ON ELEMENTARY IDEALS OF PROJECTIVE PLANES IN THE 4-SPHERE AND ORIENTED Θ-CURVES IN THE 3-SPHERE

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The concept of an infinite cyclic covering has been applied
to knot theory. In this paper that of a finite cyclic covering
is considered. This enable us to study such cases as pro-
jective planes in the 4-sphere and oriented $\theta$-curves in the
3-sphere. Some properties of elementary ideals of these
cases are examined. The technique of free differential cal-
culus is used, instead of that of coverings.

Let $L$ be a polyhedron in an $n$-sphere $S^n$ ($n \geq 1$) that does not
separate $S^n$, and let $G_L$ be the fundamental group of $S^n - L$. We
use the additive group $J_p$ of integers modulo $p$ as the coefficient group
for homology. Let $l$ be an $(n - 2)$-dimensional cycle on $L$. Let $H_p$
be the multiplicative cyclic group of order $p$, generated by $t$. Then,
there is a homomorphism $\psi$ of $G_L$ into $H_p$ such that for each $g \in G_L$,

$$g^\psi = t^{\text{link}(g, l)},$$

where $\text{link}(g, l) \in J_p$ is the linking number between $g$ and $l$ in $S^n$.

Using Fox's free differential calculus ([1], [2]), we associate to $\psi$
a sequence of elementary ideals $E_d(G_L, \psi)$ of the group $G_L$, evaluated
in the group ring $JH_p$ of $H_p$ over integers $J$. This sequence of
elementary ideals depends only on $G_L$ and $\psi$, and hence it depends
only on the position of $l$ on $L$ in $S^n$. We shall denote it by $E_d(l)$. If $l$ and $l'$ are homologous on $L$, then $E_d(l) = E_d(l')$ for every $d$.

In this paper we apply these elementary ideals $E_d(l)$ to the study
of the position of $L$ in $S^n$. The following two cases of $E_d(l)$ are
considered: (1) $L$ is a projective plane in $S^4$ and $p = 2$, and (2) $L$
is a $\theta$-curve in $S^3$ and $p = 3$.

1. Miscellanea. Let $\sigma(t) = 1 + t + \cdots + t^{p-1} \in JH_p$.

**Theorem 1.** If $\psi$ is onto, then $E_0(l) \subset (\sigma(t))$ in $JH_p$.

**Proof.** It is proved in [2] that

$$E_0(H_p, \text{id}) = (\sigma(t)),$$

where $\text{id}$ is the identity isomorphism of $H_p$. From the diagram

$$G_L \xrightarrow{\psi} H_p \xrightarrow{\text{id}} H_p,$$
where $\psi$ is onto, and Theorem 1 in [4], it follows that $E_0(l) \subset (\sigma(t))$ in $JH_\rho$.

Now assume that $\psi$ is onto, and let $E_0(l) = \sigma(t)E(l)$. Let $\omega = e^{\pi i/p}$ and let $J[\omega]$ be the ring of all complex numbers of the form $\sum_{i=0}^{p-1} a_i \omega^i$, where $a_i \in J$ ($i = 0, 1, \ldots, p - 1$). A homomorphism $*$ of $H_\rho$ into $J[\omega]$ is defined by $t^* = \omega$. We naturally extend $*$ to a ring homomorphism of $JH_\rho$ onto $J[\omega]$. Though $E_0(l)^* = (0)$, sometimes $E(l)^*$ is a nontrivial ideal in $J[\omega]$.

A trivializer of a group $G$ is a homomorphism of $G$ onto the trivial group that consists of only one element. Any trivializer will be denoted by the same notation $\circ$ in this paper. Further the group ring $JG^\circ$ will be identified with $J$.

2. Projective planes in $S^4$. Let $P$ be a polyhedral projective plane in $S^4$. By the Alexander duality theorem, the abelianization of the fundamental group $G_P$ of $S^4 - P$ is a cyclic group of order 2. We use $J_z$ as the coefficient group for homology. Let $l$ be a 2-cycle on $P$.

**Theorem 2.**

\[
\begin{align*}
E_0(l)^\circ &= (2) \text{ and} \\
E_0(l)^\circ &= (1), \text{ if } d > 0, \text{ in } J.
\end{align*}
\]

**Proof.** This follows to Theorem 2 in [4].

A projective plane $P$ has only two cycles. First let $l_0$ be the trivial one.

**Theorem 3.**

\[
\begin{align*}
E_0(l_0) &= (2) \text{ and} \\
E_0(l_0) &= (1), \text{ if } d > 0, \text{ in } JH_2.
\end{align*}
\]

**Proof.** The proof is similar to that of Theorem 3 in [4].

Now let $l$ be the nontrivial 2-cycle on $P$, i.e., the fundamental cycle for $J_z$-orientation of $P$. Since the homomorphism $\psi$ is onto in this case, by Theorem 1 we have $E_0(l) \subset (1 + t)$ in $JH_z$. Let $E_0(l) = (1 + t)E(l)$.

**Theorem 4.** $E(l)^\circ = (1)$ in $J$.

**Proof.** Since

\[
(2) = E_0(l)^\circ = (1 + t)^\circ E(l)^\circ = (2)E(l)^\circ
\]

in $J$, we have $E(l)^\circ = (1)$ in $J$.

Further $E(l)^* \subset J$ is also an invariant of $P$ in $S^4$.

**Theorem 5.** The ideal $E(l)^*$ in $J$ is generated by an odd integer.
Proof. Let $E(l)$ be generated by $a_i + b_i t$ ($i = 1, 2, \ldots, n$) in $JH_2$. Assume on the contrary that $E(l)^*$ is not generated by an odd integer. Then we have $a_i - b_i = 0 \mod 2$ for every $i$. From this it follows that $a_i + b_i = 0 \mod 2$ for every $i$. Hence we have $E(l)^* \neq (1)$ in $J$ which contradicts Theorem 4.

**Example 1.** Let $f(t)$ be an integral polynomial with $f(1) = 1$. Then, for each $f(t)$ there is a polyhedral, locally flat projective plane $P_f$ in $S'$, where the odd natural number $|f(-1)|$ is a topological invariant of $P_f$ in $S'$ (see [3]). In these example, it is easy to see that for the nontrivial 2-cycle $l$ on $P_f$ we have $E(l) = (f(t))$ in $JH_2$, where $f(t)$ is considered as an element of $JH_2$. Further we have $E(l)^* = (f(-1))$ in $J$.

3. $\theta$-curves in $S'$. Let $P$ and $Q$ be two distinct points in $S'$ and let $a_1$, $a_2$, and $a_3$ be three polygonal arcs from $P$ to $Q$, which are mutually disjoint to each other except at $P$ and $Q$. Then $L = a_1 \cup a_2 \cup a_3$ is called a $\theta$-curve in $S'$. Further, if each of these three arcs is oriented from $P$ to $Q$, then $L$ is called an oriented $\theta$-curve in $S'$.

From now on we use $J_s$ as the coefficient group for homology.

Let $L$ be a $\theta$-curve in $S'$ Then the abelianization of the fundamental group of $S' - L$ is a free abelian group of rank 2. Let $l$ be a 1-cycle on $L$.

**Theorem 6.** \[
\begin{cases}
E_0(l)^{o} = E_1(l)^{o} = (0) & \text{and} \\
E_d(l)^{o} = (1), \text{ if } d > 1, \text{ in } J.
\end{cases}
\]

**Proof.** This follows to Theorem 9 in [4].

**Theorem 7.** $E_0(l) = E_1(l) = (0)$ in $JH_2$.

**Proof.** This follows to corollary of Theorem 7 in [4].

Now let $L$ be an oriented $\theta$-curve in $S'$. Then there is a nontrivial 1-cycle $l$ on $L$ such that the coefficient of $l$ for each oriented 1-simplex of $L$ is $1 \in J_3$. The 1-cycle $l$ is called the fundamental cycle for the $J_3$-orientation of $L$. Then, $E_2(l)$ in $JH_3$ and $E_2(l)^*$ in $J[\omega]$, where $\omega = e^{2\pi i/3}$, are topological invariants of the oriented $\theta$-curve $L$ in $S'$.

**Example 2.** Let $L$ be the example of an oriented $\theta$-curve in [4], where the orientation of $L$ is given as shown in the figure in [4]. Let $l$ be the fundamental cycle for this $J_3$-orientation of $L$. Then we have
\begin{equation*}
\begin{cases}
E_0(l) = E_1(l) = (0) , \\
E_2(l) = (t^2 + t + 1, 2) \text{ and} \\
E_d(l) = (1) , \text{ if } d > 2 ,
\end{cases}
\end{equation*}
in \(JH_3\) and \(E_2(l)^* = (2)\) in \(J[\omega]\).

**Theorem 8.** Let \(f(t) \in JH_3\) with \(f(1) = 1\). Then there is an oriented \(\theta\)-curve \(L\) in \(S^3\) such that for the fundamental cycle \(l\) for the \(J_3\)-orientation of \(L\) we have

\begin{equation*}
\begin{cases}
E_0(l) = E_1(l) = (1) , \\
E_2(l) = (f(t)) \text{ and} \\
E_d(l) = (1) , \text{ if } d > 2 , \text{ in } JH_3 .
\end{cases}
\end{equation*}

**Proof.** Let \(f(\tau) \in JH\) with \(f(1) = 1\), where \(H\) is an infinite cyclic multiplicative group generated by \(\tau\). Then there is an example of a \(\theta\)-curve \(L\), and a 1-cycle \(l_1\) on \(L_1\) such that

\begin{equation*}
\begin{cases}
E_0(l_1) = E_1(l_1) = (1) , \\
E_2(l_1) = (f(\tau)) \text{ and} \\
E_d(l_1) = (1) , \text{ if } d > 2 ,
\end{cases}
\end{equation*}
in \(JH\) (see [5]). The coefficients of \(l_1\) on \(L_1\) are distributed as shown in Fig. 1. Note that arcs in the outside of the cube shown by dotted lines in the figure are possibly complicated. Now the sequence of elementary ideals remains invariant, even if \(L_1\) is “blown up” to a cube with 2 handles. Then the 1-cycle \(l_2\) on \(L_2\) as shown in Fig. 4 has the same sequence of elementary ideals to that of \(l_1\) on \(L_1\). Considering \(l_2\) in the homology of integers modulo 3, we have an example of a 1-cycle \(l\) on \(L\) as shown in Fig. 5. The 1-cycle \(l\) is the fundamental cycle of a \(J_3\)-orientation of the \(\theta\)-curve \(L\) and for each \(d\) we have \(E_d(l) = E_d(l,)'\) in \(JH_3\), where ′ is a ring homomorphism.
of \( JH \) onto \( JH_3 \) defined by \( \tau' = t \). Now the theorem can be seen easily.

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Received July 16, 1974.

Florida State University
Pacific Journal of Mathematics
Vol. 57, No. 1 January, 1975

Keith Roy Allen, *Dendritic compactification* ........................................ 1
Daniel D. Anderson, *The Krull intersection theorem* ................................. 11
George Phillip Barker and David Hilding Carlson, *Cones of diagonally dominant matrices* ................................................................. 15
David Wilmot Barnette, *Generalized combinatorial cells and facet splitting* .......... 33
Stefan Bergman, *Bounds for distortion in pseudoconformal mappings* ................. 47
Nguyễn Phuong Các, *On bounded solutions of a strongly nonlinear elliptic equation* .......... 53
Philip Throop Church and James Timourian, *Maps with 0-dimensional critical set* .............................................................. 59
G. Coquet and J. C. Dupin, *Sur les convexes ubiquitaires* ............................. 67
Kandiah Dayanithy, *On perturbation of differential operators* ............................. 85
Thomas P. Dence, *A Lebesgue decomposition for vector valued additive set functions* ................................................................. 91
John Riley Durbin, *On locally compact wreath products* .................................. 99
Allan L. Edelson, *The converse to a theorem of Conner and Floyd* ...................... 109
William Alan Feldman and James Franklin Porter, *Compact convergence and the order bidual for C(X)* .......................................................... 113
Ralph S. Freese, *Ideal lattices of lattices* ..................................................... 125
R. Gow, *Groups whose irreducible character degrees are ordered by divisibility* ..... 135
David G. Green, *The lattice of congruences on an inverse semigroup* .................. 141
John William Green, *Completion and semicompletion of Moore spaces* ................ 153
David James Hallenbeck, *Convex hulls and extreme points of families of starlike and close-to-convex mappings* .................................................. 167
Israel (Yitzchak) Nathan Herstein, *On a theorem of Brauer-Cartan-Hua type* ........ 177
Virgil Dwight House, Jr., *Countable products of generalized countably compact spaces* ................................................................................. 183
John Sollion Hsia, *Spinor norms of local integral rotations. I* ......................... 199
Hugo Junghenn, *Almost periodic compactifications of transformation semigroups* .......................................................... 207
Shin’ichi Kinoshita, *On elementary ideals of projective planes in the 4-sphere and oriented Θ-curves in the 3-sphere* ........................................ 217
Ronald Fred Levy, *Showering spaces* ........................................................... 223
Geoffrey Mason, *Two theorems on groups of characteristic 2-type* ..................... 233
Cyril Nasim, *An inversion formula for Hankel transform* .................................. 255
W. P. Novinger, *Real parts of uniform algebras on the circle* ............................... 259
T. Parthasarathy and T. E. S. Raghavan, *Equilibria of continuous two-person games* ............................................................................. 265
John Pfaltzgraff and Ted Joe Suffridge, *Close-to-starlike holomorphic functions of several variables* .......................................................... 271
Esther Portnoy, *Developable surfaces in hyperbolic space* .................................. 281
Maxwell Alexander Rosenlicht, *Differential extension fields of exponential type* ...... 289
Keith William Schrader and James Lewis Thornburg, *Sufficient conditions for the existence of convergent subsequences* .................................... 301
Joseph M. Weinstein, *Reconstructing colored graphs* ........................................ 307