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Let X be a finite dimensional complex normed linear space with unit ball $B = \{x \in X: \|x\| < 1\}$. In this paper the notion of a close-to-starlike holomorphic mapping from B into X is defined. The definition is a direct generalization of W. Kaplan's notion of one dimensional close-to-convex functions. It is shown that close-to-starlike mappings of B into X are univalent and these mappings are given an alternate characterization in terms of subordination chains.

1. Introduction. In 1952 [2] W. Kaplan defined the class of close-to-convex functions: $f(z) = z + \dots$ analytic and

$$(1.1) \quad \operatorname{Re} \{f'(z)/\phi'(z)\} > 0$$

in $|z| < 1$, for some univalent convex function $\phi(z) = az + \dots$ ($|z| < 1$). Subsequent interest in this class stems from Kaplan's observation that (1.1) implies $f(z)$ is univalent in $|z| < 1$. In this paper we present the natural generalization of close-to-convex vector valued functions in finite dimensional complex spaces. This is a continuation of recent work on vector valued holomorphic starlike and convex mappings [7], [8]. We use the notions of subordination chains of holomorphic maps in C^n and the generalized Loewner differential equation [5] to elucidate the geometry of the mappings.

2. Statement of main results. Let X be a finite dimensional complex normed linear space with dual X^* and $\mathcal{L}(X)$ the set of continuous linear operators from X into X . We let $\mathcal{H}(B)$ denote the set of functions $f(x)$ that are holomorphic in the unit ball $B = \{x \in X: \|x\| < 1\}$ with values in X . The notation $f(x) = ax + \dots$, $a \in C$, for $f \in \mathcal{H}(B)$ indicates that $Df(0) = aI$ where I is the identity in $\mathcal{L}(X)$.

For $0 \neq x \in X$ we define

$$T(x) = \{x^* \in X^*: x^*(x) = \|x\| \quad \text{and} \quad \|x^*\| = 1\},$$

and note that $T(x)$ is nonempty by the Hahn-Banach theorem. We let \mathcal{M} denote the class of functions $h(x) = x + \dots \in \mathcal{H}(B)$ such that $\operatorname{Re} x^*(h(x)) > 0$ for each $x \in B - \{0\}$ and $x^* \in T(x)$. A mapping $g(x) = x + \dots \in \mathcal{H}(B)$ is called starlike if g is univalent in B and $tg(B) \subset g(B)$ for all $0 \leq t \leq 1$.

DEFINITION 1. A mapping $f(x) = x + \dots \in \mathcal{H}(B)$ is said to be *close-to-starlike* if there exist $h(x) \in \mathcal{M}$ and a starlike map $g \in \mathcal{H}(B)$ such that

$$(2.1) \quad Df(x)(h(x)) = g(x), \quad x \in B.$$

REMARK. By this definition, close-to-starlike maps in X are a generalization of Kaplan's close-to-convex functions in C . Indeed when $X = C$ a function $h \in \mathcal{M}$ has the form $h(z) = zP(z)$ where

$$P(0) = 1, \quad \operatorname{Re} P(z) > 0 (|z| < 1)$$

(see [8, p. 576]) and (2.1) is equivalent to the condition (1.1) for the convex function $\phi(z) = \int_0^z g(x)/x \, dx$. By using the criteria for starlikeness and convexity of vector valued maps established in [7] one can easily construct examples showing that Alexander's theorem ($\phi(z)$ is convex if and only if $g(z) = z\phi'(z)$ is starlike) fails in spaces of dimension greater than one. Hence the name close-to-starlike seems most natural in our work.

A mapping $v(x) \in \mathcal{H}(B)$ is called a *Schwarz function* if $\|v(x)\| \leq \|x\|$ for all $x \in B$. A *subordination chain* ([5], [6]) is a function $f(x, t)$ from $B \times [0, \infty)$ into X such that for each $t \geq 0$, $f_t(x) = f(x, t) = e^t x + \dots$ is in $\mathcal{H}(B)$ and there exist Schwarz functions $v(x, s, t)$ such that

$$(2.2) \quad f(x, s) = f(v(x, s, t), t), \quad 0 \leq s \leq t, \quad x \in B,$$

for all $0 \leq s \leq t < \infty$. A *univalent subordination chain* is a subordination chain $f(x, t)$ such that for each $t \geq 0$, $f_t(x)$ is univalent in B .

THEOREM 1. *If $f(x) = x + \dots \in \mathcal{H}(B)$ is locally biholomorphic in B and close-to-starlike relative to the starlike function $g(x) = x + \dots$ then*

$$(2.3) \quad F(x, t) = f(x) + (e^t - 1)g(x), \quad 0 \leq t, \quad x \in B,$$

is a univalent subordination chain. Hence $f(x)$ is univalent in B .

We shall give the proof of Theorem 1 in §3 below. The subordination chain characterization (2.3) yields the linear accessibility of the images of the balls $B_r = \{x \in X: \|x\| < r\}$ ($0 < r < 1$) (compare [1] and [3]).

COROLLARY 1. *If f and g satisfy the hypotheses of Theorem 1 then for each r , $0 < r < 1$, the complement (in X) of $f(B_r)$ is the union of nonintersecting rays.*

Proof. We assume that Theorem 1 holds and therefore the rays

$$L(t; x, r) = \{f(x) + tg(x) : t \geq 0, x \text{ fixed}, \|x\| = r\}$$

are clearly disjoint and fill up the complement of $f(B_r)$.

THEOREM 2. *Suppose $f(x) = x + \dots$ is holomorphic in B and that $g(x) = x + \dots \in \mathcal{H}(B)$ is starlike. If*

$$(2.3) \quad F(x, t) = f(x) + (e^t - 1)g(x), \quad 0 \leq t, x \in B$$

is a univalent subordination chain then f is close-to-starlike relative to g .

We shall prove this theorem in §4 below. By the results in [8] a mapping $f(x) = x + \dots \in \mathcal{H}(B)$ is starlike univalent if and only if it is close-to-starlike relative to itself, i.e., (2.1) holds with $g = f$. Thus from Theorems 1 and 2 we have immediately the

COROLLARY 2. *Let $f(x) = x + \dots$ be locally biholomorphic in B . Then f is univalent and starlike in B if and only if $F(x, t) = e^t f(x)$ is a univalent subordination chain.*

This extends to higher dimensional spaces Pommerenke's one dimensional result in Folgerung 2 of [6].

3. Proof of Theorem 1. We shall give the proof in a sequence of three lemmas. We use the notation $f_r(x) = f(rx)/r$, $g_r(x) = g(rx)/r$ and $F_r(x, t) = f_r(x) + (e^t - 1)g_r(x)$ for $0 \leq r \leq 1$, $t \geq 0$. Let $R = \{r : 0 \leq r \leq 1 \text{ and } F_\rho(x, t) \text{ is a univalent subordination chain for } \rho < r\}$. Then $0 \in R$ and clearly R is closed. We wish to show that R is open so $R = [0, 1]$.

LEMMA 3.1. *If $r \in R$ then $F_r(x, t)$ is a univalent subordination chain.*

Proof. Since $f_\rho(x) = Df(0)(x) = x = g_\rho(x)$ we have $F_\rho(x, t) = x + (e^t - 1)x = e^t x$ which is clearly a univalent subordination chain. Now if $0 < r \in R$, $\rho \leq \lambda < r$ then for $s \leq t$ and $\|x\| < \rho/\lambda$ we have

$$\begin{aligned} F_\lambda(v_\rho(x, s, t), t) &= F_\lambda(x, s) = f(\lambda x)/\lambda + (e^s - 1)g(\lambda x)/\lambda \\ &= (\rho/\lambda)[(1/\rho)f(\rho(\lambda x/\rho)) + (1/\rho)(e^s - 1)g(\rho(\lambda x/\rho))] \\ &= (\rho/\lambda)F_\rho(\lambda x/\rho, s) = (\rho/\lambda)F_\rho(v_\rho(\lambda x/\rho, s, t), t) \\ &= F_\lambda((\rho/\lambda)v_\rho(\lambda x/\rho, s, t), t). \end{aligned}$$

Hence $(\rho/\lambda)v_\rho((\lambda/\rho)x, s, t)$ is independent of ρ when $\rho \leq \lambda$ and $\|x\| <$

$\rho/\lambda(v_\rho(x, s, t), v_\lambda(x, s, t))$ are the univalent Schwarz functions postulated by the fact that $F_\rho(x, t)$ and $F_\lambda(x, t)$ are univalent subordination chains. Hence we may define $v_r(x, s, t) = (\rho/r)v_\rho((r/\rho)x, s, t)$ where $\|x\| < 1$ and ρ satisfies $\|x\| < \rho/r < 1$. Then v_r is well defined in B , it is a univalent Schwarz function and

$$\begin{aligned} F_r(x, s) &= (\rho/r)F_\rho(rx/\rho, s) = (\rho/r)F_\rho(v_\rho(rx/\rho, s, t), t) \\ &= F_r((\rho/r)v_\rho(rx/\rho, s, t), t) = F_r(v_r(x, s, t), t) \end{aligned}$$

when $\|x\| < \rho/r < 1, 0 \leq s \leq t$. Therefore $F_r(x, t)$ is a univalent subordination chain.

LEMMA 3.2. *If $r \in R, r < 1$ then there exists $\varepsilon_0 > 0$ such that $F_{r+\varepsilon}(x, t)$ is a univalent function of $x \in B$ for each $t \geq 0$ and $0 \leq \varepsilon < \varepsilon_0$.*

Proof. Since $F_r(x, t)$ is a univalent subordination chain, $f_r(x)$ is univalent in the closed ball \bar{B} (for otherwise, there exist $\rho < r, x, y, t, x \neq y, \|x\| = \|y\| < \rho/r, t > 0$ such that $v_r(x, 0, t) = v_r(y, 0, t)$). Let $G(x, y)$ be the $n \times n$ determinant whose k th column is

$$A_k = \begin{cases} (x_k - y_k)^{-1}[f_r(y_1, \dots, y_{k-1}, x_k, \dots, x_n) - f_r(y_1, \dots, y_k, x_{k+1}, \dots, x_n)], & x_k \neq y_k \\ \frac{\partial}{\partial x_k} f_r(y_1, \dots, y_{k-1}, x_k, \dots, x_n), & \text{if } x_k = y_k \end{cases}$$

and define $H(x, y) = |G(x, y)| + \|f_r(x) - f_r(y)\|$ where $x, y \in B_{1+\varepsilon}, 1 + \varepsilon < 1/r$. If $x = y$ we have $H(x, x) = |\det Df_r(x)| > 0$ since f_r is biholomorphic. If $x \neq y$ and $f_r(x) \neq f_r(y)$ then $H(x, y) > 0$. If $x \neq y$ and $f_r(x) = f_r(y)$ then $\sum_{k=1}^n (x_k - y_k)A_k = f_r(x) - f_r(y) = 0$ and $H(x, y) = 0$ since the columns of $G(x, y)$ are dependent. Thus $H(x, y) = 0$ if and only if $f_r(x) = f_r(y)$ and $x \neq y$. We conclude that $H(x, y)$ has a positive minimum on $\bar{B} \times \bar{B}$ and in fact $H(x, y) > 0$ if $(x, y) \in B_{1+\varepsilon} \times B_{1+\varepsilon}$ when $0 \leq \varepsilon < \varepsilon'$ for some $\varepsilon' > 0$. This implies that $f_{r+\varepsilon}$ is univalent in B for $0 \leq \varepsilon < \varepsilon''$ for some $\varepsilon'' > 0$.

For small $\varepsilon > 0, e^{-t}F_{r+\varepsilon}(x, t)$ converges to $g_{r+\varepsilon}(x)$ uniformly in B as $t \rightarrow \infty$. Hence $F_{r+\varepsilon}(x, t)$ is univalent and starlike for $t > t_0$ for some $t_0 > 0$.

Now assume the lemma is false. Then there exist sequences $\{\varepsilon_k\}, \{t_k\}$ of positive numbers and points $\{x_k\}, \{y_k\}$ in \bar{B} such that $\varepsilon_k \rightarrow 0, x_k \neq y_k, \|x_k\| = \|y_k\| = 1, t_k < t_0$ and $F_{r+\varepsilon_k}(x_k, t_k) = F_{r+\varepsilon_k}(y_k, t_k)$. (We may assume $\|x_k\| = \|y_k\| = 1$ since by the reasoning of Ono in [4] univalence on the boundary of B implies univalence in the interior.) By choosing subsequences we may find limit points $s, u, v, 0 < s \leq t_0, \|u\| = \|v\| = 1$ such that $F_r(u, s) = F_r(v, s)$. Since $F_r(x, t)$ is a uni-

valent subordination chain we must have $u = v$. Hence

$$\begin{aligned} 0 &= \frac{F_{r+\varepsilon_k}(x_k, t_k) - F_{r+\varepsilon_k}(y_k, t_k)}{\|x_k - y_k\|} = DF_r(u, s) \left(\frac{x_k - y_k}{\|x_k - y_k\|} \right) \\ &+ (DF_{r+\varepsilon_k}(y_k, t_k) - DF_r(y_k, t_k)) \left(\frac{x_k - y_k}{\|x_k - y_k\|} \right) \\ &+ (DF_r(y_k, t_k) - DF_r(u, s)) \left(\frac{x_k - y_k}{\|x_k - y_k\|} \right) + o(x_k - y_k) \end{aligned}$$

and by using appropriate subsequences we conclude that $DF_r(u, s)$ is singular. This is a contradiction since $DF_r(u, s) = DF_r(v_r(x, s, t), t)Dv_r(x, s, t)$ is the composition of two nonsingular maps in $\mathcal{L}(X)$, and the lemma is established.

LEMMA 3.3. *Let ε_0 be as determined in Lemma 3.2. Then for $0 \leq \varepsilon < \varepsilon_0$ $F_{r+\varepsilon}(x, t)$ is a univalent subordination chain.*

Proof. We must show that for $0 \neq x \in B$ and $x^* \in T(x)$ we have

$$\operatorname{Re} x^*([DF_{r+\varepsilon}(x, t)]^{-1}(g(x))) \geq 0,$$

for then $\partial \|v_{r+\varepsilon}(x, s, t)\|/\partial t \leq 0$, $s \leq t$. It will follow that $v_{r+\varepsilon}(x, s, t) = F_{r+\varepsilon}^{-1}(F_{r+\varepsilon}(x, s), t)$ is a univalent Schwarz function.

Let $x^* \in T(x)$, $0 \neq x \in B$ and suppose

$$\operatorname{Re} x^* \{ [DF_{r+\varepsilon}(x, t)]^{-1}(g(x)) \} < 0$$

for some t . Then since the reverse inequality holds for $t = 0$ and sufficiently large t , there exist s, t, u, v , $0 < s < t < \infty$, $u, v \in X$, $\operatorname{Re} x^*(u) = \operatorname{Re} x^*(v) = 0$ such that

$$(3.1) \quad e^s g(x) = Df(x)(u) + (e^s - 1)Dg(x)(u)$$

$$(3.2) \quad e^t g(x) = Df(x)(v) + (e^t - 1)Dg(x)(v).$$

Let

$$L = \{y \in X: \operatorname{Re} x^*(y) = 0\}$$

and

$$L_1 = L \cap (Df(x))^{-1}(Dg(x)(L)) = L \cap (Dg(x))^{-1}(Df(x)(L))$$

and view L and L_1 as linear spaces over the real numbers. If $L = L_1$, then $g(x)$ is in the space $Df(x)(L) = Dg(x)(L)$ which is impossible since $\operatorname{Re} x^* \{ [Df(x)]^{-1}(g(x)) \} > 0$ by (2.1). Thus L and L_1 have real dimension $2n - 1$ and $2n - 2$ respectively where n is the complex dimension of X .

We wish to show that $u = v$ and $s = t$. Let $y_0 \in L - L_1$ and observe that we may write u and v uniquely in the form $u = ay_0 +$

$u_1, v = by_0 + v_1$ where a and b are real, $u_1, v_1 \in L_1$. Then (3.1) and (3.2) yield that

$$(3.3) \quad \begin{aligned} g(x) &= a[e^{-s}Df(x)(y_0) + (1 - e^{-s})Dg(x)(y_0)] + w_1 \\ &= b[e^{-t}Df(x)(y_0) + (1 - e^{-s})Dg(x)(y_0)] + w_2 \end{aligned}$$

where $w_1, w_2 \in Dg(x)(L_1) = Df(x)(L_1)$. We shall show that $g(x)$ has a unique representation of the form $\alpha Df(x)(y_0) + \beta Dg(x)(y_0) + w$ where $w \in Dg(x)(L_1)$ and α, β are real. To this end, we assume that

$$\alpha Df(x)(y_0) + \beta Dg(x)(y_0) \in Dg(x)(L_1)$$

for some real α, β . Then $Df(x)(\alpha y_0) = Dg(x)(w_3 - \beta y_0)$ for some $w_3 \in L_1$ and consequently $\alpha y_0 \in L_1$. This implies that $\alpha = 0$ and then $\beta y_0 = w_3 \in L_1$ and $\beta = 0$. Thus from (3.3) we conclude $ae^{-s} = be^{-t}$, $a(1 - e^{-s}) = b(1 - e^{-t})$ and therefore $a = b$ and $s = t$. This contradicts our assumption that $s < t$ and completes the proof of the lemma.

The proof of Theorem 1 is now complete for we have shown that R is a nonempty subset of $[0, 1]$ that is both open and closed. Hence $R = [0, 1]$ and $F(x, t) = F_1(x, t)$ is a univalent subordination chain by Lemma 3.1.

4. *Proof of Theorem 2.* By hypothesis there are univalent Schwarz functions $v(x, s, t)$ such that $F(x, s) = F(v(x, s, t), t)$ ($0 \leq s \leq t$) for the chain $F(x, t)$ defined in (2.3). It is clear from the form of (2.3) that the derivative

$$(4.1) \quad \frac{\partial F}{\partial t}(x, t) = \lim_{s \rightarrow t} \frac{F(x, s) - F(x, t)}{s - t}$$

exists and the convergence is uniform on compact subsets of B .

We fix $t > 0$, let $s < t$ and write

$$\begin{aligned} F(x, s) - F(x, t) &= F(x, s) - F(v(x, s, t), t) \\ &= DF(x, t)(v(x, s, t) - x) + o(v - x) \end{aligned}$$

where $o(v - x)/\|v - x\|$ tends to zero uniformly for x in a compact subset of B as $v(x, s, t) - x$ tends to zero. Thus

$$(4.2) \quad \frac{F(x, s) - F(x, t)}{s - t} = DF(x, t) \left(\frac{x - v(x, s, t)}{t - s} \right) + \frac{o(v - x)}{s - t}$$

and since $DF(x, t)$ is nonsingular we can argue (as in [8] Lemma 2) that $(x - v(x, s, t))/(t - s)$ is bounded and tends to a limit, and that $o(v(x, s, t) - x)/(s - t)$ tends to zero as s tends to t (the univalence of the chain insures that $v(x, s, t)$ tends to x as $s \rightarrow t$). Since $t - s > 0$ and

$$\begin{aligned} \operatorname{Re} x^*(x - v(x, s, t)) &= \|x\| - \operatorname{Re} x^*(v(x, s, t)) \\ &\geq \|x\| - \|v(x, s, t)\| \geq 0 \end{aligned}$$

for each $x^* \in T(x)$ it follows that the function

$$(4.3) \quad h(x, t) = \lim_{s \rightarrow t} \frac{x - v(x, s, t)}{t - s}, \quad t > 0,$$

is in the class \mathcal{M} .

From (4.1) – (4.3) we conclude that $F(x, t)$ satisfies the generalized Loewner differential equation [5]

$$(4.4) \quad \partial F(x, t)/\partial t = DF(x, t)(h(x, t)), \quad x \in B,$$

for each $t > 0$. For the specific subordination chain (2.3) it is clear that we may let t tend to zero in (4.4) to obtain

$$g(x) = Df(x)(h(x, 0)),$$

and $h(x, 0) \in \mathcal{M}$ since the properties of \mathcal{M} are preserved by local uniform convergence. This completes the proof of Theorem 2.

5. EXAMPLES. (1) Let $f(z) = z + \dots$ be close to the starlike function $g(z) = z + \dots$ where f and g are complex valued analytic functions of z in the open unit disk, $|z| < 1$. Let X be a complex finite dimensional inner product space with inner product $\langle \cdot, \cdot \rangle$ and let $x_0 \in X$, $\|x_0\| = 1$. Define the vector valued holomorphic maps

$$F(x) = \frac{f(\langle x, x_0 \rangle)}{\langle x, x_0 \rangle} x, \quad G(x) = \frac{g(\langle x, x_0 \rangle)}{\langle x, x_0 \rangle} x$$

for x in B , the unit ball in X . Then

$$DG(x) = \frac{-\langle \cdot, x_0 \rangle}{\langle x, x_0 \rangle^2} g(\langle x, x_0 \rangle) x + \frac{\langle \cdot, x_0 \rangle}{\langle x, x_0 \rangle} g'(\langle x, x_0 \rangle) x + \frac{g(\langle x, x_0 \rangle)}{\langle x, x_0 \rangle} I$$

where $I \in \mathcal{L}(X)$ is the identity. A similar formula holds for $DF(x)$. Setting $H(x) = g(\langle x, x_0 \rangle)x/(\langle x, x_0 \rangle g'(\langle x, x_0 \rangle))$ we see that $H \in \mathcal{M}$ and $DG(x)(H(x)) = G(x)$ so G is starlike [7]. Similarly if $K(x) = g(\langle x, x_0 \rangle)x/(\langle x, x_0 \rangle f'(\langle x, x_0 \rangle))$ then $K \in \mathcal{M}$ and $DF(x)(K(x)) = G(x)$ so F is close-to-starlike. Note that F and G both reduce to the identity map on the subspace orthogonal to x_0 . An interesting choice of f and g is $f(z) = (1/2) \log [(1+z)/(1-z)]$, $g(z) = z/(1+z)^2$. Then $f + (e^t - 1)g$ maps the unit disk onto the entire plane slit along two parallel rays when $0 < t < \infty$. Also $F(x) + (e^t - 1)G(x)$ has similar behavior on the one dimensional slice $\{\alpha x_0; \alpha \in \mathbb{C}, |\alpha| < 1\}$.

(2) Let $X = \mathbb{C}^2$ with the usual inner product and Euclidean norm

$$\langle x, y \rangle = \sum_{j=1}^2 x_j \bar{y}_j, \|x\| = \langle x, x \rangle^{1/2},$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in C^2 . We define the functions

$$(5.1) \quad f(x) = (2^{-1}[(1-x_1)^{-2} - 1], x_2 + ax_1x_2),$$

$$(5.2) \quad g(x) = (x_1/(1-x_1)^2, x_2[1 + 2bx_1 + bx_1^2]),$$

$$(5.3) \quad h(x) = (x_1(1-x_1), x_2[1 + 4ax_1/(2a-1)])$$

where $\|x\| < 1$, $b = a(2a+1)/(2a-1)$ and a is a complex number with small modulus. We claim that if $|a|$ is sufficiently small then: (I) $h(x)$ belongs to the class \mathcal{M} , (II) $g(x) = x + \dots \in \mathcal{H}(B)$ is starlike, (III) $f(x) = x + \dots \in \mathcal{H}(B)$ is close-to- $g(x)$, and (IV) f is not starlike.

(I) Clearly (5.3) is holomorphic in B and has the required normalization $h(x) = x + \dots$. Furthermore, if $|a|$ is sufficiently small then

$$(5.4) \quad \operatorname{Re} \langle h(x), x \rangle = |x_1|^2 \operatorname{Re} (1-x_1) + |x_2|^2 \operatorname{Re} \left(1 + \frac{4ax_1}{2a-1} \right) > 0,$$

for all $x \in B$ and $h \in \mathcal{M}$ [8, p. 577].

(II) The holomorphy and normalization of (5.2) are clear. We must show that $(Dg(x))^{-1}(g(x))$ belongs to \mathcal{M} if $|a|$ is small. Elementary computations with (5.2) yield that

$$(Dg(x))^{-1}(g(x)) = \left(x_1 \left(\frac{1-x_1}{1+x_1} \right), x_2 \left[1 - \frac{2bx_1(1-x_1)}{1+2bx_1+bx_1^2} \right] \right)$$

and therefore $\operatorname{Re} \langle (Dg(x))^{-1}(g(x)), x \rangle \geq 0$ for all $x \in B$ and small $|a|$.

(III) It is easy to verify that (5.1), (5.2), and (5.3) satisfy the equation $Df(x)(h(x)) = g(x)$ and hence that f is close-to- g .

(IV) We must show that $(Df(x))^{-1}(f(x))$ does not belong to \mathcal{M} . This follows when $|a|$ is small since

$$(Df(x))^{-1}(f(x)) = \left(\frac{x_1(2-x_1)(1-x_1)}{2}, x_2 \left[1 - \frac{ax_1(2-x_1)(1-x_1)}{1+ax_1} \right] \right),$$

and $\operatorname{Re} (2-x_1)(1-x_1) < 0$ at some points in the unit disk $|x_1| < 1$.

Finally we mention that the functions (5.1), (5.2), and (5.3) provide an example similar to the preceding one when we consider $X = C^2$ with the sup norm, $\|x\|_\infty = \max(|x_1|, |x_2|)$. In this setting the condition (5.4) for membership in \mathcal{M} is replaced by the condition $\operatorname{Re} (h_j(x)/x_j) > 0$ when $\|x\|_\infty = |x_j| > 0$ [8].

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