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VIRGIL DWIGHT HOUSE, JR.

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COUNTABLE PRODUCTS OF GENERALIZED COUNTABLY COMPACT SPACES[1]

V. DWIGHT HOUSE

In this paper two ways of generalizing compactness are studied. We may consider various types of refinements of open covers, such as countable open refinements, locally finite open refinements, etc. In another direction, countably compact spaces may be characterized as having the property that any sequence has a cluster point. Spaces which require that certain sequences have cluster points, such as Σ-spaces, $w\Delta$-spaces, and q-spaces, will be referred to as generalized countably compact spaces.

These more general properties do not behave as well as compactness with respect to products. For example, the product of two Lindelöf spaces need not even be meta-Lindelöf, and the product of two countably compact spaces need not be a q-space. The question naturally arises as to what conditions must be placed on such spaces to insure that they are better behaved with respect to products.

Let Q be a class of generalized countably compact spaces, let $X_1, X_2, \cdots$ be a sequence of spaces each of which belongs to Q. Consider the following two questions.

1. When does $\prod_{n=1}^{\infty} X_n$ belong to Q?

2. Suppose that each X_n has a covering property P which generalizes compactness. When does $\prod_{n=1}^{\infty} X_n$ have P?

In § 3 we answer the first question where Q is any of the following classes: countably compact spaces, Σ-spaces, $w\Delta$-spaces, q-spaces, β-spaces, and wN-spaces. In § 4 the second question is answered for the case where Q is the class of $w\Delta$-spaces and P is one of the following: paracompact, metacompact, subparacompact, and meta-Lindelöf.

2. **Preliminaries.** Unless otherwise stated, no separation axioms will be assumed. Undefined terms are used as defined in [16], except that paracompact spaces are always Hausdorff. The set of natural numbers will be denoted by N, and i, j, k, and n will denote elements of N. If $\mathscr{A}_1, \cdots, \mathscr{A}_n$ are collections of subsets of a set X, we let $\bigwedge_{i=1}^{n} \mathscr{A}_i$ denote the collection $\{\bigcap_{i=1}^{n} A_i \mid A_i \in \mathscr{A}_i,\ i = 1, \cdots, n\}$. A sequence $\mathscr{A}_1, \mathscr{A}_2, \cdots$ of collections is said to be *decreasing* if $\mathscr{A}_{n+1}$ refines $\mathscr{A}_n$ (written $\mathscr{A}_{n+1} < \mathscr{A}_n$), for $n = 1, 2, \cdots$. Also, if $\mathscr{A}$ is a

[1] This work was taken from the author's doctoral dissertation at Duke University. I would like to thank Dr. R. E. Hodel for his guidance in the preparation of this paper.

collection of subsets of X, and $p \in X$, then we will denote by $St(p, \mathscr{A})$ the set $\bigcup \{A \mid p \in A \in \mathscr{A}\}$, by $c(p, \mathscr{A})$ the set $\bigcap \{A \mid p \in A \in \mathscr{A}\}$ and by $ord(p, \mathscr{A})$ the number of elements of A which contain p. If $\mathscr{U}$ is a collection of open sets of a space X, and $M \subseteqq X$ such that $M \subseteqq U$, for all U in $\mathscr{U}$, then we call $\mathscr{U}$ a *base* for M, if given an open set W with $M \subset W$. Then we have $M \subseteqq U \subseteqq W$, for some U in $\mathscr{U}$.

The following conventions will be used in discussing product spaces. Recall that in the product space $\prod_{\alpha \in A} X_\alpha$, basic open sets are of the form $\bigcap_{i=1}^n p_{\alpha_i}^{-1}(U_{\alpha_i})$, where $p_\beta \colon \prod_{\alpha \in A} X_\alpha \to X_\beta$ is the projection function onto the βth coordinate space, and where each U_{α_i} is open in X_{α_i}. We will denote $\bigcap_{i=1}^n p_{\alpha_i}^{-1}(U_{\alpha_i})$ by $\langle U_{\alpha_1}, \cdots, U_{\alpha_n} \rangle$. Also, given any nonvoid open set U in $\prod_{\alpha \in A} X_\alpha$, we have that $p_\alpha(U) \neq X_\alpha$, for at most finitely many α in A. We will use $R(U)$ to denote the set of "restricted coordinates" of U, i.e., $R(U) = \{\alpha \mid \alpha \in A$, and $p_\alpha(U) \neq X_\alpha\}$. Since the elements of $\prod_{\alpha \in A} X_\alpha$ are functions from A into $\bigcup_{\alpha \in A} X_\alpha$, the symbols f, g, and h will be used to denote elements of an infinite product space.

Let X be a topological space. If every open cover of X has a locally finite (respectively, point finite; or point countable) open refinement, then X is paracompact (respectively, metacompact, or meta-Lindelöf). If every open cover of X has a σ-discrete closed refinement, then X is subparacompact [4]. (These spaces were introduced in [21] as F_σ-screenable spaces.) If for any open cover $\mathscr{U}$ of X there is a sequence $\mathscr{G}_1, \mathscr{G}_2, \cdots$ of open refinements of $\mathscr{U}$ such that given x in X, there is an n in N such that $\mathrm{ord}(x, \mathscr{G}_n)$ is finite, then X is θ-refinable [30].

Let $\mathscr{G}_1, \mathscr{G}_2, \cdots$ be a sequence of open covers of a space X having the property that given $x_n \in \mathrm{St}(p, \mathscr{G}_n)$, for all n in N and some p in X, then $\langle x_n \rangle$ has a cluster point. Such a sequence of open covers is called a *wΔ-sequence* for X, and X is called a *wΔ-space* [3]. If a sequence $\mathscr{G}_1, \mathscr{G}_2, \cdots$ of open covers of a space has the property that $\mathscr{G}_{n+1}^* < \mathscr{G}_n$, for $n = 1, 2, \cdots$, where $\mathscr{G}_{n+1}^* = \{pt(G, \mathscr{G}_{n+1}) \mid G \in \mathscr{G}_{n+1}\}$, then it is called a *normal* sequence. A space which has a normal $w\Delta$-sequence is called an *M-space* [22]. A paracompact $w\Delta$-space is an M-space. Let $\mathscr{F}_1, \mathscr{F}_2, \cdots$ be a sequence of locally finite closed covers of a space X having the property that given $x_n \in c(p, \mathscr{F}_n)$, for all n in N and some p in X, then $\langle x_n \rangle$ has a cluster point. Such a sequence is called a *Σ-net* for X, and X is called a *Σ-space* [23]. Clearly $C(p) = \bigcap_{n=1}^\infty c(p, \mathscr{F}_n)$ is countably compact. If $C(p)$ is compact for all p in X, X is called a *strong* Σ-space. Thus a Σ-space is a strong Σ-space in the presence of a property which when combined with countable compactness implies compactness. For example a θ-refinable Σ-space is a strong Σ-space by Theorem (i) of

[30], and by Proposition 3 of [1], a meta-Lindelöf Σ-space is a strong Σ-space.

Let $(X, \mathscr{T})$ be a topological space, and let $g: N \times X \to \mathscr{T}$ be a function such that $x \in \bigcap_{n=1}^{\infty} g(n, x)$, for all x in X. Consider the following conditions on g.

(a) If $\{x_n, p\} \subseteq g(n, y_n)$, for $n = 1, 2, \cdots$, then the sequence $\langle x_n \rangle$ has a cluster point.

(b) If $x_n \in g(n, p)$, for $n = 1, 2, \cdots$, then the sequence $\langle x_n \rangle$ has a cluster point.

(c) If $p \in g(n, x_n)$, for $n = 1, 2, \cdots$, then the sequence $\langle x_n \rangle$ has a cluster point.

(d) If $g(n, x_n) \cap g(n, p) \neq \phi$, for $n = 1, 2, \cdots$, then the sequence $\langle x_n \rangle$ has a cluster point.

The class of $w\Delta$-spaces can be characterized by (a). X is defined to be a q-space [19] or a β-space [12] if (b) or (c) hold respectively. A space with a function satisfying (d) is called a wN-space [13]. The relationship between the classes of spaces defined in this section is summarized in the two diagrams below.

$$\begin{tikzcd}
 & \text{paracompact} \arrow[dl] \arrow[dr] & \\
\text{metacompact} \arrow[d] \arrow[dr] & & \text{subparacompact} \arrow[dl] \\
\text{meta-Lindelöf} & \theta\text{-refinable} &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \text{countably compact} \arrow[dr] & & \text{metrizable} \arrow[dl] & \\
 & & M\text{-space} \arrow[dll] \arrow[d] \arrow[drr] & & \\
w\Delta\text{-space} \arrow[dr] \arrow[drrr] & & wN\text{-space} \arrow[dl] \arrow[dr] & & \Sigma\text{-space} \arrow[dl] \\
 & q\text{-space} & & \beta\text{-space} &
\end{tikzcd}$$

3. Countable products of generalized countably compact spaces. This section is devoted to the consideration of question (1) of §1. In [15] Ishiwata gives an example of two countably compact spaces whose product is not a q-space. This example indicates that we must restore some sort of compactness to the spaces X_n in order to insure that $\prod_{n=1}^{\infty} X_n$ belongs to Q.

In [24] Noble introduces the class C^* of all T_1 spaces X satisfying the property that every infinite subset of X meets some compact subset of X in an infinite set. We see this property again in [26], where it is proved that a product of at most χ_1 spaces in C^* again belongs to C^*. Next notice that for a Hausdorff space X, $X \in C^*$ if and only if X is countably compact and every sequence in X has a subsequence with compact closure. These remarks and the following definition prompt Definition 3.2.

DEFINITION 3.1. A topological space X is called *subsequential* if each sequence in X which has a cluster point has a convergent subsequence.

DEFINITION 3.2. A topological space X is called *weakly subsequential* if each sequence in X which has a cluster point has a subsequence with compact closure.

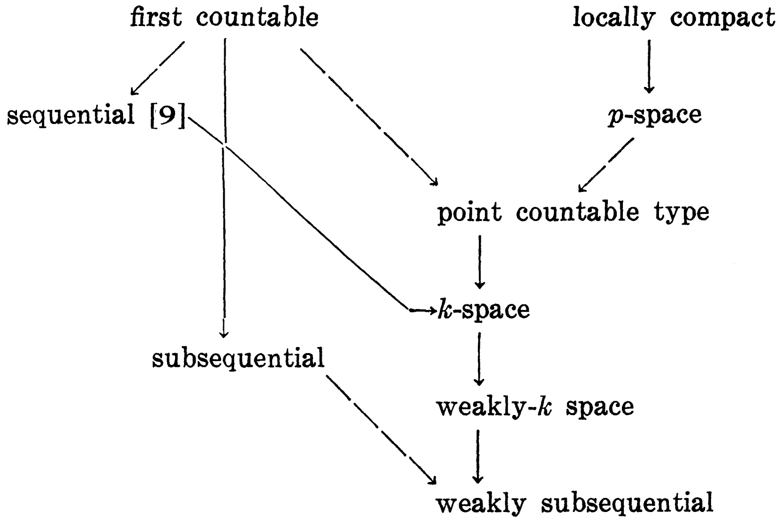
REMARK 3.3. In [14], Ishii, Tsuda, and Kunugi essentially prove that a countable product of weakly subsequential M-spaces is a weakly subsequential M-space. In this section the technique of [14] is abstracted to obtain a general theorem (Theorem 3.10) on countable products from which follow product theorems for several classes of generalized countably compact spaces, including M-spaces.

Before proving the main theorem of this section, we will first study the property of being weakly subsequential and compare it with other properties which are more familiar.

DEFINITION 3.4. A topological space X is *weakly-k* if given $F \subseteqq X$, $F \cap C$ is finite for all compact subsets C of X implies that F is closed.

DEFINITION 3.5. A topological space X is of *point countable type* if X has a cover consisting of compact subsets each of which has a countable base.

Definition 3.4 was introduced by Rishel [25] as a generalization of k-spaces [16], and Definition 3.5, which simultaneously generalizes first countable and locally compact spaces, is due to Arhangel'skiĭ [2]. Arhangel'skiĭ proved that Hausdorff spaces of point countable type are k-spaces. It will be shown that weakly-k, Hausdorff spaces are weakly subsequential. Also notice that since a countably compact space which is subsequential is sequentially compact, an uncountable product of closed unit intervals is weakly subsequential, but not subsequential. We have the following diagram for Hausdorff spaces.



THEOREM 3.6. *Let X be a weakly-k, Hausdorff space. Then X is weakly subsequential.*

Proof. Let $\langle x_n \rangle$ be a sequence in X with a cluster point p. Let $F = \{x_n \mid n \in N\}$. If for each k in N, there is an $n_k \geqq k$ such that $x_{n_k} = p$, then $\langle x_{n_k} \rangle$ is a subsequence of $\langle x_n \rangle$ with compact closure. Otherwise $F - \{p\}$ is not closed, so there is a compact subset C of X such that $(F - \{p\}) \cap C$ is infinite. Set $(F - \{p\}) \cap C = \{x_{n_k} \mid k \in N\}$, where $\langle x_{n_k} \rangle$ is a subsequence of $\langle x_n \rangle$. Then $\langle x_{n_k} \rangle$ has compact closure.

THEOREM 3.7. *Let $(X, \mathscr{T})$ be a T_1, q-space which is weakly subsequential. Then X is weakly-k.*

Proof. Let g: $N \times X \rightarrow \mathscr{T}$ satisfy condition (b) in § 2. Let $p \in \bar{F} - F$. Then we have a sequence $\langle x_n \rangle$ of distinct points such that $x_n \in g(n, p) \cap F$, for $n = 1, 2, \cdots$. Hence $\langle x_n \rangle$ has a cluster point, and thus it has a subsequence $\langle x_{n_k} \rangle$ with compact closure. Let $C = \langle x_{n_k} \rangle^-$. Then $F \cap C$ is infinite, and X is therefore weakly-k.

EXAMPLE 3.8. A paracompact, weakly subsequential space which is not weakly-k.

Let X be an uncountable set, and let $p \in X$. Open neighborhoods of p will be sets whose compliments are countable, and all other points are open. It is easy to see that X is T_1, regular, Lindelöf, and weakly subsequential. Since compact subsets of X are finite, X is not weakly-k.

EXAMPLE 3.9. A strong Σ-space which is neither a q-space nor a weakly subsequential space.

Let X be the Arens-Fort Space (Example 26 of [28]), i.e., X is the set of all ordered pairs of nonnegative integers topologized so that $\{(m, n)\}$ is open if at least one of m or n is nonzero, and basic neighborhoods of $(0, 0)$ are sets which contain all but finitely many points in all but finitely many columns in X. It is well known that X is a T_1, regular, Lindelöf space which is not first countable at $(0, 0)$. One can see that X is a Σ-space, hence a strong Σ-space. Since $\{(0, 0)\}$ is a G_δ-subset of X, according to Lutzer [17], $(0, 0)$ is not a q-point of X, for otherwise X would be first countable at $(0, 0)$. Finally, it is not hard to see that X is not weakly subsequential.

THEOREM 3.10. *Let $\{(X_j, \mathscr{T}_j) \mid j \in N\}$ be a sequence of T_1-spaces, and let $\langle f_n \rangle$ be an infinite sequence in $\prod_{j=1}^{\infty} X_j$. Suppose that for each j in N, and each subsequence $\langle f_{n_k} \rangle$ of $\langle f_n \rangle$ with $n_1 \geqq j$, there is a subsequence $\langle f_{n_{k_t}} \rangle$ of $\langle f_{n_k} \rangle$ such that $\langle f_{n_{k_t}}(j) \rangle^-$ is compact. Then $\langle f_n \rangle$ has a subsequence with compact closure.*

Proof. Since $\langle f_n \rangle$ is a subsequence of itself, there is a subsequence $\langle f_{n_{1t}} \rangle$ of $\langle f_n \rangle$ such that $\langle f_{n_{1t}}(1) \rangle^-$ is compact. We may assume that $2 \leqq n_{1t}$ for all t in N. We thus have a subsequence $\langle f_{n_{2t}} \rangle$ of $\langle f_{n_{1t}} \rangle$ such that $\langle f_{n_{2t}}(2) \rangle^-$ is compact. We may assume that $3 \leqq n_{2t}$, for all t in N. In general, suppose that we have sequences $\langle f_{n_{1t}} \rangle, \cdots, \langle f_{n_{kt}} \rangle$ such that:

(1) $\langle f_{n_{i+1,t}} \rangle$ is a subsequence of $\langle f_{n_{it}} \rangle$, for $i = 1, \cdots, k-1$.

(2) $i + 1 \leqq n_{it}$, for all t in N, and for $i = 1, \cdots, k$.

(3) $\langle f_{n_{it}}(i) \rangle^-$ is compact, for $i = 1, \cdots, k$.

Then $\langle f_{n_{kt}} \rangle$ is a subsequence of $\langle f_n \rangle$ with $k + 1 \leqq n_{k1}$. So there is a subsequence $\langle f_{n_{k+1,t}} \rangle$ of $\langle f_{n_{kt}} \rangle$ such that $\langle f_{n_{k+1,t}}(k+1) \rangle^-$ is compact. We may assume that $k + 2 \leqq n_{k+1,t}$, for all t in N. Thus we obtain a sequence of sequences $\langle f_{n_{1t}} \rangle, \langle f_{n_{2t}} \rangle, \cdots$ such that:

(1) $\langle f_{n_{i+1,t}} \rangle$ is a subsequence of $\langle f_{n_{it}} \rangle$, for all i in N.

(2) $i + 1 \leqq n_{it}$, for all t and i in N.

(3) $\langle f_{n_{it}}(i) \rangle^-$ is compact, for all i in N.

It will now be shown that the subsequence $\langle f_{n_{kk}} \rangle$ of $\langle f_n \rangle$ has compact closure. Set $C_1 = \{f_{n_{1t}}(1) \mid t \in N\}^-$. For $i \geqq 2$, set

$$C_i = \{f_{n_{it}}(i) \mid t \in N\}^- \cup \{f_{n_{jt}}(i) \mid j, t < i\} .$$

Now let $C = \prod_{i=1}^{\infty} C_i$. Clearly C is closed and compact. Since $f_{n_{kk}} \in C$, for all k in N, we see that $\langle f_{n_{kk}} \rangle^-$ is compact.

THEOREM 3.11. *Let $\{(X_j, \mathscr{T}_j) \mid j \in N\}$ be a sequence of weakly* *subsequential, T_1-spaces. Let $X = \prod_{j=1}^{\infty} X_j$, and let $\mathscr{T}$ be the product topology on X.*

(1) *If each X_j is countably compact, then X is weakly subsequential and countably compact.*

(2) *If each X_j is a $w\Delta$-space, then X is a weakly subsequential, $w\Delta$-space.*

(3) *If each X_j is a q-space, then X is a weakly subsequential, q-space.*

(4) *If each X_j is a wN-space, then X is a weakly subsequential, wN-space.*

(5) *If each X_j is a β-space, then X is a β-space.*

(6) *If each X_j is a Σ-space, then X is a Σ-space.*

Proof. Only the proofs of (2) and (6) will be included; the others will be left for the reader.

(2) For each j in N, let $g_j\colon N \times X \to \mathscr{T}_j$ be a function satisfying condition (a) in § 2. We may assume that $g_j(n+1, x) \subseteqq g_j(n, x)$, for all x in X_j and all n in N. Let $g\colon N \times X \to \mathscr{T}$ be defined as follows: $g(n, f) = \langle g_1(n, f(1)), \cdots, g_n(n, f(n))\rangle$. Suppose that $\{f, f_n\} \subseteqq g(n, h_n)$, for $n = 1, 2, \cdots$. Let $j \in N$, and let $\langle f_{n_k}\rangle$ be a subsequence of $\langle f_n\rangle$ with $n_1 \geqq j$. Since $\{f, f_{n_k}\} \subseteqq g(n_k, h_{n_k})$ and $n_k \geqq j$, for all k in N, we have $\{f(j), f_{n_k}(j)\} \subseteqq g_j(n_k, h_{n_k}(j)) \subseteqq g_j(k, h_{n_k}(j))$. So $\langle f_{n_k}(j)\rangle$ has a cluster point in X_j, and thus it also has a subsequence $\langle f_{n_{k_t}}\rangle$ such that $\langle f_{n_{k_t}}(j)\rangle^-$ is compact. By Theorem 3.10 $\langle f_n\rangle$ must have a subsequence with compact closure, thus assuring that it also has a cluster point.

To see that X is weakly subsequential, let $\langle f_n\rangle$ be a sequence with a cluster point f. Then there is an n_1 in N such that $n_1 \geqq 1$ and $f_{n_1} \in g(1, f)$. If for $i = 1, \cdots, k$, we have n_i in N such that $n_1 < n_2 < \cdots < n_k$, $n_i \geqq i$, and $f_{n_i} \in g(i, f)$, then choose n_{k+1} in N such that $n_{k+1} \geqq \max\{k+1, n_{k+1}\}$ and $f_{n_{k+1}} \in g(k+1, f)$. Then we get a subsequence $\langle f_{n_k}\rangle$ of $\langle f_n\rangle$ such that $\{f_{n_k}, f\} \subseteqq g(k, f)$, for $k = 1, 2, \cdots$. As above we see that $\langle f_{n_k}\rangle$ has a subsequence with compact closure, thus assuring that $\langle f_n\rangle$ also has a subsequence with compact closure.

(6) For each j in N, let $\mathscr{F}_1^j, \mathscr{F}_2^j, \cdots$ be a Σ-net for X_j with the property that $\mathscr{F}_n^j = \bigwedge_{k=1}^n \mathscr{F}_k^j$, for all n in N. For each n in N, set $\mathscr{F}_n = \{\prod_{j=1}^n F_j \times \prod_{j>n} X_j \mid F_j \in \mathscr{F}_n^j,\ j = 1, \cdots, n\}$. Let $f_n \in c(f, \mathscr{F}_n)$, for $n = 1, 2, \cdots$. Let $j \in N$, and let $\langle f_{n_k}\rangle$ be a subsequence of $\langle f_n\rangle$ with $n_1 \geqq j$. Since $n_k \geqq j$, for all k in N, we have $f_{n_k}(j) \in c(f(j), \mathscr{F}_{n_k}^j) \subseteqq c(f(j), \mathscr{F}_k^j)$, for $k = 1, 2, \cdots$. Thus $\langle f_{n_k}(j)\rangle$ has a cluster point. So $\langle f_{n_k}\rangle$ has a subsequence $\langle f_{n_{k_t}}\rangle$ such that $\langle f_{n_{k_t}}(j)\rangle^-$ is compact. $\langle f_n\rangle$ therefore has a subsequence with compact closure, and hence it also has a cluster point.

REMARK 3.12. Theorem 3.10 should be compared to 4.2.3 in [11]. As was mentioned earlier, in [26] Saks and Stephenson show that we may actually take a product of up to χ_1 factors in (1) of Theorem 3.11 instead of just countably many. In [13] (Proposition 3.2 and Remark 3.3) it is shown that a wN-space is countably paracompact, and that a β-space is countably metacompact. So in (4) and (5) X is respectively countably paracompact and countably metacompact. Ishiwata's example [15] shows that weakly subsequential can not be dropped from the hypothesis in (1)-(4) of Theorem 3.11, and the following example (Stone [29]) shows that we must have a countable product in (2)-(6).

EXAMPLE 3.13. Let $X = \prod_{\alpha \in A} N_\alpha$, where A is uncountable, and each N_α is a copy of the positive integers with the discrete topology. Then X is neither a q-space nor a β-space.

Proof. We will first show that X is not a q-space. Let $g\colon A \to N$ be the constant function which maps each α in A to 1. Let $V_1, V_2, \cdots$ be a sequence of open neighborhoods of g. Let $\beta \in A - \bigcup_{n=1}^{\infty} R(V_n)$. Then $P_\beta(V_n) = N_\beta$, for all n in N. For each k in N, let $f_k\colon A \to N$ be defined as follows: $f_k(\alpha) = \begin{cases} 1, & \alpha \neq \beta \\ k, & \alpha = \beta \end{cases}$. Then $f_k \in V_k$, for all k in N. Suppose f is a cluster point of $\langle f_n \rangle$. Let $W = P_\beta^{-1}(\{f(\beta)\})$. Then $f_n \notin W$, for $n \neq f(\beta)$. This is a contradiction. Thus $\langle f_n \rangle$ has no cluster point, and so X is not a q-space.

We will now show that X is not a β-space. Suppose that $g\colon N \times X \to \mathscr{T}$ is a function which satisfies condition (c) in § 2. Let f_1 in X be such that $f_1(\alpha) = 1$, for all α in A. Let V_1 be a basic open neighborhood of f_1 such that $V_1 \subseteqq g(1, f_1)$. Set $f_2(\alpha) = \begin{cases} f_1(\alpha), & \alpha \in R(V_1) \\ 2, & \alpha \notin R(V_1) \end{cases}$. Let V_2 be a basic open neighborhood of f_2 such that $V_2 \subseteqq g(2, f_2)$, and $R(V_1) \subseteqq R(V_2)$. Now suppose that we have $f_1, \cdots, f_n$ and $V_1, \cdots, V_n$ such that:

(1) V_i is a basic open neighborhood of f_i, for $i = 1, \cdots, n$;

(2) $V_i \subseteqq g(i, f_i)$, for $i = 1, \cdots, n$;

(3) $R(V_i) \subseteqq R(V_{i+1})$, for $i = 1, \cdots, n-1$;

(4) $f_{i+1}(\alpha) = \begin{cases} f_i(\alpha), & \alpha \in R(V_i) \\ i+1, & \alpha \notin R(V_i), \end{cases}$ for $i = 1, \cdots, n-1$.

Set $f_{n+1}(\alpha) = \begin{cases} f_n(\alpha), & \alpha \in R(V_n) \\ n+1, & \alpha \notin R(V_n) \end{cases}$, and let V_{n+1} be a basic open neighborhood of f_{n+1} such that $V_{n+1} \subseteqq g(n+1, f_{n+1})$, and $R(V_n) \subseteqq R(V_{n+1})$. We continue in this manner to get a sequence $\langle f_n \rangle$ and open sets $V_1, V_2, \cdots$ satisfying (1)-(4) above for all i in N. Now, suppose that $\langle f_n \rangle$ has a cluster point f. Let $\beta \in A - \bigcup_{n=1}^{\infty} R(V_n)$, and let $W = p_\beta^{-1}(\{f(\beta)\})$. Then $f_n \notin W$, for $n \neq f(\beta)$. This is a contradiction. So $\langle f_n \rangle$ has no cluster point.

To show that X is not a β-space, it suffices to find an element h in X such that $h \in g(n, f_n)$, for all n in N. Set $h(\alpha) = \begin{cases} f_n(\alpha), & \text{if } \alpha \in R(V_n), \text{ for some } n \text{ in } N \\ 1, & \text{if } \alpha \notin \bigcup_{n=1}^{\infty} R(V_n) \end{cases}$. Then $h \in V_n$, for all n in N.

4. Countable products of covering properties. Paracompactness is perhaps the most important covering property which generalizes compactness. However, the product of two paracompact spaces is not necessarily paracompact [27]. The question naturally arises as to what extra conditions can be placed upon paracompact spaces to insure that products of these spaces will also be paracompact. In [29], Stone has shown that we may as well concern ourselves only with products which are at most countable. We know that a countable product of metric spaces is a metric space, and is therefore paracompact by [29]. Frolík [10] has shown that a countable product of paracompact absolute G_δ (i.e., being G_δ in its Stone-Čech compactification) spaces is paracompact. Arhangel'skiĭ [2] and Morita [22] have improved upon Frolík's result by showing that a countable product of paracompact $w\Delta$-spaces is paracompact. In a different direction, Ceder [7] has shown that a stratifiable space is paracompact and that a countable product of stratifiable spaces is stratifiable. The best result so far is that of Nagami [23] which generalizes all of the above mentioned results. Nagami's theorem is that a countable product of paracompact Σ-spaces is paracompact.

in this section we will utilize Nagami's technique of proof to obtain countable product theorems for other covering properties such as metacompactness, subparacompactness, and the meta-Lindelöf property. In this direction Nagami [23] has shown that a countable product of strong Σ-spaces is a strong Σ-space. Since a regular, strong Σ-space is subparacompact, it follows that a countable product of regular, θ-refinable (or metaLindelöf) Σ-spaces is subparacompact. In connection with this problem we introduce the class of strong $w\Delta$-spaces whose difinition is suggested by Nagami [23] and Burke and Stoltenberg [6]. (Also see Michael's discussion of mod-k networks [20].)

DEFINITION 4.1. Let X be a topological space. A decreasing sequence $\mathscr{G}_1, \mathscr{G}_2, \cdots$ of open covers of X is called a *strong $w\Delta$-sequence* for X if

(1) $C(p) = \bigcap_{n=1}^{\infty} \mathrm{St}(p, \mathscr{G}_n)$ is compact, for all p in X;

(2) $\{\mathrm{St}(p, \mathscr{G}_n) \mid n = 1, 2, \cdots\}$ is a base for $C(p)$, for all p in X.

A space with a strong $w\Delta$-sequence is called a *strong $w\Delta$-space*.

REMARK 4.2. It is clear that a developable space is a strong $w\Delta$-space, and a strong $w\Delta$-space is a space of point countable type. Thus a Hausdorff strong $w\Delta$-space is weakly subsequential. We also have the following theorem whose proof is left for the reader.

THEOREM 4.3. *Let $\mathscr{G}_1, \mathscr{G}_2, \cdots$ be a strong $w\Delta$-sequence for X. Then it is also a $w\Delta$-sequence for X.*

EXAMPLE 4.4. A weakly subsequential, countably compact space which is neither a strong $w\Delta$-space nor a strong Σ-space.

Let $W = [0, \Omega)$, the space of all ordinals less than the first uncountable ordinal Ω with the order topology. It is well known that W is countably compact and first countable. We will show that W is not a strong $w\Delta$-space first. To do this we will need two lemmas about W which are based on ideas due to J. H. Roberts.

LEMMA 4.5. *Suppose that $g: N \times W \to \mathscr{T}$ is a function which has as its values open sets in W of the form $g(n, \alpha) = (\beta, \alpha]$, for all α in W. Then for fixed m in N, there exists ν_m in W such that $\nu_m \in g(m, \alpha)$, for uncountably many α in W.*

LEMMA 4.6. *Let g be as in Lemma 4.5. Then there is a ν in W such that for each n in N, $\nu \in g(n, \alpha)$, for uncountably many α in W.*

Now suppose that $\mathscr{G}_1, \mathscr{G}_2, \cdots$ is a strong $w\Delta$-sequence for W. Define $g: N \times W \to \mathscr{T}$ by $g(n, \alpha) = (\beta, \alpha]$, where $(\beta, \alpha] \subseteqq G$, for some G in $\mathscr{G}_n$ containing α, for all α in W. Let ν be as in Lemma 4.6. Since $C(\nu)$ is compact, there is a γ in W such that $C(\nu) \subseteqq [0, \gamma)$. Thus there is an n in N such that $\mathrm{St}(\nu, \mathscr{G}_n) \subseteqq [0, \gamma)$. By Lemma 4.6, we have an $\alpha > \gamma$ such that $\nu \in g(n, \alpha)$. But $\nu \in g(n, \alpha)$ implies that $\alpha \in \mathrm{St}(\nu, \mathscr{G}_n)$. Hence $\alpha < \gamma$; a contradiction. So W is not a strong $w\Delta$-space.

To show that W is not a strong Σ-space, we need to know that a regular, strong Σ-space is subparacompact [20]. Then it is easy to see that W is not a strong Σ-space. For it is well known that W is a regular, T_1, countably compact space which is not compact. Thus by Therem (i) of [30], and by the fact that a subparacompact space is θ-refinable, W can not be subparacompact, and hence not a strong Σ-space.

Before we can prove our theorems on countable products and covering properties, we must have the following two lemmas which are based on notes by J. Vaughan.

LEMMA 4.7. *If* $\mathscr{A}$ *is a collection of subsets of* X, $p \in X$, *and* $f: X \to Y$ *is a function, then we have* $f(\mathrm{St}(p, \mathscr{A})) \subseteqq \mathrm{St}(f(p), f(\mathscr{A}))$, *where* $f(\mathscr{A}) = f\{(A) \mid A \in \mathscr{A}\}$.

LEMMA 4.8. *Let* $\mathscr{G}_1^n, \mathscr{G}_2^n, \cdots$ *be a sequence of covers of* X_n, *for* $n = 1, 2, \cdots$. *For each* j *in* N, *let*

$$\mathscr{G}_j = \left\{\prod_{n=1}^{j} G_j^n \times \prod_{n>j} X_n \mid G_j^n \in \mathscr{G}_j^n, n = 1, \cdots, j\right\} .$$

(1) $\mathscr{G}_j$ *covers* $\prod_{n=1}^\infty X_n$, *for each* j *in* N.

(2) $P_n(\mathscr{G}_j) = \begin{cases} \mathscr{G}_j^n, & n \leqq j \\ \{X_n\}, & n > j \end{cases}$.

(3) $\mathrm{St}(f, \mathscr{G}_j) = \prod_{n=1}^j \mathrm{St}(f(n), \mathscr{G}_j^n) \times \prod_{n>j} X_n$.

(3′) $c(f, \mathscr{G}_j) = \prod_{n=1}^j c(f(n), \mathscr{G}_j^n) \times \prod_{n>j} X_n$.

(4) $\bigcap_{j=1}^\infty \mathrm{St}(f, \mathscr{G}_j) = \prod_{n=1}^\infty (\bigcap_{j=1}^\infty \mathrm{St}(f(n), \mathscr{G}_j^n))$, *if* $\mathscr{G}_{j+1}^n < \mathscr{G}_j^n$, *for all* n *and* j *in* N.

(4′) $\bigcap_{j=1}^\infty c(f, \mathscr{G}_j) = \prod_{n=1}^\infty (\bigcap_{j=1}^\infty c(f(n), \mathscr{G}_j^n))$, *if* $\mathscr{G}_j^n = \bigwedge_{k=1}^j \mathscr{G}_k^n$, *for all* j *and* n *in* N.

Proof. The proofs of (1), (2), and (3) are straightforward and are left to the reader.

(4) Let $g \in \bigcap_{j=1}^\infty \mathrm{St}(f, \mathscr{G}_j)$. We have

$$\begin{aligned} P_n\left(\bigcap_{j=1}^\infty \mathrm{St}(f, \mathscr{G}_j)\right) &\subseteqq \bigcap_{j=1}^\infty P_n(\mathrm{St}(f, \mathscr{G}_j)) \subseteqq \bigcap_{j=1}^\infty \mathrm{St}(f(n), P_n(\mathscr{G}_j)) \\ &= \left[\bigcap_{j=n}^\infty \mathrm{St}(f(n), \mathscr{G}_j^n)\right] \cap \left[\bigcap_{j=1}^{n-1} \mathrm{St}(f(n), \{X_n\})\right] \\ &= \left[\bigcap_{j=n}^\infty \mathrm{St}(f(n), \mathscr{G}_j^n)\right] \cap X_n = \bigcap_{j=1}^\infty \mathrm{St}(f(n), \mathscr{G}_j^n) . \end{aligned}$$

So $g(n) \in \bigcap_{j=1}^\infty \mathrm{St}(f(n), \mathscr{G}_j^n)$, and thus $g \in \prod_{n=1}^\infty (\bigcap_{j=1}^\infty \mathrm{St}(f(n), \mathscr{G}_j^n))$. Now, let $g \in \prod_{n=1}^\infty (\bigcap_{j=1}^\infty \mathrm{St}(f(n), \mathscr{G}_j^n))$. Fix j. Then for $1 \leqq n \leqq j$, we have $g(n) \in \mathrm{St}(f(n), \mathscr{G}_j^n)$. For $n = 1, \cdots, j$, choose G_j^n in $\mathscr{G}_j^n$ so that $g(n), f(n) \in G_j^n$. Let $G = \prod_{n=1}^j G_j^n \times \prod_{n>j} X_n$. Then $f, g \in G \in \mathscr{G}_j$, and thus we have $g \in \mathrm{St}(f, \mathscr{G}_j)$.

We now need a well known generalization of a theorem of Wallace (5.12 of [16]). The proof of Lemma 4.9 can be obtained by generalizing the the proof in [16].

LEMMA 4.9. *For each* α *in an index set* A, *let* D_α *be a compact subset of a topological space* X_α. *Let* $D = \prod_{\alpha \in A} D_\alpha$, *and let* U *be an open subset of* $\prod_{\alpha \in A} X_\alpha$ *with* $D \subseteqq U$. *Then there is an* n *in* N *and open sets* $B_{\alpha_1}, \cdots, B_{\alpha_n}$ in $X_{\alpha_1}, \cdots, X_{\alpha_n}$ *respectively such that* $D \subseteqq \langle B_{\alpha_1}, \cdots, B_{\alpha_n}\rangle \subseteqq U$.

THEOREM 4.10. *A countable product of strong $w\Delta$-spaces is a strong $w\Delta$-space.*

Proof. For each n in N, let $\mathscr{G}_1^n, \mathscr{G}_2^n, \cdots$ be a strong $w\Delta$-sequence for X_n. For each j in N, let

$$\mathscr{G}_j = \left\{\prod_{n=1}^{j} G_j^n \times \prod_{n>j} X_n \mid G_j^n \in \mathscr{G}_j^n,\ n = 1, \cdots, j\right\}.$$

We will show that $\mathscr{G}_1, \mathscr{G}_2, \cdots$ is a strong $w\Delta$-sequence for $\prod_{n=1}^{\infty} X_n$. Let $f \in \prod_{n=1}^{\infty} X_n$. Then $C(f) = \bigcap_{j=1}^{\infty} \mathrm{St}(f, \mathscr{G}_j) = \prod_{n=1}^{\infty} (\bigcap_{j=1}^{\infty} \mathrm{St}(f(n), \mathscr{G}_j^n)) = \prod_{n=1}^{\infty} C(f(n))$ is compact, since it is the product of compact sets. Now, let U be open in $\prod_{n=1}^{\infty} X_n$ such that $C(f) \subseteqq U$. Then we have a k in N, and open sets V_n in X_n, for $n = 1, \cdots, k$ such that $\prod_{n=1}^{\infty} C(f(n)) \subseteqq \prod_{n=1}^{k} V_n \times \prod_{n>k} X_n \subseteqq U$. For $n = 1, \cdots, k$, we have an N_n in N such that $C(f(n)) \subseteqq \mathrm{St}(f(n), \mathscr{G}_{N_n}^n) \subseteqq V_n$. Let $m = \max\{N_1, \cdots, N_k, k\}$. Then $C(f) = \prod_{n=1}^{\infty} C(f(n)) \subseteqq \mathrm{St}(f, \mathscr{G}_m) \subseteqq U$. Thus $\{\mathrm{St}(f, \mathscr{G}_j) \mid j = 1, 2, \cdots\}$ is a base for $C(f)$.

COROLLARY 4.11. *A countable product of strict p-spaces is a strict p-space.*

Proof. This follows immediately from Theorem 2.2 of [6].

Before we get to the main theorem of this section, we must have two lemmas about covers of spaces.

LEMMA 4.12. *Let $\mathscr{U}$ be an open cover of a space X, and let $\mathscr{U}'$ be the collection of all finite unions of elements of $\mathscr{U}$.*

(1) *If $\mathscr{U}'$ has a point countable open refinement, then so does $\mathscr{U}$.*

(2) *If $\mathscr{U}'$ has a σ-point finite open refinement, then so does $\mathscr{U}$.*

(3) *If $\mathscr{U}'$ has a σ-locally finite open refinement, then so does $\mathscr{U}$.*

(4) *If $\mathscr{V}$ is an open cover of X such that $\overline{\mathscr{V}} < \mathscr{U}$, $\mathscr{V}'$ is the collection of all finite unions of elements of $\mathscr{V}$, and $\mathscr{V}'$ has a σ-locally finite closed refinement, then so does $\mathscr{U}$.*

LEMMA 4.13. *Let X be a topological space, and let $\mathscr{G}_1, \mathscr{G}_2, \cdots$ be a strong $w\Delta$-sequence for X.*

(1) *If each $\mathscr{G}_n$ has a point finite open refinement, then X is metacompact.*

(2) *If each $\mathscr{G}_n$ has a locally finite open refinement, and X is T_1 and regular, then X is paracompact.*

(3) *If each $\mathscr{G}_n$ has a point countable open refinement, then X is meta-Lindelöf.*

(4) *If each* $\mathscr{G}_n$ *has a* σ*-locally finite closed refinement, and* X *is regular, then* X *is subparacompact.*

Proof. Let $\mathscr{U}$ be an open cover of X, and let $\mathscr{U}'$ be the collection of all finite unions of elements in $\mathscr{U}$.

(1) For each n in N, let $\mathscr{G}'_n$ be a point finite open refinement of $\mathscr{G}_n$, and let $\mathscr{V}_n = \{G \mid G \in \mathscr{G}'_n$, and $G \subseteqq U$, for some U in $\mathscr{U}'\}$. Clearly $\mathscr{V} = \bigcup_{n=1}^{\infty} \mathscr{V}_n$ is a σ-point finite open collection, each element of which is contained in an element of $\mathscr{U}'$. Now let $p \in X$. Then $C(p) \subseteqq U$, for some U in $\mathscr{U}'$. Thus $C(p) \subseteqq \mathrm{St}(p, \mathscr{G}_n) \subseteqq U$, for some n in N. So $\mathscr{V}$ covers X. By (2) of Lemma 4.12, $\mathscr{U}$ has a σ-point finite open refinement. Now by Remark 3.12, we see that X is countably metacompact since it is a $w\Delta$-space. Hence X is metacompact.

(2) This argument is similar to that in (1). We get a σ-locally finite open refinement of $\mathscr{U}$. Since X is T_1 and regular, it is paracompact by a theorem of Michael [18].

(3) This argument is easy and is left to the reader.

(4) Let $\mathscr{V}$ be an open cover of X such that $\overline{\mathscr{V}} < \mathscr{U}$, and let $\mathscr{V}'$ be the collection of all finite unions of elements of $\mathscr{V}$. Let $\mathscr{F}_n$ be a σ-locally finite closed refinement of $\mathscr{G}_n$, for each n in N. Set $\mathscr{H}_n = \{F \mid F \in \mathscr{F}_n$, and $F \subseteqq V$, for some V in $\mathscr{V}'\}$. Clearly $\mathscr{H} = \bigcup_{n=1}^{\infty} \mathscr{H}_n$ is a σ-locally finite closed collection, each element of which is contained in an element of $\mathscr{V}'$. $\mathscr{H}$ covers X by an argument similar to that in (1). By (4) of Lemma 4.12, $\mathscr{U}$ has a σ-locally finite closed refinement. So X is subparacompact by Theorem 1.2 of [4].

THEOREM 4.14. *Let* $X_1, X_2, \cdots$ *be a sequence of strong* $w\Delta$*-spaces, and let* $X = \prod_{n=1}^{\infty} X_n$.

(1) *If each* X_n *is metacompact, then* X *is metacompact.*

(2) *If each* X_n *is paracompact, then* X *is paracompact.*

(3) *If each* X_n *is meta-Lindelöf, then* X *is meta-Lindelöf.*

(4) *If each* X_n *is subparacompact and regular, then* X *is subparacompact.*

Proof. Let $\mathscr{G}_1^n, \mathscr{G}_2^n, \cdots$ be a strong $w\Delta$-sequence for X_n, for $n = 1, 2, \cdots$. Let $\mathscr{G}_j = \{\prod_{n=1}^{j} G_j^n \times \prod_{n>j} X_n \mid G_j^n \in \mathscr{G}_j^n$, for $n = 1, \cdots, j\}$ for each j in N. Then $\mathscr{G}_1, \mathscr{G}_2, \cdots$ is a strong $w\Delta$-sequence for X.

(1) For each n and j in N, let $\mathscr{V}_j^n$ be a point finite open refinement of $\mathscr{G}_j^n$. Set $\mathscr{V}_j = \{\prod_{n=1}^{j} V_j^n \times \prod_{n>j} X_n \mid V_j^n \in \mathscr{V}_j^n$, for $n = 1, \cdots, j\}$, for each j in N. Then clearly each $\mathscr{V}_j$ is a point finite open refinement of $\mathscr{G}_j$. So by (1) of Lemma 4.13, X is metacompact.

The arguments for (2) and (3) are similar to (1).

(4) For each n and j in N, let $\mathscr{F}_j^n = \bigcup_{i=1}^{\infty} \mathscr{F}_{ji}^n$ be a closed refinement of $\mathscr{G}_j^n$, where each $\mathscr{F}_{ji}^n$ is discrete. For each j and i in N, set $\mathscr{F}_{ji} = \{\prod_{n=1}^{j} F_{ji}^n \times \prod_{n>j} X_n \mid F_{ji}^n \in \mathscr{F}_{ji}^n, \text{ for } n = 1, \cdots, j\}$. Then each $\mathscr{F}_{ji}$ is a locally finite closed collection, and $\mathscr{F}_j = \bigcup_{i=1}^{\infty} \mathscr{F}_{ji}$ refines $\mathscr{G}_j$, for $j = 1, 2, \cdots$. By (4) of Lemma 4.13, X is subparacompact.

COROLLARY 4.15. *Let $X_1, X_2, \cdots$ be a sequence of $w\Delta$-spaces and let $X = \prod_{n=1}^{\infty} X_n$.*

(1) *If each X_n is regular and metacompact, then X is metacompact.*

(2) *If each X_n is paracompact, then X is paracompact.*

(3) *If each X_n is regular and subparacompact, then X is subparacompact.*

Proof. By Remark 1.9 of [5], we see that a regular, θ-refinable, $w\Delta$-space is a strong $w\Delta$-space. So each X_n in (1)-(3) is actually a strong $w\Delta$-space.

REMARK 4.16. Note that (2) of Corollary 4.15 is the theorem of Arhangel'skiĭ and Morita. Theorem 5 of [8] is also the same as (1) of Corollary 4.15. This can be seen by checking that the proof of Theorem 2.2 of [6] also works for the Wallman compactification.

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