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For each triple α, β, ω where $\alpha > 0$ and $\beta \geqq \aleph_0$ are cardinal numbers and $\omega > 0$ is an ordinal number, $S^{\omega}_{\alpha,\beta}$ is defined. It is proved that $S^{\omega}_{\alpha,\beta}$ is Hausdorff, paracompact and zero-dimensional. Various topological properties of $S^{\omega}_{\alpha,\beta}$ are discussed and are used to give examples.

It this paper the showering space $S^{\omega}_{\alpha,\beta}$ is defined where α and β are cardinal numbers and ω is an ordinal number. An important characteristic of the showering spaces is that various topological properties are determined by set theoretic properties of the indices α, β, and ω. Thus, for example, whether or not $S^{\omega}_{\alpha,\beta}$ is Baire is essentially a question of whether or not ω is a sequential ordinal. Theorems of this nature are given in §1–§5. Specific examples using these results, such as an almost P-space which contains a closed copy of the space of rationals, are given in §6.

It has come to my attention that the special cases $S^{\omega_0+\omega_0}_{\aleph_0,\aleph_0}$ and $S^{\omega_1}_{\aleph_0,\aleph_0}$ are defined in [3] and [1].

1. Definition and basic properties of showering spaces.

Given spaces will be assumed to be completely regular (Hausdorff). Let $\alpha > 0$ be a fixed cardinal number and let $\omega > 0$ be a fixed ordinal number. As usual we identify a cardinal with the initial ordinal of that cardinal. If γ and δ are ordinals, $[\gamma, \delta)$ will denote the half-open interval $\{\rho \mid \gamma \leqq \rho < \delta\}$ and $[\gamma, \delta]$ will denote the closed interval $\{\rho \mid \gamma \leqq \rho \leqq \delta\}$. If A is a set, $|A|$ will denote the cardinal of A. Many of the proofs of this section, particularly Theorem 1.6, were given for special cases in [1] and [3].

For each $\gamma < \omega$, let $R_\gamma = [0, \alpha)^{[0,\gamma)}$. In particular, $R_0 = \{\varnothing\}$. When we are discussing $\varnothing$ as the element of R_0, we will usually denote it p_0, so $R_0 = \{p_0\}$. For $0 < \gamma < \omega$, the elements of R_γ are nets valued in $[0, \alpha)$ and indexed by the initial segment $[0, \gamma)$ of ordinals. Let $S^{\omega}_{\alpha} = \bigcup_{\gamma<\omega} R_\gamma$. Define a relation $\prec$ on S^{ω}_{α} by $(x_\lambda)_{\lambda<\gamma_1} \prec (y_\lambda)_{\lambda<\gamma_2}$ if $\gamma_1 \leqq \gamma_2$ and $x_\lambda = y_\lambda$ for all $\lambda < \gamma_1$. In particular, $p_0 \prec x$ for all $x \in S^{\omega}_{\alpha}$. It is easy to verify that $\prec$ is a partial ordering on S^{ω}_{α}.

PROPOSITION 1.1. (1) *$(S^{\omega}_{\alpha}, \prec)$ is a tree.*

(2) *If $\gamma \leqq \delta < \omega$ and $y \in R_\delta$, then there is exactly one $x \in R_\gamma$ such that $x \prec y$.*

(3) *If $\gamma + 1 < \omega$, $p \in R_\gamma$, and $A_p = \{x \in R_{\gamma+1} \mid p \prec x\}$, then $|A_p| = \alpha$.*

(4) *If* $x \in R_{\gamma_1}$, $y \in R_{\gamma_2}$, $x \prec y$, *then* $\gamma_1 \leqq \gamma_2$ *and* $\gamma_1 = \gamma_2$ *if and only if* $x = y$.

(5) *If* $\gamma \leqq \gamma_0 < \omega$, $x \in R_\gamma$, *then there is a* $y \in R_{\gamma_0}$ *such that* $x \prec y$.

Proof. (1) Suppose $x \in S^\omega_\alpha$. We must show that $\{y \mid y \prec x\}$ is well-ordered. Suppose $a \prec x$, $b \prec x$, $a = (a_\lambda)_{\lambda<\gamma_1}$, $b = (b_\lambda)_{\lambda<\gamma_2}$, $x = (x_\lambda)_{\lambda<\gamma_3}$, with $\gamma_1 \leqq \gamma_2$. Then $\gamma_1 \leqq \gamma_2 \leqq \gamma_3$ and if $\gamma < \gamma_1$, $a_\lambda = x_\lambda = b_\lambda$, so $a \prec b$. Hence $\{y \mid y \prec x\}$ is a chain. If $\varnothing \neq A \subseteqq \{y \mid y \prec x\}$, we must show that there is a $q \in A$ such that $q \prec y$ for all $y \in A$. Let γ_0 be the smallest ordinal such that there is an element of A of the form $(y_\lambda)_{\lambda<\gamma_0}$. If $q = (q_\lambda)_{\lambda<\gamma_0} \in A$ and $(z_\lambda)_{\lambda<\gamma_1} \in A$, and if $x = (x_\lambda)_{\lambda<\gamma_2}$, then $\gamma_0 \leqq \gamma_1 \leqq \gamma_2$ and if $\lambda < \gamma_0$, $q_\lambda = x_\lambda = z_\lambda$, so $q \prec (z_\lambda)_{\lambda<\gamma_1}$. Hence, A has a smallest element.

(2) If $y = (y_\lambda)_{\lambda<\delta}$, and $x = (y_\lambda)_{\lambda<\gamma}$, $x \in R_\gamma$, $x \prec y$, and x is the only element of R_γ such that $x \prec y$.

(3) If $p = (p_\lambda)_{\lambda<\gamma}$, $A_p = \{(x_\lambda)_{\lambda\leqq\gamma} \mid x_\lambda = p_\lambda \text{ for all } \lambda < \gamma\}$. Therefore, $|A_p| = |[0, \alpha)| = \alpha$.

(4) The only statement that needs proof is that if $\gamma_1 = \gamma_2$, $x = y$. But if $x = (x_\lambda)_{\lambda<\gamma_1}$, $y = (y_\lambda)_{\lambda<\gamma_1}$, then $x_\lambda = y_\lambda$ for all $\lambda < \gamma_1$, so $x = y$.

(5) If $x = (x_\lambda)_{\lambda<\gamma}$, let $y = (y_\lambda)_{\lambda<\gamma_0}$ be defined by $y_\lambda = x_\lambda$ if $\lambda < \gamma$ and $y_\lambda = 0$ if $\gamma \leqq \lambda < \gamma_0$. Then $y \in R_{\gamma_0}$ and $x \prec y$.

DEFINITION. $(S^\omega_\alpha, \prec)$ is called the *showering tree of type* α, ω. If $\gamma + 1 < \omega$ and $p \in R_\gamma$, A_p will denote the set $\{x \in R_{\gamma+1} \mid p \prec x\}$. If $\gamma + 1 = \omega$ and $p \in R_\gamma$, let $A_p = \varnothing$.

REMARK. Any tree can be order-embedded in a showering tree. Specifically, if T is a tree, $\omega = \sup\{\lambda \mid T \text{ has a chain of order type } \lambda\}$, and $\alpha = \sup\{\gamma \mid \text{some element of } T \text{ has } \gamma \text{ immediate successors}\}$, then T can be order-embedded in S^ω_α.

We will now introduce a topology on S^ω_α. For $p \in S^\omega_\alpha$ and $A \subseteqq A_p$, let $\tilde{U}_p(A) = \{x \in S^\omega_\alpha \mid p \prec x$ but it is not the case that there is an $a \in A$ such that $a \prec x\}$. If $p \in R_\gamma$ and $A \subseteqq R_{\gamma+1}$, $\tilde{U}_p(A \cap A_p)$ will be denoted $U_p(A)$. Note that $U_p(\varnothing)$ is just the set of successors of p. Now let $\beta \geqq \aleph_0$ be a fixed cardinal number. For $p \in S^\omega_\alpha$, let $\mathscr{U}_p = \{U_p(A) \mid |A| < \beta\}$.

LEMMA 1.2. *If* $p \in R_\gamma$, $A \subseteqq R_{\gamma+1}$, *and* $q \in A_p \backslash A$, *then* $U_q(\varnothing) \subseteqq U_p(A)$.

Proof. Suppose $x \in U_q(\varnothing)$. Then $q \prec x$, so q is the unique element of $R_{\gamma+1}$ such that $q \prec x$. Therefore, there is no $a \in$ A such that $a \prec x$. Hence, $x \in U_p(A)$.

THEOREM 1.3. *$\{\mathscr{U}_p \mid p \in S_\alpha^\omega\}$ is a system of neighborhood bases for a topology on S_α^ω.*

Proof. Clearly, $p \in U$ for each $U \in \mathscr{U}_p$. If $U_p(A)$, $U_p(\tilde{A}) \in \mathscr{U}_p$, $U_p(A) \cap U_p(\tilde{A}) = U_p(A \cup \tilde{A})$, and since $|A \cup \tilde{A}| \leqq |A| + |\tilde{A}| < \beta + \beta = \beta$, $U_p(A) \cap U_p(\tilde{A}) \in \mathscr{U}_p$. If $U_p(A) \in \mathscr{U}_p$ and $s \in U_p(A)\backslash\{p\}$, choose $q \in A_p$ such that $q \prec s$. Such a q exists by Proposition 1.1, $q \notin A$. Hence $U_q(\phi) \subseteqq U_p(A)$ by Lemma 1.2. But since $q \prec s$, $U_s(\varnothing) \subseteqq U_q(\varnothing)$. Therefore $U_s(\varnothing) \subseteqq U_p(A)$. Since $U_s(\varnothing) \in \mathscr{U}_s$, this completes the proof.

DEFINITION. Let $S_{\alpha,\beta}^\omega$ denote the topological space obtained from S_α^ω by taking as a neighborhood base at $p \in S_\alpha^\omega$ the collection $\mathscr{U}_p$. $S_{\alpha,\beta}^\omega$ is called the *showering space of type α, β, ω.*

LEMMA 1.4. *If p, $q \in R_\gamma$, $p \neq q$, then $U_p(\varnothing) \cap U_q(\varnothing) = \varnothing$. If $s \in R_\delta$, $\gamma < \delta$, and t is the unique element of $R_{\gamma+1}$ such that $t \prec s$, then $U_p(\{t\}) \cap U_s(\varnothing) = \varnothing$.*

Proof. The first statement is immediate from (2) of Proposition 1.1. For the second statement, we observe that $U_p(\{t\}) \cap U_t(\varnothing) = \varnothing$, and since $U_s(\varnothing) \subseteqq U_t(\varnothing)$, $U_p(\{t\}) \cap U_s(\varnothing) = \varnothing$.

LEMMA 1.5. (1) *Every set of the form $U_p(A)$ is closed in $S_{\alpha,\beta}^\omega$* (irrespective of the cardinal of A).

(2) *Every element of $\mathscr{U}_p$ is open in $S_{\alpha,\beta}^\omega$ for all p.*

Proof. (1) Suppose $p \in R_\gamma$ and $q \notin U_p(A)$. Suppose $q \in R_\delta$. If $\delta < \gamma$, choose $s \in R_{\delta+1}$ such that $s \prec p$; then by Lemma 1.4, $U_q(\{s\}) \cap U_p(\varnothing) = \varnothing$, so $U_q(\{s\}) \cap U_p(A) = \varnothing$. Therefore, we may assume $\gamma \leqq \delta$. If $\gamma = \delta$, $q \neq p$ so $U_q(\varnothing) \cap U_p(\varnothing) = \varnothing$ by Lemma 1.4; hence $U_q(\varnothing) \cap U_p(A) = \varnothing$. If $\gamma < \delta$, choose $t \in R_{\gamma+1}$ such that $t \prec q$. Then $t \in A$ or $t \notin A_p$. If $t \in A$, $U_t(\varnothing) \cap U_p(A) = \varnothing$, so $U_q(\varnothing) \cap U_p(A) = \varnothing$. If $t \notin A_p$, $t \in A_s$ for some $s \neq p$. $U_s(\varnothing) \cap U_p(\varnothing) = \varnothing$, so $U_q(\varnothing) \cap U_p(A) = \varnothing$. Thus q has a neighborhood which misses $U_p(A)$. Therefore, $U_p(A)$ is closed.

(2) This is immediate from Lemma 1.2 and the fact that if $q \prec t$, $U_t(\varnothing) \subseteqq U_q(\varnothing)$.

THEOREM 1.6. *$S_{\alpha,\beta}^\omega$ is Hausdorff, zero-dimensional* (that is, ind $S_{\alpha,\beta}^\omega = 0$), *and paracompact. In fact, every open cover of $S_{\alpha,\beta}^\omega$ has a discrete open refinement which covers $S_{\alpha,\beta}^\omega$.*

Proof. That $S_{\alpha,\beta}^\omega$ is Hausdorff is immediate from Lemma 1.4. $S_{\alpha,\beta}^\omega$ is zero-dimensional by Lemma 1.5. We now prove that $S_{\alpha,\beta}^\omega$ is paracompact. Suppose $\mathscr{V}$ is an open cover of $S^{\omega}_{\alpha,\beta}$. Choose $A \subseteqq A_{p_0}$ such that $|A| < \beta$ and $U_{p_0}(A) \subseteqq V$ for some $V \in \mathscr{V}$. Let $\mathscr{V}_0 = \{U_{p_0}(A)\}$. Suppose inductively that $\mathscr{V}_\gamma$ is defined for $\gamma < \gamma_0 < \omega$ such that $\bigcup_{\gamma<\gamma_0} \mathscr{V}_\gamma$ is an open refinement of $\mathscr{V}$ which covers $\bigcup_{\gamma<\gamma_0} R_\gamma$ and whose elements are pairwise disjoint. Let $\widetilde{\mathscr{V}}_{\gamma_0} = \bigcup_{\gamma<\gamma_0} \mathscr{V}_\gamma$. For each $p \in R_{\gamma_0} \backslash \bigcup \{V \mid V \in \widetilde{\mathscr{V}}_{\gamma_0}\}$, choose $B_p \subseteqq A_p$ such that $|B_p| < \beta$ and such that $U_p(B_p) \subseteqq V$ for some $V \in \mathscr{V}$. Let $\mathscr{V}_{\gamma_0} = \{U_p(B_p) \mid p \in R_{\gamma_0} \backslash \bigcup \{V \mid V \in \widetilde{\mathscr{V}}_{\gamma_0}\}\}$. Then $\bigcup_{\gamma<\gamma_0+1} \mathscr{V}_\gamma$ is an open refinement of $\mathscr{V}$ which covers $\bigcup_{\gamma<\gamma_0+1} R_\gamma$ and whose elements are pairwise disjoint. Let $\widetilde{\mathscr{V}} = \bigcup_{\gamma<\omega} \mathscr{V}_\gamma$. Then $\widetilde{\mathscr{V}}$ is an open refinement of $\mathscr{V}$ by pairwise disjoint sets which covers $S^{\omega}_{\alpha,\beta}$. Hence, $S^{\omega}_{\alpha,\beta}$ is paracompact.

REMARK 1.7. A slight modification of the proof of Theorem 1.6 shows that for any α, $S^{\omega_0}_{\alpha,\aleph_0}$ and $S^{\omega_0}_{\alpha,\aleph_1}$ are Lindelöf.

2. Isolated points and the Baire property. In this section we characterize the Baire showering spaces.

PROPOSITION 2.1. (1) *If $\beta > \alpha$, $S^{\omega}_{\alpha,\beta}$ is discrete* (and hence Baire).

(2) *If $\omega = \gamma + 1$, every point of R_γ is isolated.*

(3) *If $\alpha \geqq \beta$, $\gamma + 1 < \omega$, and $p \in R_\gamma$, then p is not isolated. In particular, if $\alpha \geqq \beta$ and ω is a limit ordinal, $S^{\omega}_{\alpha,\beta}$ is dense-in-itself.*

Proof. (1) $\{p\} = U_p(A_p)$, and if $\alpha < \beta$, $|A_p| = \alpha < \beta$, so $U_p(A_p)$ is open.

(2) If $A_p = \varnothing$, $U_p(\varnothing) = \{p\}$, so p is isolated.

(3) If $p \in R_\gamma$, $U_p(A) \in \mathscr{U}_p$, then $|A| < \beta \leqq \alpha$, so there is an $x \in A_p \backslash A$. Then $x \in U_p(A)$. Hence, p is not isolated.

THEOREM 2.2. *Suppose $\alpha \geqq \beta$.*

(1) *If $\omega = \gamma + 1$, then the set R_γ of isolated points is dense in $S^{\omega}_{\alpha,\beta}$. Hence, $S^{\omega}_{\alpha,\beta}$ is Baire.*

(2) *Suppose ω is a limit ordinal. Then the following are equivalent:*

(i) *ω is not a sequential ordinal, that is, if $\omega_i < \omega$ for $i = 1, 2, \cdots$, then $\sup \omega_i < \omega$.*

(ii) *$S^{\omega}_{\alpha,\beta}$ is Baire.*

(iii) *$S^{\omega}_{\alpha,\beta}$ is second category in itself.*

Proof. (1) Suppose $p \in S^{\omega}_{\alpha,\beta} \backslash R_\gamma$ and suppose $U_p(A) \in \mathscr{U}_p$. Since $\alpha \geqq \beta$, there is an $x \in A_p \backslash A$. By Proposition 1.1, there is a $y \in R_\gamma$ such that $x \prec y$. Then $y \in U_p(A)$.

(2) (i) implies (ii). Suppose ω is not a sequential ordinal. Let

$\{D_i\}_{i=1}^{\infty}$ be a decreasing sequence of open dense subsets of $S^{\omega}_{\alpha,\beta}$. Suppose $p \in S^{\omega}_{\alpha,\beta}$ and $U_p(A) \in \mathscr{U}_p$. We must show $U_p(A) \cap \bigcap_{i=1}^{\infty} D_i \neq \emptyset$. $U_p(A) \cap D_1 \neq \emptyset$, so there is a set $U_{x_1}(B_1)$ such that $|B_1| < \beta$ and $U_{x_1}(B_1) \subseteqq U_p(A) \cap D_1$. Choose $y^1 \in A_p \backslash B_1$. Then $U_{y^1}(\emptyset) \subseteqq U_p(A) \cap D_1$. Inductively, suppose $y^1, y^2, \cdots, y^{n-1}$ are chosen so that $U_{y^k}(\emptyset) \subseteqq U_p(A) \cap D_k$, and $y^1 \prec y^2 \prec \cdots \prec y^{n-1}$. There is a set $U_{x_n}(B_n) \in U_{y^{n-1}}$ such that $U_{x_n}(B_n) \in \mathscr{U}_{y^{n-1}}(\emptyset) \cap D_n$. Choose $y^n \in A_{x_n} \backslash B_n$. Then $U_{y^n}(\emptyset) \subseteqq U_p(A) \cap D_n$, and $y_{n-1} \prec y_n$. Hence, we have inductively defined a sequence $\{y^n\}_{n=1}^{\infty}$ such that $y^1 \prec y^2 \prec \cdots$ and such that $y^n \in U_p(A) \cap D_n$. Suppose $y^n \in R_{\gamma_n}$ and let $\delta = \sup\{\gamma_n \mid n = 1, 2 \cdots\}$. $\delta < \omega$ since ω is not sequential. Let $y = (\tilde{y}_\lambda)_{\lambda<\delta} \in R_\delta$ be defined by $\tilde{y}_\lambda = y^n_\lambda$ where $\lambda < \gamma_n$. y is well-defined since $y_k \prec y_{k+1}$ for all k. $y^n \prec y$ for all n. Hence $y \in \bigcap_{n=1}^{\infty} U_y n(\emptyset) \subseteqq U_p(A) \cap \bigcap_{n=1}^{\infty} D_n$. Thus $S^{\omega}_{\alpha,\beta}$ is Baire.

(ii) implies (iii) is trivial.

(iii) implies (i). If ω is sequential, there are ordinals $\gamma_1, \gamma_2, \cdots$, such that $\gamma_k < \omega$ for all k and such that $\omega = \sup_k \gamma_k$. Let $C_k = \bigcup_{\lambda<\gamma_k} R_\lambda$. Then each C_k is closed, and by Proposition 1.1(5), each C_k has empty interior. But $S^{\omega}_{\alpha,\beta} = \bigcup_{k=1}^{\infty} C_k$. Hence, $S^{\omega}_{\alpha,\beta}$ is not second category in itself.

3. *P*-spaces and almost *P*-spaces. If X is a space, a point $x \in X$ is called a *P-point* if every G_δ containing x is a neighborhood of x. A space X is called a *P-space* if every point of X is a P-point. X is called an almost *P-space* if every nonempty G_δ of X has nonempty interior. Every P-space is clearly an almost P-space. In Proposition 2.1 it was proved that if $\alpha < \beta$, $S^{\omega}_{\alpha,\beta}$ is discrete and hence a P-space. In this section we characterize the P-spaces and the almost P-spaces among the nondiscrete showering spaces. We note that since $S^{1}_{\alpha,\beta}$ consists of a single point, the case $\omega = 1$ will not be of interest to us.

PROPOSITION 3.1. *Suppose* $\alpha \geqq \beta$ *and* $\omega > 1$. *Then the following are equivalent*:

(i) $S^{\omega}_{\alpha,\beta}$ *is a P-space.*

(ii) $S^{\omega}_{\alpha,\beta}$ *has a non-isolated P-point.*

(iii) β *is not a sequential cardinal.*

Proof. (i) implies (ii) is trivial since by Proposition 2.1, p_0 is not isolated.

(ii) implies (iii). Let p be a non-isolated P-point. Suppose $\beta_i < \beta$ for $i = 1, 2, \cdots$. Let $B_1, B_2, \cdots$ be pairwise disjoint subsets of A_p such that $|B_k| = \beta_k$. Then $\bigcap_{k=1}^{\infty} U_p(B_k) = U_p(\bigcup_{k=1}^{\infty} B_k)$ is a neighborhood of p. Therefore, $\sum_{k=1}^{\infty} \beta_k = \sum_{k=1}^{\infty} |B_k| = |\bigcup_{k=1}^{\infty} B_k| < \beta$. Hence β is not sequential.

(iii) implies (i). Suppose $p \in S^{\omega}_{\alpha,\beta}$ and suppose $p \in \bigcap_{i=1}^{\infty} U_p(B_i)$ where each $U_p(B_i) \in \mathscr{U}_p$, and $B_i \subsetneqq A_p$ for each i. Then

$$\bigcap_{i=1}^{\infty} U_p(B_i) = U_p\Big(\bigcup_{i=1}^{\infty} B_i\Big),$$

and $|\bigcup_{i=1}^{\infty} B_i| \leqq \sum_{i=1}^{\infty} \beta_i < \beta$ since $\beta_i < \beta$ for each i and β is not sequential. Hence $\bigcap_{i=1}^{\infty} U_p(B_i)$ is a neighborhood of p. Thus, $S^{\omega}_{\alpha,\beta}$ is a P-space.

PROPOSITION 3.2. *Suppose $\alpha \geqq \beta$ and $\omega > 1$. Then $S^{\omega}_{\alpha,\beta}$ is an almost P-space if and only if $\sum_{i=1}^{\infty} \beta_i < \alpha$ whenever $\beta_i < \beta$ for each i.*

Proof. Suppose there were $\beta_1, \beta_2, \cdots$ such that $\beta_i < \beta$ for each i but $\sum_{i=1}^{\infty} \beta_i = \alpha$. Choose pairwise disjoint subsets $B_1, B_2, \cdots$ of A_{p_0} such that $\bigcup_{i=1}^{\infty} B_i = A_{p_0}$ and $|B_i| = \beta_i$. Then $\bigcap_{i=1}^{\infty} U_{p_0}(B_i) = U_{p_0}(\bigcup_{i=1}^{\infty} B_i) = U_{p_0}(A_{p_0}) = \{p_0\}$ which has empty interior by Proposition 2.1(3). Thus $S^{\omega}_{\alpha,\beta}$ is not almost P-space. For the converse, suppose $\sum_{i=1}^{\infty} \beta_i < \alpha$ whenever each $\beta_i < \beta$. Then for each $p \in S^{\omega}_{\alpha,\beta}$, if $U_p(B_i) \in \mathscr{U}_p$ where $B_i \subsetneqq A_p$ for each i, $\bigcap_{i=1}^{\infty} U_p(B_i) = U_p(\bigcup_{i=1}^{\infty} B_i)$. $|\bigcup_{i=1}^{\infty} B_i| \leqq \sum_{i=1}^{\infty} |B_i| \leqq \sum_{i=1}^{\infty} \beta_i < \alpha$. Hence there is a $q \in A_p \backslash \bigcup_{i=1}^{\infty} B_i$; $U_q(\varnothing) \subsetneqq \bigcap_{i=1}^{\infty} U_p(B_i)$, so $\operatorname{int} \bigcap_{i=1}^{\infty} U_p(B_i) \neq \varnothing$.

COROLLARY 3.3. *Suppose $\omega > 1$. Then the following are equivalent:*

(i) *$S^{\omega}_{\alpha,\alpha}$ is a P-space.*
(ii) *$S^{\omega}_{\alpha,\alpha}$ is an almost P-space.*
(iii) *α is not a sequential cardinal.*

4. Subspaces and autohomeomorphisms. In this section we state results about the embedding of showering spaces in other showering spaces and about autohomeomorphisms of showering spaces, but proofs will be omitted or only sketched since they are straightforward.

PROPOSITION 4.1. *Suppose $\omega \leqq \tilde{\omega}$ and $\alpha \leqq \tilde{\alpha}$. Then $S^{\omega}_{\alpha,\beta}$ is a closed subspace of $S^{\tilde{\omega}}_{\tilde{\alpha},\beta}$.*

LEMMA 4.2. *Suppose $[0, \omega)$ is order-isomorphic to $[\gamma, \omega)$ where $\gamma < \omega$, and suppose $p \in R_\gamma$. Then there is an order-preserving homeomorphism $f\colon S^{\omega}_{\alpha,\beta} \to U_p(\varnothing)$.*

Proof. Suppose $p = (p_\lambda)_{\lambda<\gamma}$. Define f by $f((x_\lambda)_{\lambda<\delta}) = (y_\lambda)_{\lambda<\gamma+\delta}$ where $y_\lambda = \begin{cases} p_\lambda & \text{if } \lambda < \gamma \\ x_{\tilde{\lambda}} & \text{if } \lambda = \gamma + \tilde{\lambda}. \end{cases}$ Then f is one-to-one and onto, and order-preserving. Thus, since the topology on $S^{\omega}_{\alpha,\beta}$ and the relative topology on $U_p(\emptyset)$ are each determined by the partial order $\prec$ and β, f is a homeomorphism.

LEMMA 4.3. *If* $[0, \omega)$, $[\gamma_1, \omega)$, *and* $[\gamma_2, \omega)$ *are mutually order-isomorphic, and if* $p \in R_{\gamma_1}$, $q \in R_{\gamma_2}$, *then there is a homeomorphism* F *of* $S^{\omega}_{\alpha,\beta}$ *such that* F^2 *is the identity and* $F(p) = q$(and hence $F(q) = p$).

Proof. By Lemma 4.2, there are order-preserving homeomorphisms $f_1: S^{\omega}_{\alpha,\beta} \to U_q(\emptyset)$ and $f_2: S^{\omega}_{\alpha,\beta} \to U_q(\emptyset)$. Let $f = f_2 \circ f_1^{-1}$. Then

$$f: U_p(\emptyset) \longrightarrow U_q(\emptyset)$$

is an order-preserving homeomorphism. There are essentially two cases: Case (i). $U_p(\emptyset) \cap U_q(\emptyset) = \emptyset$. Define F by $F | S^{\omega}_{\alpha,\beta} \backslash (U_p(\emptyset) \cup U_q(\emptyset))$ is the identity map, $F | U_p(\emptyset) = f$, $F | U_q(\emptyset) = f^{-1}$. Case (ii). $U_p(\emptyset) \supsetneqq U_q(\emptyset)$. Define F by $F | (S^{\omega}_{\alpha,\beta} \backslash U_p(\emptyset)) \cup U_{f(q)}(\emptyset)$ is the identity map,

$$F | U_p(\emptyset) \backslash U_q(\emptyset) = f | U_p(\emptyset) \backslash U_q(\emptyset),\ F | U_q(\emptyset) \backslash U_{f(q)}(\emptyset)$$
$$= f^{-1} | U_q(\emptyset) \backslash U_{f(q)}(\emptyset) .$$

DEFINITION. A space X is *bihomogeneous* if for each $p, q \in X$ there is a homeomorphism $f: X \to X$ such that $f(p) = q$ and $f(q) = p$.

PROPOSITION 4.4. *If* $[0, \omega)$ *is order-isomorphic to* $[\gamma, \omega)$ *for each* $\gamma < \omega$, *then* $S^{\omega}_{\alpha,\beta}$ *is bihomogeneous.*

5. First countability and developability. In this section we give a necessary and sufficient condition for a showering space to be first countable. We also give a necessary condition for a nondiscrete showering space to be developable. It will be shown in §6 that this condition is also sufficient for $S^{\omega}_{\alpha,\beta}$ to be developable and is in fact equivalent to the metrizability of $S^{\omega}_{\alpha,\beta}$. We recall that if $\alpha < \beta$ or if $\omega = 1$, $S^{\omega}_{\alpha,\beta}$ is discrete and hence metrizable. For this reason, these cases will not be of interest to us here.

PROPOSITION 5.1. *Suppose* $\omega > 1$ *and* $\alpha \geqq \beta$. *Then the following are equivalent:*

(i) $\alpha = \beta = \aleph_0$.
(ii) $S^{\omega}_{\alpha,\beta}$ *is first countable.*
(iii) $S^{\omega}_{\alpha,\beta}$ *has a nonisolated point of first countability.*

Proof. (i) implies (ii). If $\alpha = \beta = \aleph_0$, and $p \in S^{\omega}_{\alpha,\beta}$ is not isolated, let $A_p = \{a_1, a_2, \cdots\}$. Then $\{U_p(\{a_1, \cdots, a_n\})\}_{n=1}^{\infty}$ is a base at p.

(ii) implies (iii) is trivial.

(iii) implies (i). If $\alpha > \beta$, $S^{\omega}_{\alpha,\beta}$ is an almost P-space by Proposition 3.2, and hence $S^{\omega}_{\alpha,\beta}$ is not first countable at any nonisolated point. Therefore, if $S^{\omega}_{\alpha,\beta}$ is first countable at a nonisolated point p_1, $\alpha = \beta$. We must show that $\beta = \aleph_0$. Recall that β is always assumed to be infinite. Suppose $\beta > \aleph_0$. Let $\{U_{p_1}(B_i)\}_{i=1}^{\infty}$ be a countable base at p_1, with $B_i \subsetneqq A_{p_1}$, $|B_i| < \beta(=\alpha)$ for each i. Choose $x_i \in A_{p_1}\backslash B_i$ for each i. Let $B = \{x_i \mid i = 1, 2, \cdots\}$. Then $B \subsetneqq A_{p_1}$ and $|B| \leqq \aleph_0 < \beta$ so $U_{p_1}(B) \in \mathscr{U}_{p_1}$. But for each i, $x_i \in U_{p_1}(B_i)\backslash U_{p_1}(B)$, so $U_{p_1}(B)$ contains no $U_{p_1}(B_i)$, contradicting the assumption that $\{U_{p_1}(B_i)\}_{i=1}^{\infty}$ is a base at p_1.

REMARK 5.2. It is not difficult to prove that if $\omega > 1$ and $\alpha \geqq \beta$, then every point of $S^{\omega}_{\alpha,\beta}$ is a G_δ if and only if α is a sequential cardinal and $\alpha = \beta$. (Compare to Corollary 3.3.) Hence there are showering spaces which fail at each point to be first countable but which have countable pseudo character. $S^{\omega_0}_{\aleph_{\omega_0},\aleph_{\omega_0}}$ is such a space.

PROPOSITION 5.3. *Suppose* $\alpha \geqq \beta$. *If* $S^{\omega}_{\alpha,\beta}$ *is developable,* $\alpha = \beta = \aleph_0$ *and* $\omega \leqq \omega_0$.

Proof. Any developable space is first countable, so by Proposition 5.1, $\alpha = \beta = \aleph_0$. Suppose $\omega > \omega_0$. By Proposition 4.1, $S^{\omega_0+1}_{\alpha,\beta}$ is a subspace of $S^{\omega}_{\alpha,\beta}$, so it suffices to prove that $S^{\omega_0+1}_{\aleph_0,\aleph_0}$ is not developable. Suppose $\{\mathscr{D}_n\}_{n=1}^{\infty}$ is a countable collection of open covers of $S^{\omega_0+1}_{\aleph_0,\aleph_0}$. Choose $D_0 \in \mathscr{D}_1$ such that $p_0 \in D_0$. Choose $B_1 \subsetneqq A_{p_0}$, $|B_1| < \aleph_0$ such that $U_{p_0}(B_1) \subsetneqq D_0$. Choose $p_1 \in A_{p_0}\backslash B_1$. Then $U_{p_1}(\varnothing) \subsetneqq D_0$. Choose $D_1 \in \mathscr{D}_2$ such that $p_1 \in D_1$. Chose $B_2 \subsetneqq A_{p_1}$, $|B_2| < \aleph_0$ such that $U_{p_1}(B_2) \subsetneqq D_1$. Choose $p_2 \in A_{p_1}\backslash B_2$. Then $U_{p_2}(\varnothing) \subsetneqq D_1$. Inductively, suppose $p_0, p_1, \cdots, p_n$, and $D_0, \cdots, D_{n-1}$ are chosen so that $D_{k-1} \in \mathscr{D}_k$, $p_k \in R_k$, and $U_{p_k}(\varnothing) \subsetneqq D_{k-1}$ for $k = 1, 2, \cdots n$. Choose $D_n \in \mathscr{D}_{n+1}$ such that $p_n \in D_n$. Choose $B_{n+1} \subsetneqq A_{p_n}$, $|B_{n+1}| < \aleph_0$ such that $U_{p_n}(B_{n+1}) \subsetneqq D_n$. Choose $p_{n+1} \in A_{p_n}\backslash B_{n+1}$. Then $U_{p_{n+1}}(\varnothing) \subsetneqq D_n$. Now $p_0 \prec p_1 \prec p_2 \prec \cdots$ and $p_k \in R_k$, so there is a unique $q \in R_{\omega_0}$ such that $p_k \prec q$ for each k. By Proposition 2.1(2), q is isolated in $S^{\omega_0+1}_{\aleph_0,\aleph_0}$, but $q \in D_n$ for $n = 0, 1, \cdots$, so $\mathrm{St}\,(q, \mathscr{D}_n) \supsetneqq D_{n-1} \neq \{q\}$. Hence, $\{\mathrm{St}\,(q, \mathscr{D}_n)\}_{n=1}^{\infty}$ is not a base at q. Therefore, $S^{\omega_0+1}_{\aleph_0,\aleph_0}$ is not developable.

6. EXAMPLES. In this section we apply the results of previous sections to get several examples.

EXAMPLE 6.1. $S^{\omega_0}_{\aleph_0,\aleph_0}$ is homeomorphic to the space Q of rationals.

Proof. $S^{\omega_0}_{\aleph_0,\aleph_0}$ is easily seen to be countable. It is first countable by Proposition 5.1. Hence it is second countable and thus metrizable. $S^{\omega_0}_{\aleph_0,\aleph_0}$ is dense-in-itself by Proposition 2.1. Therefore, by a theorem of Sierpinski [5], $S^{\omega_0}_{\aleph_0,\aleph_0}$ is homeomorphic to Q.

COROLLARY 6.2. *If $\alpha > \beta$, the following are equivalent:*
(i) *$S^{\omega}_{\alpha,\beta}$ is metrizable.*
(ii) *$S^{\omega}_{\alpha,\beta}$ is developable.*
(iii) *$\alpha = \beta = \aleph_0$ and $\omega \leqq \omega_0$.*

Proof. This follows from Proposition 5.3 and Example 6.1.

EXAMPLE 6.3. The space Q of rationals is a closed subspace of a dense-in-itself almost P-space.

Proof. Q is homeomorphic to $S^{\omega_0}_{\aleph_0,\aleph_0}$ by Example 6.2. $S^{\omega_0}_{\aleph_0,\aleph_0}$ is homeomorphic to a closed subspace of $S^{\omega_0}_{\aleph_1,\aleph_0}$ by Proposition 4.1, and by Propositions 2.1 and 3.2, $S^{\omega_0}_{\aleph_1,\aleph_0}$ is a dense-in-itself almost P-space.

The following example is given in [4] where the proof may be found.

EXAMPLE 6.4. If ω is not sequential, the first countable, paracompact space $S^{\omega}_{\aleph_0,\aleph_0}$ contains no dense developable subspace.

EXAMPLE 6.5. For each uncountable cardinal α, there is a P-space of cardinal α which is first category in itself and an almost P-space of cardinal α which has no P-points and which is first category in itself.

Proof. $|S^{\omega_0}_{\alpha}| = \alpha$, so by Proposition 3.1 and Theorem 2.2, $S^{\omega_0}_{\alpha,\aleph_1}$ is the required P-space, and by Propositions 3.1 and 3.2, and Theorem 2.2, $S^{\omega_0}_{\alpha,\aleph_0}$ is the required almost P-space.

LEMMA 6.6. *If X is a Lindelöf P-space and $f: X \rightarrow \boldsymbol{R}$ is continuous, $|f(X)| \leqq \aleph_0$.*

Proof. $\{f^{-1}(r) \mid r \in \boldsymbol{R}\}$ is an open cover for X. Therefore, it has a countable subcover.

EXAMPLE 6.7. If $\alpha > \aleph_0$, every continuous function $f: S^{\omega_0}_{\alpha,\aleph_1} \rightarrow \boldsymbol{R}$ has countable image.

Proof. This is immediate from Example 6.5, Lemma 6.6, and Remark 1.7.

EXAMPLE 6.8. Let $X = S^{2}_{2^c,\aleph_1} \times S^{2}_{c,\aleph_1} \backslash (p_0, p_0)$, where $c = 2^{\aleph_0}$. Then X is a nonnormal P-space.

Proof. By Proposition 3.1 and the fact that P-spaces are closed under subspaces and finite products, X is a P-space. Let $A = (\{p_0\} \times S^2_{c,\aleph_1}) \cap X$ and $B = (S^2_{2^c,\aleph_1} \times \{p_0\}) \cap X$. Then A and B are disjoint closed subsets of X. But a proof similar to the usual proof that the Tychanoff plank is not normal shows that A and B are not completely separated. Hence, X is not normal.

EXAMPLE 6.9. Let $Y = S^{\omega_0}_{2^c,\aleph_1} \times S^{\omega_0}_{c,\aleph_1}$. Then Y is a dense-in-itself P-space such that for any $p \in Y$, $Y \backslash \{p\}$ is nonnormal.

Proof. By Propositions 2.1, 3.1, and 4.4, Y is a product of dense-in-themselves homogeneous P-spaces, so Y is a dense-in-itself homogeneous P-space. If X is as in Example 6.8, X is a closed subspace of $Y \backslash \{(p_0, p_0)\}$, so $Y \backslash \{p_0, p_0\}$ is nonnormal. By the homogeneity of Y, for any $p \in Y$, $Y \backslash \{p\}$ is nonnormal.

REFERENCES

1. C. E. Aull, *Topological spaces with a σ-point finite base*, Proc. Amer. Math. Soc., **29** (1971), 411-416.
2. L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, 1960.
3. A. S. Miscenko, *Spaces with point countable bases*, Soviet Math. Dokl., **3** (1962), 855-858.
4. G. M. Reed, *Concerning first countable spaces*, III, preprint.
5. W. Sierpinski, *Sur une propriété topologique des ensembles dénombrables denses en soi*, Fund. Math., **1** (1920), 11-16.
6. S. Willard, *General Topology*, Addison-Wesley, 1970.

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