COMPLETION AND SEMICOMPLETION OF MOORE SPACES

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Two properties are given which characterize those meta-compact Moore spaces that are completable and two which characterize those which are semicompletable. The relationship among these properties in nonmetacompact spaces is investigated. Two new characterizations of Moore-closed spaces are given as is a characterization of developability. Some unification of technique is achieved among the various ways of completing or semicompleting a Moore space.

In 1962, O. H. Alzoobaee stated two properties A and B of developments and showed that every Moore space having a development with property (A) B is (semicompletable) completable and every (semicomplete) complete Moore space has a development with property (A) B. In the present work, it is shown that every *metacompact* (semicompletable) completable Moore space has a development with property (A) B. Related properties A' and B' are investigated. It is shown that a Moore space is Moore-closed iff *every* strong development has property A and iff every strong development has property B. Finally, a developable topological space is shown to have a sequence $\langle G_n \rangle$ of open covers such that for each point p and open set U containing p, there is an n such that only one member of G_n contains p and it is a subset of U.

Complete Moore spaces were introduced in [7] and it was shown in [12] and [17] that a metrizable space is metrically topologically complete iff it is a complete Moore space. Semicompleteness (sometimes called Rudin completeness), a weaker property, was introduced in [14] and shown to be equivalent to completeness in metrizable spaces. Unlike the theories of metrizable or uniformizable spaces, there exist Moore spaces that cannot be embedded in any semicomplete Moore space [14], [9] and there exist semicomplete Moore spaces that cannot be completed [14]. The question naturally arises then of characterizing (semi-) completable Moore spaces. Whipple [15] has characterized completable Moore spaces in terms of Cauchy sequences and Creede [3] has what might be called an external characterization in terms of the Wallman compactification. An external characterization of semicompletability is given by J. N. Reed in [11]. Stronger completeness properties are investigated in [4] and [5]. The properties of Alzoobaee mentioned above were introduced in [1] and reported

in [2] and are internal properties. The above mentioned characteriza-
tion of developability has a bearing on the idea of θ-base [16] and
is motivated in part by [13].

1.2. *Terminology.* A *development* for a topological space S is
a sequence $G = G_1, G_2, \cdots$ of open covers of S such that for each point
x and neighborhood U of x, there is an n such that st $(x, G_n) \subseteq U$. The
development G will be called *monotonic* if $G_n \supseteq G_{n+1}$ for each n and
nested if in addition, $\bar{G}_{n+1}$ refines G_n. A Moore space is a regular
Hausdorff developable space. A Moore space is (1) complete (2) semi-
complete iff there is a monotonic development G for S such that if
$M_1, M_2, \cdots$ is a nonincreasing sequence of closed point sets and $g_1, g_2, \cdots$
is a sequence such that for each n, $g_n \in G_n$, then (1) if $M_n \subseteq \bar{g}_n$ for
each n, then $\cap M_n \neq \varnothing$, (2) if $\bar{g}_{n+1} \subseteq g_n$ for each n, then $\cap \bar{g}_n \neq \varnothing$. It
is shown in [14] that if the closure symbol is omitted in (1) or in its
first occurrence in (2), then the resulting statements are equivalent
to those given. If G is a monotonic development for a Moore space
S then $F = \{f_n : n \in N\}$ is (1) a G-filtersubbase, (2) a G-filterbase, (3)
a G-nest [2] iff for each n, $f_n \in G_n$ and (1) F is a filtersubbase (i.e.,
the intersection of each finite subcollection of F is nonempty), (2)
$f_n \supseteq f_{n+1}$ for each n, (3) $f_n \supseteq \bar{f}_{n+1}$ for each n, respectively. A filter-
subbase F intersects a set M iff every set in F intersects M and two
filterbases are said to intersect iff every set in one intersects every
set in the other. If G is a development for the Moore space S, then
G is said to have property A', A, B', B according as G is monotonic
and (A') if U and V are members of G_1 and U is a proper subset of
V and F is a G-nest intersecting U, then there is a positive integer
n such that if $W \in G_n$ and F intersects W, then $W \subseteq V$.

(A) A is obtained from A' by replacing the phrase "U is a proper
subset of V" with the phrase "$\bar{U} \subseteq V$".

(B')   If $U \in G_1$ and F is a G-filtersubbase some member of which
is a proper subset of U, then there is a positive integer n such that
if $W \in G_n$ and F intersects W, then $\bar{W} \subseteq U$.

(B) B is obtained from B' by replacing the phrase "some member
of which is a proper subset of U" with the phrase "the closure of
some member of which is included in U".

It should be observed that if G has property A' or B', then G_1
is a *nested basis* for the topology of S; that is, if U and V belong to
G_1 and U is a proper subset of V, then $\bar{U} \subseteq V$. The argument for
Theorem 9 of [5] shows that every metacompact Moore space has a
nested basis. In 3.3 a nonmetacompact Moore space having a nested
basis is indicated.

1.3. **Preliminaries.** If G is a nested development for the Moore

space S and α is a sequence of points and U is an open set, then α is said to be *G-interior* to U [15] iff (1) α is G-Cauchy (for each n, some member of G_n contains a final segment of α) and (2) for some n, every member of G_n containing a final segment of α is a subset of U. The open set U is G-interior to the open set V iff every G-Cauchy sequence having a final segment in U is G-interior to V. In [15] it is proved that a Moore space S is completable iff there is a nested development G for S such that if α is a G-interior sequence of the open set U, then for each n there is a G-interior subset of U belonging to G_n and containing a final segment of α. A developmet with this property will be said to have Whipple's property. It is easily demonstrated that if G is a nested development with Whipple's property, then (1) for each G-Cauchy sequence α, there is a G-nest $G(\alpha) = \{g_n(\alpha): n \in N\}$ such that $g_{n+1}(\alpha)$ is G-interior to $g_n(\alpha)$ and contains a final segment of α, (2) $G(\alpha)$ intersects $G(\beta)$ iff $G(\alpha)$ and $G(\beta)$ are equivalent filterbases. Finally, some unification of technique among [15], [1], [2], [11], and the more general [8] (see [6] for additional references and results using the technique of [8]) is obtained by observing that if G is a nested development with Whipple's property, H is a collection of free nonintersecting G-nests such that every free G-nest intersects a member of H and for each open set U in S, $U_H^* = U \cup \{F \in H: U$ includes a member of $F\}$, then S_H^*, with $\{U_H^*: U$ open in $S\}$ as a basis, is a complete Moore space having S as a dense subspace and G_H^*, defined in the obvious fashion, is a complete monotonic development for S_H^*.

2. Completion and semicompletion in arbitrary Moore spaces.

LEMMA 2.1. *If S is a complete* (semicomplete) *Moore space and G is a monotonic development for S, then there is a complete* (semicomplete) *nested development H for S such that for each n, $H_n \subseteqq G_n$.*

The argument for the complete case is entirely trivial and for the semicomplete case is a straightforward application of Rudin's lemma ([14], Theorem 2).

LEMMA 2.2. *The property that every x of the Moore space S has a neighborhood having no point other than x as a limit point implies that S is semicomplete but not that S is complete.*

Proof. Under the stated hypothesis, there is a nested development G for S such that if $U \in G_1$ then either U is singleton or for some point x of U, x is the only limit point of U. Suppose for each n, $g_{n+1} \subseteqq g_n \in G_n$. If for some n, g_n is singleton, then $\cap g_i = g_n$. Suppose

for each n, g_n has only one limit point, x_n. If for some i and j, $x_i \neq x_j$, then for each $n > i + j$, no nonsingleton member of G_n is a subset of $g_i \cap g_j$, contrary to supposition. Thus, $x_n = x_1$ for each n and $\cap g_n = \{x_1\}$. Thus, G is a semicomplete development for S.

Theorem 9 of [14] provides an example of a semicomplete Moore space that is not complete. If this example is modified by requiring each point above the x-axis to be open, then the resulting space satisfies the hypotheses of this lemma and Rudin's argument, with only the obvious modifications, shows the space is not complete.

THEOREM 2.3. *If D is an open* (semi-) *complete subspace of the* (semi-) *completable space S, there is a* (semi-) *completion of S having D as an open subset.*

Proof (Complete Case). Let S' denote a completion of S and for each subset A of S, let $\bar{A}$ denote the closure of A in S'. Then $D \cup \overline{S - D}$ is a completion of S having D as an open subset. For clearly D is open in this space. Let $\mathrm{Cl}\, A = $ closure in $D \cup \overline{S - D}$ of A.

Since $\overline{S - D}$ is closed in S', there exist sets $U_1, U_2, \cdots$ open in S' such that $\overline{S - D} = \bigcap_i U_i$. Let $F = F_1, F_2, \cdots$ denote a complete development for D such that for each n and each $f \in F_n$, $\mathrm{Cl}_D f = \mathrm{Cl}\, f$.

Let $G = G_1, G_2, \cdots$ denote a complete development for S'. For each i, let

$$H_i = \{g \cap (D \cup \overline{S - D}): g \in G_i \quad \text{and} \quad \bar{g} \subseteqq U_i\} .$$

Finally, for each i, let $K_i = F_i \cup H_i$. Clearly, $K = K_1, K_2, \cdots$ is a development for $D \cup \overline{S - D}$. Suppose $M_1, M_2, \cdots$ is a nonincreasing sequence of closed point sets in $D \cup \overline{S - D}$ and for each i, $M_i \subseteqq \mathrm{Cl}\, k_i$ and $k_i \in K_i$.

Case 1. For infinitely many i, $k_i \in H_i$. Then $\bigcap_i \bar{M}_i \subseteqq \bigcap_i \bar{k}_i$ and since $H_1, H_2, \cdots$ is a complete development for S', $\bigcap_i \bar{M}_i \neq \varnothing$. Since $k_i \in H_i$ implies $\bar{k}_i \subseteqq U_i$, it follows that $\bigcap_i \bar{M}_i \subseteqq \bigcap_i U_i = \overline{S - D}$. Hence, $\bigcap_i \bar{M}_i = \bigcap_i M_i \neq \varnothing$.

Case 2. For infinitely many i, $k_i \in F_i$. Then for each such i, $\mathrm{Cl}\, M_i \subseteqq \mathrm{Cl}\, k_i \subseteqq \mathrm{Cl}_D k_i$, so M_i is closed in D. The completeness of F implies $\bigcap_i \mathrm{CL}\, M_i = \bigcap M_i \neq \varnothing$.

2.4. **Acknowledgment.** The proofs of 2.3 and 2.5 and the last paragraph of the proof of 4.1 which appear here are considerably simpler than the author's original arguments and are due to the referee.

THEOREM 2.5. *If M is a closed (semi-)complete subspace of the (semi-)completable Moore space S, there is a (semi-)completion of S having M as a closed subset.*

Proof (Complete Case). Suppose M is a closed, complete subspace of S and S' is a completion of S. There is a set V open in S' such that $V \cap S = S - M$. Now $\mathrm{Cl}_{S'} M$ is a complete space and since M is also complete, M is a G_δ in S'. Let $U_1, U_2, \cdots$ denote sets open in S' such that $\bigcap_i U_i \cap \mathrm{Cl}_{S'} M = M$. Then M is closed in $V \cup \bigcap_i U_i$ which is a G_δ in S' and hence complete.

COROLLARY 2.6. *If M is a subspace of the (semi-)complete Moore space S, then in order that there exists a (semi-)completion T of S such that $\mathrm{Cl}_T M = \mathrm{Cl}_S M$, it is necessary and sufficient that $\mathrm{Cl}_S M$ be (semi-)complete.*

LEMMA 2.7. *If S is a Moore space for which at least one development has property B', then S is completable.*

Proof. There is a nested development G for S having property B'. Suppose α is a G-interior sequence of the open set D, so that for some n, every member of G_n containing a final segment of α is a subset of D. If for some point x, $\{x\}$ is open and contains a final segment of α, then $\{x\} \in G_i$ for every i and is G-interior to D. Suppose there is no singleton open set containing a final segment of α. There exist $f_1 \in G_n$ and f_2 in G_{n+1} such that f_2 is a proper subset of f_1 and contains a final segment of α. Suppose β is a G-Cauchy sequence having a final segment in f_2. Suppose further that for each m, there exists $t_m \in G_m$ and containing a final segment of β and intersecting $S - D$. Let $f_i = t_i$ for $i > 2$ and $F = \{f_i : i \in N\}$. F is a G-filtersubbase containing f_2 and f_2 is a proper subset of f_1. Since G has property B', there exists $m > 2$ such that if $W \in G_m$ and F intersects W, then $\overline{W} \subseteq f_1$. But $f_m \in G_m$ and F intersects f_m and f_m intersects $S - f_1$. This is a contradiction. Therefore, β is G-interior to f_1. It follows that G has Whipple's property and S is completable.

THEOREM 2.8. *If some development for the Moore space S has property B', then some development for S has property B.*

Proof. There exists a monotonic development G for S having property B'. Let $M = \{x \in S$: there is a neighborhood of x having no point other than x as a limit point$\}$. M is an open semicomplete 2.2 subspace of a completable 2.7 space. By Theorem 5 of [14], M is complete. Let $R = \{U \in G_i : U \cap (S - M) \neq \varnothing$ and U is closed$\}$. For

each U in R, let x_U and y_U denote distinct limit points of U and let $U_x = U - \{x_U\}$ and $U_y = U - \{y_U\}$. Let $C_n = \{U \in G_n : U$ is singleton or U has only one limit point and is closed$\}$. C is a monotonic development for the subspace M. With the aid of 2.1 it may be seen that there exists a complete monotonic development D for M such that (1) $D_n \subseteq C_n$, (2) every nonsingleton member of D_{n+1} is a proper subset of some member of D_n. Since $\bar{D}_{n+1}$ refines D_n, it follows that D has property B for the subspace M and since $D_n \subseteq C_n$, D also has property B'. Let $H_n = D_n \cup \{U \in G_n - R : U \cap (S - M) \neq \varnothing\} \cup \{U_t : U \in R \cap G_n$ and t is x or $y\}$. There is a nested development K for S such that for each n, $K_n \subseteq H_{n+1}$.

NOTATION. If $U \in K_1 \cap G_1$, let $U' = U$. If $U \in K_1 - G_1$, let U' denote a set in R such that $U' - \{t_U\} = U$ for some $t \in \{x, y\}$. If $F \subseteq K_1$, let $F' = \{f' : f \in F\}$.

Suppose U and V belong to K_1 and not both are subsets of M and $\bar{U} \subseteq V$. V is not closed since no member of $K_1 - D_1$ is a subset of any member of D_1. Since $U' = \bar{U} \subseteq V$, it follows that U' is a proper subset of V and of V'.

Suppose $V \in K_1 - D_1$ and F is a K-filtersubbase the closure of some member f of which is a subset of V. As shown above, f' is a proper subset of V'. Thus, there is an integer n such that if $W \in G_n$ and F' intersects W, then $\bar{W} \subseteq V'$. Suppose for each $i \geq n$ there is a set $W_i \in K_i$ such that F intersects W_i and $\bar{W}_i$ is not a subset of V. F' intersects W_i', so $\bar{W}_i' = \bar{W}_i \subseteq V'$. There is a point p such that $V' - \{p\} = V$ and it follows that $p \in \bar{W}_i$ for each i. Since $\bar{K}_{i+1}$ refines K_i, there exists $Z_i \in K_i$ containing $\bar{W}_i$ and thus containing p. It follows that $Z = \{Z_i : i > n - 2\}$ is a local base at p and contains a K-nest. $\bar{f} \subseteq V$ and $p \notin V$, so $p \notin \bar{f}$ and there is an $m > n$ such that $Z_m' \cap f = \varnothing$. There exists $j > m$ such that Z_j' either is $\{p\}$ or is a proper subset of Z_m'.

Case 1. Z_j' is a proper subset of Z_m'. Since G has property B', there is an $i > j$ such that if $W \in G_i$ and Z intersects W, then $\bar{W} \subseteq Z_m'$. In particular, $\bar{f}_i \subseteq Z_m'$. But $f \cap Z_m' = \varnothing$ and $f \cap f_i \neq \varnothing$. This is a contradiction.

Case 2. $Z_j' = \{p\}$. There is an $i > j$ such that $\mathrm{st}\,(p, G_i) = \{p\}$. Since F intersects $\{p\}$, it is necessary that $p \in \bar{f}$. This is again a contradiction. Therefore, there is an $i > n$ such that if $W \in K_i$ and F intersects W, then $\bar{W} \subseteq V$.

Suppose $V \in D_2$ and F is a K-filtersubbase such that for some f in F, $\bar{f} \subseteq V$. There exists U in D_1 such that either U is singleton

and is V or U has V as a proper subset.

Case 1. V is a proper subset of U. U and V belong to G_1, so there is an n such that if F' intersects W' and $W' \in G_n$, then $\bar{W}' \subseteqq U$.   If $W \in K_n$ and F intersects W, then $\bar{W} \subseteqq U$, so that $W \in D_1$.   Thus, F contains a D-filtersubbase having f as a member.   Since D has property B, there is an $i > n$ such that if $W \in D_i$ and F intersects W, then $\bar{W} \subseteqq V$.

Case 2. $U = \{x\}$ for some point x of M.   For some i, st $(x, G_i) = \{x\}$.   If $W \in G_i$ and $x \in \bar{W}$, then $W = \{x\}$.   Thus, $f_i = \{x\}$ and if $W \in G_i$ and is intersected by F, then $W = \{x\} \subseteqq U = V$.

Therefore, K has property B.

THEOREM 2.9.   *If some development for the Moore space S has property A', then some development for S has property A.*

This may be established by a minor alteration of the proof of Theorem 2.8.

THEOREM 2.10.   *In order that a Moore space S be (semi-)completable, it is sufficient that some development for S have property $(A')B'$.*

This follows from Theorems 2.8 and 2.9 above and 3.7 and 4.4 of [2].

THEOREM 2.11.   *Every (semi-)complete Moore space having a nested basis has a development with property $(A')B'$.*

Proof.   Suppose D is a nested basis for the (semi-)complete Moore space S.   There exists a nested (semi-)complete development G for S such that $G_1 \subseteqq D$.   Such a development must have property $(A) B$ and from this and the fact that G_1 is a nested basis, $(A')B'$ follows.

2.12.   It is easily demonstrated that B' implies A' for any monotonic development.   (This parallels Theorem 4.2 of [2].)   It may be shown along the lines of the argument for Theorem 10 of [5] that if a Moore space has a completion having a nested basis of regular open sets, then some development for S has property B'.   It may also be of interest that an argument like that for Theorem 9 of [5] may be used to obtain a characterization of metrizable spaces related to property C of [2] in the same way that B' is related to B.   For the next result, a Moore-closed space [5] is a Moore space that is a

closed subspace of every Moore space containing it. A development
G is strong [15] iff it is monotonic and for each point x and neigh-
borhood U of x, there is a positive integer n such that if $W \in G_n$ and
$x \in \bar{W}$, then $W \subsetneq U$. It is easily verified that a (semi-)complete mono-
tonic development for a Moore space S has property (A) B iff it is
strong, and that every nested development is strong (but not con-
versely). $1 \to 3$ of the next result provides a striking contrast to
Theorem 3 of [5] since it follows from hypothesis 3 that every strong
development is complete but it does not follow that S is compact,
while the property that *every* monotonic development is complete
characterizes compactness. From a private communication with G. M.
Reed, the author has learned that Reed, independently and somewhat
earlier than the date of 2.13, characterized Moore-closed spaces as
the Moore spaces in which every strong development is complete.

THEOREM 2.13. *If S is a Moore space, the following are equi-
valent*:

(1) *S is Moore-closed.*
(2) *Every strong development for S has property A.*
(3) *Every strong development for S has property B.*

Proof. $(1 \to 3)$. Suppose S is Moore-closed, G is a strong develop-
ment for S, $U \in G_1$ and F is a G-filtersubbase the closure of some
member of which is a subset of U. $\{\bigcap_{i \leq n} f_i : n \in N\}$ is a countable
open filterbase and by the argument for Theorem 2 of [5] S is feebly
compact, so there is a point p in $\bigcap_n \bar{f}_n$. Suppose D is an open set
containing p. Since G is strong, there is a positive integer n such
that if $k \geq n$, then $\bar{f}_k \subsetneq D$. Thus, F converges to p and if W is a
set intersected by F, then $p \in \bar{W}$. Therefore, there is an m such that
if $W \in G_m$ and F intersects W, then $\bar{W} \subsetneq U$.

$(3 \to 2)$ is a consequence of Theorem 4.2 of [2].

$(2 \to 1)$. Suppose S is a Moore space that is not Moore-closed.
There exist a Moore space S' having S as a dense subspace, a point
p and a strong development G' for S' such that $S' - S = \{p\}$ and
p is a limit point of S. There exist sequences $\{x_i\}$ and $\{y_i\}$ of points
of S and G'-nests $\{R_i\}, \{T_{ni}\}_i$, and $\{Q_{ni}\}_i$ for each n such that (a) x
and y converge to p, (b) $x_n \in T_{ni+1} \subsetneq \bar{T}_{ni+1} \subsetneq T_{ni} \subsetneq R_n - \bar{R}_{n+1}$, (c) $y \in$
$Q_{ni+1} \subsetneq \bar{Q}_{ni+1} \subsetneq Q_{ni} \subsetneq R_n - \bar{R}_{n+1}$, (d) $p \in R_n$, (e) $T_{n1} \cap Q_{n1} = \varnothing$ for each
n. Let $U = R_2 - \bigcup_n \bar{T}_{n1} - \{p\}$, $V = R_1 - \bigcup_n \bar{T}_{n2} - \{p\}$, $f_n = \bigcup_{i \geq n} T_{ii} \cup$
Q_{ii} and $F = \{f_i : i > 1\}$. For each n, let $G_n = \{g \in G'_n : p \in \bar{g}$ and either
$g \cap f_{n+1} = \varnothing$ or $g \subsetneq T_{kn}$ or of Q_{kn} for some $k \geq n\}$. For $n > 1$, let
$H_n = G_n \cup \{f_k : k \geq n\}$ and let $H_1 = H_2 \cup \{U, V\}$. H is a strong develop-
ment for S, $\bar{U} \subsetneq V$ (in S), U and V belong to G_1 and F is an H-nest
intersecting both U and $S - V$. Thus, H does not have property A.

2.14. "Strong" cannot be omitted from (2) of 2.13. For consider the space Δ of Example 9, page 66 of [7]. Let α and β be distinct members of Z and suppose "region" means what is intended in the cited example or one of these sets $W_n = \{x:$ for some $k \geq n$, x is the kth term of α or of $\beta\}$. These new regions can be used to construct a monotonic development G for Δ that is not strong. But Δ is Moore-closed (see the section headed "Examples" in [5]) so by Theorem 2 of [5] G is semicomplete. From 2.12, G cannot have property A.

3. Metacompact spaces.

THEOREM 3.1. *If S is a metacompact Moore space, the following are equivalent*:
(1) *S is completable.*
(2) *Some development for S has property B.*
(3) *Some development for S has property B'.*

Proof. In view of Theorem 2.8 above and 4.4 of [2], all that remains to be shown is that $(1 \to 3)$. Under the hypothesis of (1), there is a nested development K for S having Whipple's property. There exists a sequence $B_1, B_2, \cdots$ of point-finite open collections covering S minimally such that for each n, B_{n+1} refines B_n and K_{n+1}. Let $C_1 = B_1$ and for each n, let C_{n+1} denote the collection of all members U of B_{n+1} such that for every V in $\bigcup_{i \leq n} C_i$, either $U \cap (S - V) \neq \varnothing$ or U is K-interior to V.

Suppose for some m, $\bigcup_{i \geq m} C_i$ does not cover S and p is a point not covered by this collection. Let $U_1, U_2, \cdots, U_n$ denote the members of $\bigcup_{i < m} C_i$ containing p and let $U = \bigcap_{i \leq n} U_i$. The constant sequence p is K-interior to U. By Whipple's property, there is a set R in K_m containing p which is K-interior to U. There is a $k \geq m$ such that st $(p, G_k) \subseteq R$. There is a set V in B_k containing p. $V \subseteq R$, so V is K-interior to U. Therefore, either $V \in C_k$ or there is a set in $\bigcup_{m < i < k} C_i$ containing p. In either case a contradiction is obtained.

Let $H_1 = \bigcup_i C_i$ and for each n, let $H_{n+1} = \{U \in \bigcup_{i > n} C_i : U$ is K-interior to some set in $H_n\}$. H is a nested development for S. For clearly H_1 covers S and suppose p is a point and n a positive integer such that H_n covers S. There exists U in H_n containing p and $m > n$ such that st $(p, K_m) \subseteq U$. There is a set V in $\bigcup_{i \geq m} C_i$ containing p. $V \cap (S - U) = \varnothing$, so V is K-interior to U. $V \in H_m \subseteq H_{n+1}$ and it follows that H_{n+1} covers S. That H is nested follows from the fact that every member of H_{n+1} is K-interior to some member of H_n.

(i) If U and V are distinct members of H_1 and $U \subset V$, then U is K-interior to V. For $V \in C_n$ for some n and $U \in C_k$ for some k. If $k \leq n$, then there is a set W in B_k containing V. But $U \subset V \subseteq W$

and U and W are thus distinct members of B_k. This contradicts the fact that B_k covers S minimally. Therefore, $k > n$ and by construction of C_k, U is K-interior to V.

(ii) H has Whipple's property. This may be shown along the lines of the argument for Theorem 3 of [15].

There exists a collection T' such that (1) if $F \in T'$, then $F = H(\alpha)$ for some H-Cauchy sequence α (cf. 1.3), (2) if α is an H-Cauchy sequence, then $H(\alpha)$ intersects some member of T', (3) no two members of T' intersect. Let T denote the collection of all free filterbases in T'. As observed in 1.3, S_T^* is a completion of S and H_T^* is a complete monotonic development for S_T^*.

(iii) H_T^* is nested and has property B. For if $U^* \in H_{n+1}^*$ (the subscript T will be dropped here), there is a set V in H_n such that U is K-interior to V. Then $\bar{U} \subseteqq V$ (in S) and if for some $H(\alpha)$ in T, $H(\alpha)$ intersects U (i.e., $H(\alpha) \in \mathrm{Cl}\, U^*$ (in S^*)), then $H(\alpha)$ is equivalent to $H(\beta)$ for some H-Cauchy sequence β having a final segment in U and therefore, $H(\alpha) \in V^*$. So, $\mathrm{Cl}_{S^*} U^* \subseteqq V^*$. So H^* is nested and complete and this suffices to show H^* has property B.

(iv) H has property B'. For suppose $U \in H_1$ and F is an H-filtersubbase such that some member V of F is a proper subset of U. By (i), V is K-interior, hence H-interior, to U. By the argument for (iii), $\mathrm{Cl}_{S^*} V^* \subseteqq U^*$. Since H^* has property B, there is a positive integer n such that if $W^* \in H_n^*$ and F^* intersects W^*, then $\mathrm{Cl}_{S^*} W^* \subseteqq U^*$. If $W \in H_n$ and F intersects W, then F^* intersects W^*. Therefore, H has property B'.

3.2.   There is a metacompact Moore space that is not semicompletable. In [10], G. M. Reed makes the very fine observation that the first nonsemicompletable Moore space constructed in [9] is metacompact.

3.3.   There exist nonmetacompact Moore spaces having property B'. The space $\varDelta$ cited in 2.14 is not metacompact and its topology has a basis of sets both open and closed. Now every topology with such a basis has a nested basis and so, by Theorem 2.11, it has property B'. By the same reasoning, the example given in the proof of Lemma 2.2 has property A'.

THEOREM 3.4. *If S is a metacompact Moore space, the following are equivalent:*

(1) *S is semicompletable.*
(2) *Some development for S has property A.*
(3) *Some development for S has property A'.*

Proof (1 → 3). (This again is all that need be shown.) Suppose

S' is a semicompletion of the metacompact Moore space S, G' is a nested semicomplete development for S' and $G_n = \mathrm{Tr}_S(G'_n)$. G is a monotonic development for S. By Rudin's lemma, there exist $H_n \subseteqq G_n$ and $H'_n \subseteqq G'_n$ such that (a) H_n and H'_n cover S and $H_n \supseteqq H_{n+1}$ and $H'_n \supseteqq H'_{n+1}$, (b) if $h_n \in H_n$ and $h_n \supseteqq h_{n+1}$, then there exists an increasing sequence $\{p_n\}$ of positive integers and a sequence $\{h'_n\}$ of sets such that $h'_n \in H'_n$ and contains $\mathrm{Cl}\,(h'_n \cup h_{p_n})$, where $\bar{M}$ and $\mathrm{Cl}\,M$ mean the closure of M in S and S', respectively. There exists a sequence B_1, B_2, $\cdots$ of point-finite open collections covering S minimally such that B_1 refines H_1 and $\bar{B}_{n+1}$ refines B_n and H_{n+1}. Let $C_1 = B_1$ and for each n, let C_{n+1} denote the collection of all members U of B_{n+1} such that for each V in $\bigcup_{i \leq n} C_i$, either (1) $U \cap (S - V) \neq \varnothing$ or (2) $\bar{U} \subseteqq V$ and if F is an H-nest intersecting U there is an i such that if $W \in H_i$ and is intersected by F, then $W \subseteqq V$. Finally, let $I_n = \bigcup_{i \geq n} C_i$.

Suppose for some m, I_m does not cover S and p is a point not covered by this collection. Let V_1, V_2, $\cdots$, V_n denote the members of $\bigcup_{i \leq m} C_i$ containing p. There is a set V' in H'_1 containing p such that $V = V' \cap S \subseteqq \bigcap_{i \leq n} V_i$. There is a set W' in H'_1 such that $p \in W'$ and $\mathrm{Cl}\,W' \subseteqq V'$. Let $W = W' \cap S$, so $\bar{W} \subseteqq V$. There is a set R in $\bigcup_{i \geq m} B_i$ containing p such that $\mathrm{Cl}\,R \subseteqq W'$. Thus, $\bar{R} \subseteqq V$. Suppose F is an H-nest intersecting R and F'' is an H'-nest generated by F as in (b). F'' intersects W' and $\mathrm{Cl}\,W' \subseteqq V'$. Since $H'_n \subseteqq G'_n$ and G' has property A for S', it follows that there exists an integer q such that if $D' \in G'_q$ and F'' intersects D', then $D' \subseteqq V'$. Suppose $D \in H_q$ and F intersects D. By construction, $D = D' \cap S$ for some $D' \in G'_q$. Since F intersects D, F'' intersects D' and $D' \subseteqq V'$. Thus, $D \subseteqq V$. It follows that either $R \in I_m$ or there is some other set in I_m containing p. This is a contradiction. Therefore, I_m covers S for each m.

Thus, I is a monotonic development for S such that if U and V belong to I_1 and U is a proper subset of V and F is an H-nest intersecting U, then there is a positive integer n such that if $W \in H_n$ and F intersects W, then $W \subseteqq V$. Rudin's lemma may be applied again resulting in statements a' and b' obtained from a and b by replacing G' with H, G with I, H with J, and H' with J' and in which closure always means with respect to S.

J is a monotonic development for S having property A'. For suppose U and V belong to J_1, U is a proper subset of V and F is a J-nest intersecting U. U and V belong to I_1 and by (b') there is an H-nest F'' generated by F. F'' intersects U, so there is a positive integer n such that if $W \in H_n$ and F'' intersects W, then $W \subseteqq V$. Suppose $W \in J_n$ and F intersects W. By construction, there is a set Z in H_n including W. F'' intersects Z and $Z \subseteqq V$, so $W \subseteqq V$.

4. **Developable spaces.**

THEOREM 4.1. *The topological space S is developable iff there is a sequence $G_1, G_2, \cdots$ of open covers of S such that for each point p and neighborhood U of p, there is a positive integer n such that there is only one set in G_n containing p and it is a subset of U.*

Proof. That a space with such a sequence is developable is trivial. Suppose S is developable. By Theorem 4 of [16], there is a sequence $H_1, H_2, \cdots$ of open collections (not covers) such that if p is a point and U is an open set containing p, there is a positive integer n such that $\{V \in H_n : p \in V\}$ is a nonempty finite set containing a subset of U. For each n and k, let $H_{nk} = \{V: V$ is the intersection of exactly k members of $H_n\}$. (This technique may be found in [13].) Suppose p is a point and U is an open set containing p. There exist n and k such that H_n has exactly k members containing p and one of them is a subset of U. H_{nk} has only one member containing p and it is a subset of U.

Since every closed set in a developable space is a G_δ, for each n and k there exists a sequence $\langle V_{nki} : i \in N \rangle$ such that $\bigcap_i V_{nki} = S - \bigcup H_{nk}$. Let $H_{nki} = H_{nk} \cup \{V_{nki}\}$. The sequence H_{nki} is the desired development.

4.2. The technique used in 4.1 can be used to obtain a simpler proof of Theorem 4 of [16] which does not depend on the lemma stated there.

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