ON ELEMENTARY IDEALS OF PROJECTIVE PLANES IN THE 4-SPHERE AND ORIENTED Θ-CURVES IN THE 3-SPHERE

Shin’ichi Kinoshita
ON ELEMENTARY IDEALS OF PROJECTIVE PLANES IN THE 4-SPHERE AND ORIENTED $\theta$-CURVES IN THE 3-SPHERE

SHIN'ICHI KINOSHITA

The concept of an infinite cyclic covering has been applied to knot theory. In this paper that of a finite cyclic covering is considered. This enable us to study such cases as projective planes in the 4-sphere and oriented $\theta$-curves in the 3-sphere. Some properties of elementary ideals of these cases are examined. The technique of free differential calculus is used, instead of that of coverings.

Let $L$ be a polyhedron in an $n$-sphere $S^n$ ($n > 1$) that does not separate $S^n$, and let $G_L$ be the fundamental group of $S^n - L$. We use the additive group $J_p$ of integers modulo $p$ as the coefficient group for homology. Let $l$ be an $(n - 2)$-dimensional cycle on $L$. Let $H_p$ be the multiplicative cyclic group of order $p$, generated by $t$. Then, there is a homomorphism $\psi$ of $G_L$ into $H_p$ such that for each $g \in G_L$,

$$g^{\psi} = t^{\text{link}(g, l)},$$

where $\text{link}(g, l) \in J_p$ is the linking number between $g$ and $l$ in $S^n$.

Using Fox's free differential calculus ([1], [2]), we associate to $\psi$ a sequence of elementary ideals $E_d(G_L, \psi)$ of the group $G_L$, evaluated in the group ring $JH_p$ of $H_p$ over integers $J$. This sequence of elementary ideals depends only on $G_L$ and $\psi$, and hence it depends only on the position of $l$ on $L$ in $S^n$. We shall denote it by $E_d(l)$. If $l$ and $l'$ are homologous on $L$, then $E_d(l) = E_d(l')$ for every $d$.

In this paper we apply these elementary ideals $E_d(l)$ to the study of the position of $L$ in $S^n$. The following two cases of $E_d(l)$ are considered: (1) $L$ is a projective plane in $S^4$ and $p = 2$, and (2) $L$ is a $\theta$-curve in $S^3$ and $p = 3$.

1. Miscellanea. Let $\sigma(t) = 1 + t + \cdots + t^{p-1} \in JH_p$.

**THEOREM 1.** If $\psi$ is onto, then $E_0(l) \subseteq (\sigma(t))$ in $JH_p$.

**Proof.** It is proved in [2] that

$$E_0(H_p, \text{id}) = (\sigma(t)),$$

where $\text{id}$ is the identity isomorphism of $H_p$. From the diagram

$$G_L \xrightarrow{\psi} H_p \xrightarrow{\text{id}} H_p,$$

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where \( \psi \) is onto, and Theorem 1 in [4], it follows that \( E_0(l) \subset \langle \sigma(t) \rangle \) in \( JH_p \).

Now assume that \( \psi \) is onto, and let \( E_0(l) = \sigma(t)E(l) \). Let \( \omega = e^{2\pi i/p} \) and let \( J[\omega] \) be the ring of all complex numbers of the form \( \sum_{i=0}^{p-1} a_i\omega^i \), where \( a_i \in J \) (\( i = 0, 1, \ldots, p - 1 \)). A homomorphism \( \ast \) of \( H_p \) into \( J[\omega] \) is defined by \( t^\ast = \omega \). We naturally extend \( \ast \) to a ring homomorphism of \( JH_p \) onto \( J[\omega] \). Though \( E_0(l)^\ast = (0) \), sometimes \( E(l)^\ast \) is a nontrivial ideal in \( J[\omega] \).

A trivializer of a group \( G \) is a homomorphism of \( G \) onto the trivial group that consists of only one element. Any trivializer will be denoted by the same notation \( \circ \) in this paper. Further the group ring \( JG^\circ \) will be identified with \( J \).

2. Projective planes in \( S^4 \). Let \( P \) be a polyhedral projective plane in \( S^4 \). By the Alexander duality theorem, the abelianization of the fundamental group \( G_P \) of \( S^4 - P \) is a cyclic group of order 2. We use \( J_2 \) as the coefficient group for homology. Let \( l \) be a 2-cycle on \( P \).

**Theorem 2.**

\[
\begin{align*}
\{ E_0(l) \}^\circ &= (2) \text{ and} \\
\{ E_d(l) \}^\circ &= (1), \text{ if } d > 0, \text{ in } J.
\end{align*}
\]

**Proof.** This follows to Theorem 2 in [4].

A projective plane \( P \) has only two cycles. First let \( l_0 \) be the trivial one.

**Theorem 3.**

\[
\begin{align*}
\{ E_0(l_0) \} &= (2) \text{ and} \\
\{ E_d(l_0) \} &= (1), \text{ if } d > 0, \text{ in } JH_2.
\end{align*}
\]

**Proof.** The proof is similar to that of Theorem 3 in [4].

Now let \( l \) be the nontrivial 2-cycle on \( P \), i.e., the fundamental cycle for \( J_2 \)-orientation of \( P \). Since the homomorphism \( \psi \) is onto in this case, by Theorem 1 we have \( E_0(l) \subset (1 + t) \) in \( JH_2 \). Let \( E_0(l) = (1 + t)E(l) \).

**Theorem 4.** \( E(l)^\circ = (1) \) in \( J \).

**Proof.** Since

\[
(2) = E_0(l)^\circ = (1 + t)^\circ E(l)^\circ = (2)E(l)^\circ
\]

in \( J \), we have \( E(l)^\circ = (1) \) in \( J \).

Further \( E(l)^\ast \subset J \) is also an invariant of \( P \) in \( S^4 \).

**Theorem 5.** The ideal \( E(l)^\ast \) in \( J \) is generated by an odd integer.
Proof. Let $E(l)$ be generated by $a_i + b_i t$ ($i = 1, 2, \ldots, n$) in $JH_2$. Assume on the contrary that $E(l)^*$ is not generated by an odd integer. Then we have $a_i - b_i = 0 \mod. 2$ for every $i$. From this it follows that $a_i + b_i = 0 \mod. 2$ for every $i$. Hence we have $E(l)^* \neq (1)$ in $J$ which contradicts Theorem 4.

Example 1. Let $f(t)$ be an integral polynomial with $f(1) = 1$. Then, for each $f(t)$ there is a polyhedral, locally flat projective plane $P_f$ in $S^4$, where the odd natural number $|f(-1)|$ is a topological invariant of $P_f$ in $S^4$ (see [3]). In these example, it is easy to see that for the nontrivial 2-cycle $l$ on $P_f$ we have $E(l) = (f(t))$ in $JH_2$, where $f(t)$ is considered as an element of $JH_2$. Further we have $E(l)^* = (f(-1))$ in $J$.

3. $\theta$-curves in $S^3$. Let $P$ and $Q$ be two distinct points in $S^3$ and let $a_1$, $a_2$, and $a_3$ be three polygonal arcs from $P$ to $Q$, which are mutually disjoint to each other except at $P$ and $Q$. Then $L = a_1 \cup a_2 \cup a_3$ is called a $\theta$-curve in $S^3$. Further, if each of these three arcs is oriented from $P$ to $Q$, then $L$ is called an oriented $\theta$-curve in $S^3$. From now on we use $J_3$ as the coefficient group for homology.

Let $L$ be a $\theta$-curve in $S^3$. Then the abelianization of the fundamental group of $S^3 - L$ is a free abelian group of rank 2. Let $l$ be a 1-cycle on $L$.

Theorem 6. \[ \begin{cases} E_0(l)^* = E_1(l)^* = (0) \text{ and} \\ E_2(l)^* = (1), \text{ if } d > 1, \text{ in } J. \end{cases} \]

Proof. This follows to Theorem 9 in [4].

Theorem 7. $E_0(l) = E_1(l) = (0)$ in $JH_2$.

Proof. This follows to corollary of Theorem 7 in [4].

Now let $L$ be an oriented $\theta$-curve in $S^3$. Then there is a nontrivial 1-cycle $l$ on $L$ such that the coefficient of $l$ for each oriented 1-simplex of $L$ is $1 \in J_3$. The 1-cycle $l$ is called the fundamental cycle for the $J_3$-orientation of $L$. Then, $E_0(l)$ in $JH_2$ and $E_2(l)^*$ in $J[\omega]$, where $\omega = e^{2\pi i/3}$, are topological invariants of the oriented $\theta$-curve $L$ in $S^3$.

Example 2. Let $L$ be the example of an oriented $\theta$-curve in [4], where the orientation of $L$ is given as shown in the figure in [4]. Let $l$ be the fundamental cycle for this $J_3$-orientation of $L$. Then we have
in $JH_3$ and $E_0(l)^* = (2)$ in $J[\omega]$.

**Theorem 8.** Let $f(t) \in JH_3$ with $f(1) = 1$. Then there is an oriented $\theta$-curve $L$ in $S^3$ such that for the fundamental cycle $l$ for the $J_3$-orientation of $L$ we have

$$
\begin{align*}
E_0(l) &= E_1(l) = (0) , \\
E_2(l) &= (t^2 + t + 1, 2) \quad \text{and} \\
E_d(l) &= (1) , \quad \text{if} \quad d > 2 , \\
\end{align*}
$$

in $JH_3$.

**Proof.** Let $f(\tau) \in JH$ with $f(1) = 1$, where $H$ is an infinite cyclic multiplicative group generated by $\tau$. Then there is an example of a $\theta$-curve $L_1$ and a 1-cycle $l_1$ on $L_1$ such that

$$
\begin{align*}
E_0(l_1) &= E_1(l_1) = (1) , \\
E_2(l_1) &= (f(\tau)) \quad \text{and} \\
E_d(l_1) &= (1) , \quad \text{if} \quad d > 2 , \\
\end{align*}
$$

in $JH$ (see [5]). The coefficients of $l_1$ on $L_1$ are distributed as shown in Fig. 1. Note that arcs in the outside of the cube shown by dotted lines in the figure are possibly complicated. Now the sequence of elementary ideals remains invariant, even if $L_1$ is “blown up” to a cube with 2 handles. Then the 1-cycle $l_2$ on $L_2$ as shown in Fig. 4 has the same sequence of elementary ideals to that of $l_1$ on $L_1$. Considering $l_2$ in the homology of integers modulo 3, we have an example of a 1-cycle $l$ on $L$ as shown in Fig. 5. The 1-cycle $l$ is the fundamental cycle of a $J_3$-orientation of the $\theta$-curve $L$ and for each $d$ we have $E_d(l) = E_d(l)^*$ in $JH_3$, where $'$ is a ring homomorphism.
of $JH$ onto $JH_5$ defined by $\tau' = t$. Now the theorem can be seen easily.

**References**

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Received July 16, 1974.

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