of order units and their associated seminorm topology. For a realcompact space X it is shown that the semiorder-unit (sou) topology on $C(X)$, the lattice of continuous real-valued functions on X, is the topology of compact convergence if and only if X is a union of open compact sets (Theorem 1). To describe the topology of compact convergence *via* sou's for an arbitrary realcompact space requires the material of § 2. In that section, an extension of $C(X)$ which contains an ample number of sou's is introduced. This is the space $\widetilde{C(X)}$, all limits in the order bidual of order convergent nets from $C(X)$. That $\widetilde{C(X)}$ is a sublattice of the bidual is a consequence of Theorem 2, which establishes that a vector lattice together with order convergence is a convergence vector lattice. The main result, developed in § 3, describes the topology of compact convergence as the sou topology on $\widetilde{C(X)}$ restricted on $C(X)$ for any realcompact X (Theorem 3). The final section is devoted to characterizing the continuous convergence structure on $C(X)$ (Theorem 4) *via* the bidual and unbounded order convergence.

1. The semiorder-unit topology. We recall that an element u of a partially ordered vector space V is said to be an *order unit* if for each v in V there is a $\lambda > 0$ such that $v \leqq \lambda u$. If X is a compact space and u is an order unit in $C(X)$, the functional p defined by $p(f) = \bigwedge \{\lambda > 0 : |f| \leqq \lambda u\}$ is a norm on $C(X)$ generating the topology of uniform convergence. In particular, u can be chosen to be the constant function 1, in which case p is the usual supremum norm.

We wish to provide an analogous characterization of $C(Y)$ with the topology of compact convergence when Y is a completely regular (Hausdorff) space. We first note that the vector lattices $C(Y)$ and $C(\upsilon Y)$ are lattice-isomorphic where υY denotes the Hewitt realcompactification of Y (see [7], p. 118). We will therefore identify the vector lattices $C(Y)$ and $C(\upsilon Y)$, and reserve the letter X to denote realcompact spaces. We observe that if there is an order unit u in $C(X)$ then u must be bounded, since there is a $\lambda > 0$ such that $u^2 \leqq \lambda u$. It follows that $C(X)$ has order units if and only if each continuous function on X is bounded—that is, if and only if X is compact. This last equivalence follows from the fact that a realcompact space is compact if and only if it is pseudo-compact (see [7], p. 79). The following concept may prove useful in vector lattices which lack order units.

DEFINITION 1. Let V be a vector lattice. We call a positive element u in V a *semiorder-unit* (sou) if for each v in V there is a $\lambda > 0$ such that $v \wedge nu \leqq \lambda u$ for all n in the set N of positive integers.

It is easy to verify that every order unit in a vector lattice is a sou. Analogously to the way a seminorm is associated to an order unit, we associate a seminorm to a sou. We state this as a proposition whose proof is routine.

PROPOSITION 1. *Let u be a sou in vector lattice V. The functional p defined by*

$$p(v) = \bigwedge \{\lambda > 0 : |v| \wedge nu \leqq \lambda u \ \textit{for all}\ n \in N\}$$

for v in V is a seminorm on V. If u is an order unit then this functional p is the usual seminorm associated to u (i.e. $p(v) = \bigwedge \{\lambda > 0 : |v| \leqq \lambda u\}$).

If u and u' are sou's in a vector lattice with the property that there exist real numbers α and β such that $\alpha u \leqq u' \leqq \beta u$, then it follows that their associated seminorms are equivalent. Although the seminorms associated to all order units in a vector lattice are equivalent, two sou's may have associated seminorms which are not equivalent.

DEFINITION 2. Let V be a vector lattice. By the *sou topology on V* we will mean the locally convex topology generated by the collection of seminorms associated to the family of all sou's in V.

If X is a discrete space, all characteristic functions of finite subsets of X are nonzero sou's in $C(X)$. It is easy to verify that the seminorms associated to this subcollection of sou's generate the topology of compact convergence. More generally, we have the following theorem.

THEOREM 1. *Let X be realcompact. The sou topology on $C(X)$ coincides with the topology of compact convergence if and only if X is a union of open compact sets.*

Proof. We begin by showing that the sou topology on $C(X)$ is always coarser than the topology of compact convergence. Let u be a sou in $C(X)$. Since u^2 is in $C(X)$ there is a $\delta > 0$ such that $u^2 \wedge nu \leqq \delta u$ for all $n \in N$. It follows that u is bounded by δ on X. Similarly, there is an $\varepsilon > 0$ such that $\sqrt{u} \wedge nu \leqq \varepsilon u$, which implies $u(x) \geqq 1/\varepsilon^2$ if $u(x) \neq 0$. The set $S = \{x \in X: u(x) \neq 0\}$ is open and closed; we will show that S is compact. Since S is closed and X is realcompact, it is sufficient to verify that every f in $C(X)$ is bounded on S (see [7], p. 126). Given f in $C(X)$, there exists a $\lambda > 0$ such that $f \wedge nu \leqq \lambda u$ for all n. In particular, for x in S it follows that $f(x) \leqq \lambda\delta$, and thus S is compact. The fact that u is bounded and bounded away from zero on S implies that the seminorm associated to u is equivalent to the seminorm $\|\cdot\|_S$ defined by

$$\|f\|_S = \bigvee \{|f(x)|: x \in S\} .$$

Thus the sou topology on $C(X)$ is coarser than the topology of compact convergence.

Let us assume that $X = \bigcup A_\alpha$, where each A_α is an open compact set. To show that the topology of compact convergence is coarser than the sou topology, we consider a compact subset K of X. Now K is contained in some finite union $\bigcup_{i=1}^n A_{\alpha_i}$, and the characteristic function of $\bigcup_{i=1}^n A_{\alpha_i}$ is a sou in $C(X)$. The seminorm associated to this characteristic function is $\|\cdot\|_{\cup_{i=1}^n A_{\alpha_i}}$ and dominates $\|\cdot\|_K$, as desired.

To prove the converse, let us assume that the sou topology on $C(X)$ is the topology of compact convergence. We will show that each x in X is contained in an open compact set. For x in X there are a finite number of sou's $u_1, \cdots, u_n$ with associated seminorms $p_1, \cdots, p_n$ satisfying $\bigvee_{i=1}^n p_i \geqq \|\cdot\|_{\{x\}}$. (Note that for u a sou with associated seminorm p we have εu a sou with associated seminorm p/ε for any $\varepsilon > 0$.) We claim that x is in the set

$$\{y \in X: u_i(y) \neq 0 \text{ for some } i = 1, \cdots, n\} .$$

For otherwise, there would exist a function f in $C(X)$ vanishing on this set with $f(x) = 1$, implying that $(\bigvee_{i=1}^{n} p_i)(f) = 0$ whereas $\| f \|_{\{x\}} = 1$. Thus $u_j(x)$ is nonzero for some j $(1 \leqq j \leqq n)$. It follows from the remarks at the beginning of the proof that $\{y \in X: u_j(y) \neq 0\}$ is an open compact set containing x, as desired.

In particular, if X is compact then the sou topology on $C(X)$ is the norm topology. We noted previously that, alternatively, this topology is generated by any order unit.

2. **The order bidual and $C(X)$.** It is clear from Theorem 1 that the topology of compact convergence on $C(X)$ is not the sou topology for many important spaces—for example $C(\boldsymbol{R})$, where $\boldsymbol{R}$ denotes the reals. $C(\boldsymbol{R})$ lacks characteristic functions for compact subsets; in fact, $C(\boldsymbol{R})$ has no nonzero sou's. To continue our study, we will consider $C(X)$ as a subspace of its order bidual $C(X)^{00}$. (For a vector lattice V, we denote by V^0 the vector lattice generated by the positive linear functionals on V, see [14], p. 24).

We recall that $C(X)^{00}$ is an order-complete vector lattice, and we will identify $C(X)$ with its natural embedding as a sublattice of $C(X)^{00}$. This embedding is a lattice isomorphism (see [14], p. 156).

We will utilize the following theorem, due to Hewitt (see [8], p. 179).

THEOREM A. *Let Y be a completely regular space. Let $C_{CO}(Y)$ denote $C(Y)$ together with the topology of compact convergence and $C_{CO}(Y)'$ denote its continuous dual. Then $C(Y)^0$ coincides with $C_{CO}(Y)'$ if and only if Y is realcompact.*

It will be convenient to utilize the following consequence of Theorem A.

THEOREM B. *Let X be a realcompact space and ϕ a positive linear functional on $C(X)$. There exists a compact subset K of X and a positive linear functional ϕ' on $C(K)$ such that $\phi = \phi' \circ r$, where r is the restriction mapping from $C(X)$ to $C(K)$.*

Proof. By Theorem A, ϕ is continuous with respect to the topology of compact convergence. Thus there exist a compact set K in X and an $\alpha > 0$ such that $|\phi(f)| \leqq \alpha \| f \|_K$ for all f in $C(X)$. This, together with the fact that r is onto, allows one to define the mapping ϕ' as follows: for $f' \in C(K)$ let $\phi'(f') = \phi(f)$, where $f \in C(X)$ and $r(f) = f'$. Clearly ϕ' is a nonnegative element in $C(K)^0$ and $\phi = \phi' \circ r$.

We remark that if X is locally compact as well as realcompact then $C(X)^{00}$ is precisely the space M defined by Mack in [13] (see p. 227).

Given a compact subset K of X the restriction mapping r from $C(X)$ into $C(K)$ induces a linear mapping r^* from $C(K)^0$ into $C(X)^0$ defined by $r^*(\phi) = \phi \circ r$ for all ϕ in $C(K)^0$. Similarly, r^* induces a linear mapping r^{**} from $C(X)^{00}$ into $C(K)^{00}$ defined by $r^{**}(F) = F \circ r^*$ for all F in $C(X)^{00}$.

We recall that an ideal I in a vector lattice V is called a *band* if the suprema in V of subsets of I are also in I.

LEMMA 1. (a) *The mapping r^* is a lattice isomorphism onto a band L in $C(X)^0$.*

(b) *The mapping r^{**} is a lattice homomorphism and there is a band M in $C(X)^{00}$ such that the restriction of r^{**} to M is an isomorphism onto $C(K)^{00}$. In fact, M is the set of members of $C(X)^{00}$ which vanish on the orthogonal complement (in $C(X)^0$) of L.*

Proof. The proof of (b) follows from (2.4), p. 331 and (2.5), p. 332 in [10]. To prove (a) we note that $C(X)/I$ is isomorphic to $C(K)$, where

$$I = \{f \in C(X): f(K) = 0\}$$

(see [11], p. 39). Thus $C(K)^0$ is isomorphic to $(C(X)/I)^0$, which is in turn isomorphic to the ideal $J = \{\phi \in C(X)^0: \phi(I) = 0\}$. Since J is a direct summand of $C(X)^0$ whose natural embedding map "is" r^*, we have the result.

A subset A of a vector lattice V is said to be *directed upward* (*downward*) if for a and b in A there is an element in A greater than or equal to (less than or equal to) both a and b.

LEMMA 2. (1) *Let $\{f_\alpha\}$ be a subset of $C(X)$ and $\phi \geqq 0$ in $C(X)^0$. If $\{f_\alpha\}$ is directed upward (downward) and bounded above (below) in $C(X)^{00}$, then in $C(X)^{00}$*

$$[\bigvee_\alpha f_\alpha](\phi) = \bigvee_\alpha [f_\alpha(\phi)]$$
$$([\bigwedge_\alpha f_\alpha](\phi) = \bigwedge_\alpha [f_\alpha(\phi)]) .$$

(2) *Let F and G be in $C(X)^{00}$ and ϕ_x be the point-evaluation functional at x in X. Then*

$$(F \vee G)(\phi_x) = F(\phi_x) \vee G(\phi_x)$$

and

$$(F \wedge G)(\phi_x) = F(\phi_x) \wedge G(\phi_x) .$$

Proof. For the proof of (1), see (2.2) of [9].

To prove (2), it is sufficient to show that $[F^+](\phi_x) = [F(\phi_x)]^+$, where $^+$ again denotes supremum with zero. We write

$$[F^+](\phi_x) = \bigvee \{F(\phi): 0 \leqq \phi \leqq \phi_x\} .$$

Now $0 \leqq \phi \leqq \phi_x$ implies $|\phi(f)| \leqq ||f||_{\{x\}}$ for all f in $C(X)$. Arguing as in the proof of Theorem B we see that $\phi = k\phi_x$ for some $0 \leqq k \leqq 1$. Thus

$$[F^+](\phi_x) = \bigvee \{kF(\phi_x): 0 \leqq k \leqq 1\} ,$$

which simplifies to $[F(\phi_x)]^+$.

The next lemma relates the order of $C(X)^{00}$ to the point-evaluation functionals.

LEMMA 3. (a) *If F and G belong to $C(X)^{00}$ and A and B are subsets of $C(X)$ such that $F = \bigvee \{f: f \in A\}$ and $G = \bigvee \{g: g \in B\}$, then $F \leqq G$ if and only if $\bigvee \{f(x): f \in A\} \leqq \bigvee \{g(x): g \in B\}$ for all x in X.*

(b) *If F and G belong to $\widetilde{C(X)}$, then $F \leqq G$ if and only if $F(\phi_x) \leqq G(\phi_x)$ for all x in X.*

Proof. (a) We can assume that A and B are directed sets by including suprema of finite subsets. Thus the sufficiency follows by Lemma 2. For the necessity, we note that by using Dini's theorem and Theorem A one can prove as in [9], (5.5) on p. 73, that if f is in $C(X)$ and D is a subset of $C(X)$ then $f = \bigvee \{h: h \in D\}$ if and only if $f(x) = \bigvee \{h(x): h \in D\}$ for all x in X. The proof of part (a) can be completed by interpreting (6.3), p. 76 in [9], in this setting.

(b) The sufficiency is clear. On the other hand, suppose $F(\phi_x) \leqq G(\phi_x)$ for all x in X. There exist nets $\{f_\alpha\}$ and $\{g_\alpha\}$ in $C(X)$ such that $F = \bigvee_\alpha \bigwedge_{\beta \geqq \alpha} f_\beta$ and $G = \bigwedge_\alpha \bigvee_{\beta \geqq \alpha} g_\beta$ (see [14], p. 44). It follows that for all α and α',

$$\Big(\bigwedge_{\beta \geqq \alpha} f_\beta\Big)(\phi_x) \leqq \Big(\bigwedge_{\beta \geqq \alpha'} g_\beta\Big)(\phi_x) .$$

By (6.5) in [9], p. 76, we conclude that

$$\bigwedge_{\beta \geqq \alpha} f_\beta \leqq \bigvee_{\beta \geqq \alpha'} g_\beta$$

so that $F \leqq G$.

We now demonstrate that $C(X)^{00}$ is "rich" in sou's. Let K be a compact subset of X. We define an element e_K in $C(X)^{00}$ by

$$e_K = \bigwedge \{f \in C(X): f \geqq 0 \text{ and } f(K) = 1\} ,$$

the infimum being taken in $C(X)^{00}$. We remark that the family of functions used in defining e_K satisfies the hypotheses of Lemma 2.

PROPOSITION 2. *For every compact suset K of X, the element e_K is a sou in $C(X)^{00}$.*

Proof. It is clear from Lemma 1 and the fact that $C(X)^0$ is order complete that $C(K)^0$ is a direct summand of $C(X)^0$. We will show that e_K vanishes on the orthogonal complement W of $C(K)^0$ (in $C(X)^0$). Let $\phi \geqq 0$ be in W. By Theorem B, ϕ is a nonnegative regular Borel measure with compact support K_ϕ. Thus ϕ is the sum of two nonnegative measures ϕ_1 and ϕ_2 supported on $K_\phi \cap K$ and $K_\phi \backslash K$ respectively. Since ϕ_1 is in $C(K)^0$, we obtain $\phi = \phi_2$ so that $\phi(K) = 0$. Since ϕ is regular, for any $\varepsilon > 0$ there is a closed set F contained in $K_\phi \backslash K$ such that $\phi(K_\phi \backslash F) < \varepsilon$. Let g be a function in $C(X)$ with $0 \leqq g \leqq 1$, $g(K) = 1$ and $g(F) = 0$. By the definition of e_K, we have $0 \leqq e_K(\phi) \leqq g(\phi) \leqq \| g \|_{K_\phi} \phi(K_\phi \backslash F) < \varepsilon$. Thus e_K is in the ideal M defined in Lemma 1. We will complete the proof that e_K is a sou in $C(X)^{00}$ by showing that it is an order unit in M. Let $\mathscr{A} = \{f \in C(X): f \geqq 0 \text{ and } f(K) = 1\}$. For ϕ in $C(K)^0$ and r the restriction map from $C(X)$ into $C(K)$,

$$\begin{aligned}(r^{**}e_K)(\phi) &= e_K(r^*\phi) = (\bigwedge \{f: f \in \mathscr{A}\})(r^*\phi) \\ &= \bigwedge \{f(r^*\phi): f \in \mathscr{A}\} = \bigwedge \{\phi(rf): f \in \mathscr{A}\} = \phi(1)\ ,\end{aligned}$$

the third step being a consequence of Lemma 2. Thus $r^{**}(e_K)$ is the constant function 1 in $C(K)^{00}$, so that e_K is an order unit in M.

For § 3 we wish to consider not $C(X)^{00}$ but a sublattice of $C(X)^{00}$ which contains $C(X)$ and the sou's e_K discussed above. This sublattice will be defined in terms of order convergence in $C(X)^{00}$. Recall that a net $\{x_\alpha\}$ in a vector lattice V is said to *order converge to zero* if there is a collection M of nonnegative elements of V directed downward with $\bigwedge \{m: m \in M\} = 0$ such that for each m in M there is an α' satisfying $|x_\alpha| \leqq m$ for $\alpha \geqq \alpha'$ (see [14]). Order convergence to other points of V is defined by translation. We denote by $\widetilde{C(X)}$ all elements in $C(X)^{00}$ which are order convergence limits of nets in $C(X)$. (When X is compact, $\widetilde{C(X)}$ is the sublattice U defined by S. Kaplan in [9].) It is clear that $\widetilde{C(X)}$ contains $C(X)$ and the sou's e_K.

By the *order convergence adherence* of a subset W of a vector lattice Z we will mean the set of all elements of Z which are limits under order convergence of nets in W. The following theorem is a consequence of the continuity of the vector lattice operations with respect to order convergence (see Theorem 14, [2], p. 362).

THEOREM 2. *The space* $\widetilde{C(X)}$, *the order convergence adherence of* $C(X)$ *in* $C(X)^{00}$, *is a sublattice of* $C(X)^{00}$ *containing as sou's all elements*

$$e_K = \bigwedge \{f \in C(X): f \geqq 0 \text{ and } f(K) = 1\}$$

for compact subsets K *of* X.

We remark that one can prove the more general result that any archimedean vector lattice together with its order convergence is a convergence vector lattice. (By a convergence vector lattice one means a convergence vector space (see [1]) with the property that the lattice operations are continuous.)

3. The topology of compact convergence on $C(X)$. Let $\widetilde{C(X)}$ denote the vector space $\widetilde{C(X)}$ of § 2 together with its sou topology. In this section we investigate the topology τ induced on $C(X)$ as a subspace of $\widetilde{C(X)}$. We first observe that τ is finer than the topology of compact convergence. Indeed, for K a compact subset of X and e its associated sou (the element e_K in Corollary 1) we verify that $\|f\|_K \leqq p_e(f)$ for all f in $C(X)$, where $\|f\|_K = \bigvee \{|f(x)|: x \in K\}$ and p_e is the seminorm associated to e (see Proposition 1). Let f be in $C(X)$. By definition of p_e, $|f| \wedge ne \leqq p_e(f)e$ for all n. By Lemma 2 we obtain $|f(x)| \wedge ne(\phi_x) \leqq p_e(f)e(\phi_x)$. In particular, $\|f\|_K \leqq p_e(f)$ since $e(\phi_x) = 1$ for x in K (see Lemma 2).

The central purpose of the section is to establish that τ coincides with the topology of compact convergence on $C(X)$. It is important for this goal that all sou's in $\widetilde{C(X)}$ are "similar" to the e's discussed above. Although stated for $\widetilde{C(X)}$, the following proposition is valid for any sublattice of $C(X)^{00}$ which contains $C(X)$.

PROPOSITION 3. *Let* X *be realcompact and* E *a sou in* $\widetilde{C(X)}$. *Then*

(1) *there is a real number* M *such that* $E(\phi_x) \leqq M$ *for all* x *in* X, *and*

(2) *the closure in* X *of* $\{x \in X: E(\phi_x) \neq 0\}$ *is compact, where* ϕ_x *denotes the point-evaluation functional at* x *in* X.

Proof. Let A denote $\{x \in X: E(\phi_x) \neq 0\}$. To prove (2) we assume that $\bar{A}$ is not compact. By $8E$ and 1.20 in [7], there is a function f in $C(X)$ and a sequence $\{x_n\}$ in A such that $f(x_n) = nE(\phi_{x_n})$. It follows from Lemma 2 and the fact that E is sou that

$$nE(\phi_{x_n}) = f(x_n) \wedge nE(\phi_{x_n}) \leqq \lambda E(\phi_{x_n})$$

for some $\lambda > 0$ and all $n \in N$, a contradiction. To prove (1), we assume to the contrary that there is a sequence $\{x_n\}$ in A such that $E(\phi_{x_n}) \geqq n^3$. Arguing as in the proof of Proposition 5.7 (i) in [13], p. 234, we define a measure

$$\mu = \sum_{n=1}^{\infty} \frac{1}{n^2} \phi_{x_n} .$$

Since $\bar{A}$ is compact, it follows from Theorem B that μ is in $C(X)^0$. But $E(\mu) \geqq E(\phi_{x_n})/n^2 \geqq n$ for all n, a contradiction.

The crucial fact relating the sou's in $C(X)$ to the topology of compact convergence on $C(X)$ is contained in the following proposition. We recall that $\|f\|_K = \bigvee \{|f(x)|: x \in K\}$.

PROPOSITION 4. *For X a realcompact space and E a sou in $\widetilde{C(X)}$, let K denote the closure in X of $\{x \in X: E(\phi_x) \neq 0\}$ and p_E be the seminorm associated to E (in Proposition 1). Then for all f in $C(X)$,*

$$p_E(f) \leqq \| f \|_K p_E(1) .$$

Proof. Let f in $C(X)$ be given. For x in K,

$$\begin{aligned} f(x) \wedge nE(\phi_x) &\leqq (\| f \|_K \cdot 1) \wedge nE(\phi_x) \\ &\leqq p_E[(\| f \|_K)1]E(\phi_x) \\ &= \| f \|_K p_E(1) E(\phi_x) . \end{aligned}$$

Since $E(\phi_x)$ is zero for x not in K we obtain by Lemma 2 that for all x in X, $(f \wedge nE)(\phi_x) \leqq \| f \|_K p_E(1) E(\phi_x)$. Thus by Lemma 3 (b), $(f \wedge nE) \leqq \|f\|_K p_E(1) E$. Now it follows from the definition of $p_E(1)$ that $p_E(f) \leqq \|f\|_K p_E(1)$.

The space $\widetilde{C(Y)}$, the order convergence adherence of $C(Y)$ in $C(Y)^{00}$, can be defined for any completely regular space Y and is order isomorphic to $\widetilde{C(\upsilon Y)}$, where υY is the Hewitt realcompactification of Y. However, $C(\upsilon Y)$ with the topology of compact convergence is homeomorphic to $C(Y)$ with the topology of compact convergence if and only if Y is realcompact. Thus, in view of Proposition 4 and the remarks at the beginning of this section, we have proved the following theorem.

THEOREM 3. *Let Y be a completely regular topological space and let the subspace $\widetilde{C(Y)}$ of $C(Y)^{00}$ have its sou topology. The topology induced on $C(Y)$ as a subspace of $\widetilde{C(Y)}$ coincides with the* *topology of compact convergence if and only if Y is realcompact.*

A more explicit description of the seminorms in question is given in the following proposition.

PROPOSITION 5. *Let X be realcompact.*

(1) *If K is a compact subset of X with associated sou e in* $\widetilde{C(X)}$, *then* $p_e(f) = \|f\|_K$ *for all f in* $C(X)$.

(2) *If E is a sou in* $\widetilde{C(X)}$ *and* $A = \{x \in X: E(\phi_x) \neq 0\}$ *then* $p_E(\cdot)$ *is equivalent to* $\|\cdot\|_{\bar{A}}$ *on* $C(X)$; *i.e., there exist positive real numbers* α *and* β *such that* $\alpha\|f\|_{\bar{A}} \leqq p_E(f) \leqq \beta\|f\|_{\bar{A}}$ *for all f in* $C(X)$.

Proof. By the remarks of the first paragraph of this section and Proposition 4 we have $\|f\|_K \leqq p_e(f) \leqq p_e(1)\|f\|_K$. For $\phi \geqq 0$ in $C(X)^0$, we can write $\phi = \phi_1 \circ r + \phi_2 \circ r$ with $e(\phi_1 \circ r) = \phi_1(1)$ and $e(\phi_2 \circ r) = 0$, as in the proof of Proposition 2. Thus $(1 \wedge ne)(\phi) \leqq \phi_1(1) = e(\phi)$, so that $p_e(1) \leqq 1$, establishing (1). For (2), we observe that $|f(x)| \wedge nE(\phi_x) \leqq p_E(f)E(\phi_x)$ by Lemma 2. By proposition 3 there is an $M > 0$ such that $E(\phi_x) \leqq M$, and $E(\phi_x) \neq 0$ for x in A. Thus

$$\bigvee \{|f(x)|: x \in A\} \leqq p_E(f)M\,,$$

which together with Proposition 4 implies (2).

4. The continuous convergence structure on $C(X)$.

In this section we provide an order-theoretic description of the continuous convergence structure on $C(X)$ which extends some results of Kutzler [12].

For any space Y, we recall that the continuous convergence structure (see [1]) on $C(Y)$ is the coarsest convergence structure σ on $C(Y)$ such that the evaluation map ω from $C_\sigma(Y) \times Y$ into the reals, defined by $\omega(f, x) = f(x)$, is continuous. The space $C(Y)$ together with the continuous convergence structure is denoted by $C_c(Y)$. We say that a net converges to a function f in $C_c(Y)$ if its filter of final sections converges to f. It is obvious that a filter in $C_c(Y)$ converges if and only if its associated net converges.

We recall [5], [6] that a net $\{x_\alpha\}$ in a vector lattice V *unbounded order converges to zero* if each bounded net $\{y_\alpha\}$ (i.e. $|y_\alpha| \leqq v$ for some $v \in V$ and all α) with $|y_\alpha| \leqq |x_\alpha|$ order converges to zero. (This means that there exists a subset M of V directed downward with infimum zero such that for each $m \in M$ there is an α_m satisfying $|y_\alpha| \leqq m$ for $\alpha \geqq \alpha_m$.) Given a sublattices W of V, we will say that net $\{x_\alpha\}$ in W *unbounded order converges to zero in W as a subspace* *of* V if each net $\{y_\alpha\}$ in W which is bounded in W (i.e. $|y_\alpha| \leqq w$ for some $w \in W$ and all α) with $|y_\alpha| \leqq |x_\alpha|$ has the following property: there exists a subset M of W directed downward with infimum zero in V such that for each $m \in M$ there is an α_m satisfying $|y_\alpha| \leqq m$ for $\alpha \geqq \alpha_m$. Again, convergence to other points is defined by translation.

The following theorem is a consequence of two results in [12]: *Satz* 1.1 and *Satz* 1.4. For convenience, we include a complete proof of the theorem.

THEOREM 4. *Let* Y *be a completely regular topological space and* υY *its Hewitt realcompactification. A net converges in* $C_c(\upsilon Y)$ *if and only if it unbounded order converges in* $C(\upsilon Y)$ *as a subspace of* $C(\upsilon Y)^{00}$ *(or, by identification, in* $C(Y)$ *as a subspace of* $C(Y)^{00}$.)

Proof. Let X denote υY. We first suppose that net $\{f_\alpha\}$ unbounded order converges to zero in $C(X)$ as a subspace of $C(X)^{00}$. Corresponding to the bounded net $\{|f_\alpha| \wedge 1\}$ there is a subset M of $C(X)$ directed downward with infimum zero in $C(X)^{00}$ such that for each $m \in M$ there is an α_m satisfying $|f_\alpha| \wedge 1 \leqq m$ for $\alpha \geqq \alpha_m$. For $p \in X$ and $0 < \varepsilon < 1$, Lemma 3 (a) implies that $m(p)$ is less than $\varepsilon/2$ for some m in M. Since m is continuous, there is a neighborhood U_p of p such that for $\alpha \geqq \alpha_m$,

$$(|f_\alpha| \wedge 1)(U_p) = |f_\alpha|(U_p) \subseteqq [0, \varepsilon) .$$

This implies that $\{f_\alpha\}$ converges to zero in $C_c(X)$. Conversely, suppose that $\{f_\alpha\}$ is a net convergent to zero in $C_c(X)$. If $\{g_\alpha\}$ is a net bounded by a function g_0 in $C(X)$ and satisfying $|g_\alpha| \leqq |f_\alpha|$, then clearly $\{g_\alpha\}$ converges to zero in $C_c(X)$. Thus for $p \in X$ and $\varepsilon > 0$ there is a neighborhood U_p of p such that $g_\alpha(U_p) \subseteqq (-\varepsilon, \varepsilon)$ for all α beyond some α'. By the complete regularity of X there exists a function $h_{p,\varepsilon} \geqq \varepsilon$ in $C(X)$ with value ε at p and values greater than or equal to g_0 on $X \backslash U_p$. The set M of all infima of finite subcollections of $\{h_{p,\varepsilon}: p \in X$ and $\varepsilon > 0\}$ is directed downward, and for each $m \in M$ there is an α_m such that $|g_\alpha| \leqq m$ for $\alpha \geqq \alpha_m$. Lemma 3 (a) implies that $\bigwedge \{m: m \in M\}$ is zero in $C(X)^{00}$, completing the proof.

In contrast to Theorem 3, we have the following.

COROLLARY 1. *Let* Y *be a completely regular topological space. The space* Y *is realcompact if and only if the topology of compact convergence is the finest locally convex topology* τ *on* $C(Y)$ *with the* *property that every net which unbounded order converges in $C(Y)$ as a subspace of $C(Y)^{00}$ also converges in τ.*

Proof. This is an immediate consequence of the fact that the topology of compact convergence is the finest locally convex topology on $C(Y)$ coarser than the continuous convergence structure (see [4]).

COROLLARY 2. *Let X be realcompact. Unbounded order convergence of nets in $C(X)$ as a subspace of $C(X)^{00}$ defines a topology if and only if X is locally compact. This topology is the topology of compact convergence.*

Proof. It is known (see [3], p. 329) that the continuous convergence structure on $C(X)$ defines a topology if and only if X is locally compact.

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