SUFFICIENT CONDITIONS FOR THE EXISTENCE
OF CONVERGENT SUBSEQUENCES

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Let $R$ be the real numbers, $S \subset R$ and $E$ be an ordered
topological vector space. Sufficient conditions are given that
a sequence $\{y_k\}$, $y_k: S \to E$, will have a subsequence $\{h_k\}$ such
that for each $t \in S$, $\{h_k(t)\}$ is either eventually monotone or
else is convergent. In case $E$ is a Banach space, sufficient
conditions are given that $\{y_k\}$ have a subsequence $\{h_k\}$ so that
$\{h_k(t)\}$ converges for each $t \in S$. Finally, if $E = R$, the concept
of $\{y_k\}$ being equioscillatory is defined and it is shown that
a necessary and sufficient condition for $\{y_k\}$ to have a sub-
sequence that converges at every point of $S$ is that $\{y_k\}$ have
a subsequence which is pointwise bounded and equioscillatory.
An application of these results to differential equations is
treated briefly.

1. Introduction. The existence of solutions to initial and boun-
dary value problems for both ordinary and partial differential equations
is frequently shown by obtaining a convergent subsequence from a
sequence of functions and showing that the limit function is the
desired solution. For example, in the proof of the Picard-Lindelöf
Theorem [1, Theorem 1.1, p. 8] and the Cauchy-Peano Existence
Theorem [1, Theorem 2.1, p. 10] such techniques are used. The
question arises then, for a given sequence of functions, what conditions
suffice to allow extraction of a pointwise convergent subsequence.
For a sequence $\{y_k\}$ with $y_k: I \to R$, where $I$ is a real interval, there
are many results which provide sufficient conditions for the existence
of a convergent subsequence; for example, the Helly Selection Theorem
and the Theorem of Ascoli.

Let $\{y_k\}$ be a sequence of functions from a nonempty subset $S$
of the real numbers $R$ into an ordered topological vector space $E$.
Then we are interested in finding sufficient conditions that $\{y_k\}$ have
a subsequence $\{h_k\}$ such that for each $s \in S$, $\{h_k(s)\}$ is a convergent
sequence. Theorem 2.2 yields a subsequence $\{h_k\}$ such that for each
$s \in S$, $\{h_k(s)\}$ is either eventually monotone or else is convergent. By
adding conditions which will make these eventually monotone sub-
sequences converge, the desired convergence result can be obtained.
Such a result is given by Corollary 2.3. Furthermore, when $E = R$,
we obtain a necessary and sufficient condition for a sequence $\{y_k\}$,
$y_k: S \to R$, to have a subsequence which converges for each $s \in S$.
This is stated in Corollary 2.5.
In §3 an application to differential equations is given. A more
detailed description of the applications to boundary value problems
for ordinary differential equations may be found in [4].

2. Primary results. We begin this section with the definition
of a proper pair.

DEFINITION 2.1. Let $S$ be a nonempty subset of real numbers
and $f$ be a function, $f: S \to E$, where $E$ is an ordered vector space
with positive cone $K$. Consider the set $\mathcal{P}$ of all finite nonempty
partitions $P = \{x_1, x_2, \ldots, x_n\}$ of $S$ where $n \geq 1$, $x_i \in S$ for $i = 1, 2, \ldots, n$ and $x_1 < x_2 < \cdots < x_n$. If $f(s) \neq \theta$ for some $s \in S$, we say that
$(f, P)$ is a proper pair if $(-1)^i f(x_i) > \theta$ for $i = 1, 2, \ldots, n$ or else
$(-1)^i f(x_i) < \theta$ for $i = 1, 2, \ldots, n$. If $f(s) = \theta$ for all $s \in S$ we say
that $(f, P)$ is a proper pair if $P$ contains exactly one point.

THEOREM 2.2. Let $S$ be a nonempty subset of real numbers
and $\{y_k\}$ be a sequence of functions, $y_k: S \to E$ where $E$ is a sequen-
tially complete ordered locally convex space with positive cone $K$.
For each $t \in S$ assume that $\{y_k(t)\}$ is an eventually comparable sequence.
Assume, for each $s \in S$, that $E$ has a nested countable basis of circled
sets at $\theta$ denoted by $\{U_s(n)\}$. For each $t \in S$ and each positive integer
$n$ assume that there are nonnegative integers $N(n, t), H(n, t)$ and a
number $\delta(n, t) > 0$ such that for all $k, j \leq H(n, t)$ if $(y_k - y_j, P)$ is
a proper pair then $P$ contains at most $N(n, t)$ points $x$ such that
$y_k(x) - y_j(x) \in U_s(n)$ and $t - \delta(n, t) < x < t + \delta(n, t)$. Then $\{y_k\}$ con-
tains a subsequence $\{h_k\}$ such that for each $t \in S, \{h_k(t)\}$ is either
eventually monotone or else is convergent.

Proof. If $y_k(t)$ and $y_j(t)$ are comparable for all $k, j \geq M(t)$ and
$M(t)$ is the smallest positive integer having this property then let
$A_i = \{t: t \in S, M(t) = i\}$ for $i = 1, 2, \ldots$. For any $t \in A_i$ we have
$y_k(t)$ and $y_j(t)$ comparable for $k, j \geq i$. We will prove the theorem
assuming that $y_k(t)$ and $y_j(t)$ are comparable for all $t \in S$ and then
a standard diagonalization argument where $S$ is replaced by $A_1, A_2,
\ldots$ yields the desired result.

We note that we may assume $S$ is bounded because if the theorem
is true for bounded sets a standard diagonalization argument yields
the result for unbounded sets. Also, we may assume $S$ is a closed
interval because if the theorem is true for closed intervals, $I$, then
we may choose $I$ to be a closed interval containing the bounded set
$S$ and define a sequence of functions $\{z_k\}, z_k: S \to E$ by
\[
\begin{align*}
z_k(t) &= y_k(t) & \text{for } t \in S \\
\theta &= \text{for } t \notin S
\end{align*}
\]
then the sequence \( \{ z_k \} \) satisfies the hypotheses of the theorem on \( I \) and the result would follow for bounded sets \( S \).

Furthermore, because of the compactness of \( S \), we may assume that for each positive integer \( n \) there are nonnegative integers \( N(n) \), \( H(n) \) such that for all \( k, j \geq H(n) \) if \( (y_k - y_j, P) \) is a proper pair then \( P \) contains at most \( N(n) \) points \( x \) such that \( y_k(x) - y_j(x) \in U_x(n) \).

Let \( \{ J_\ell \} \) be an enumeration of all nonempty open subintervals of \( S \) with rational endpoints. Applying a slight modification of Corollary 2.2 of [3] to \( J_\ell \), observe that either there is a subsequence of \( \{ y_k \} \), again denoted by \( \{ y_k \} \), such that \( \{ y_k \} \) is monotone on \( J_\ell \), or else there is a subsequence of \( \{ y_k \} \), again denoted by \( \{ y_k \} \), such that for \( k \neq j \), \( y_k(t) > y_j(t) \) and \( y_k(t) < y_j(t) \) hold for some \( t, \tau \in J_\ell \). Now repeat the process described in the previous sentence consecutively on the intervals \( J_\ell, J_{\ell+1}, \ldots \) and then take the diagonal subsequence, denoted by \( \{ y_k \} \) again. This sequence has the property that on \( J_\ell \) it is eventually monotone or else for every \( k \neq j \) sufficiently large, depending on \( \ell \), there is \( t, \tau \in J_\ell \) such that \( y_k(t) > y_j(t) \) and \( y_k(t) < y_j(t) \).

Now using \( J_\ell \) and \( U_{\ell}(1) \) it follows from a slight modification of Corollary 2.3 of [3] that either there is a subsequence of \( \{ y_k \} \), again denoted by \( \{ y_k \} \), such that, for \( k \neq j \), \( y_k(t) - y_j(t) \in U_{\ell}(1) \) for all \( t \in J_\ell \) or else there is a subsequence, again denoted by \( \{ y_k \} \), such that for \( k \neq j \) there is a \( t \in J_\ell \) with \( y_k(t) - y_j(t) \in U_{\ell}(1) \). Now repeat the process described in the preceding sentence for \( U_{\ell}(2), U_{\ell}(3), \ldots \) and then take the diagonal subsequence, denote it by \( \{ y_k \} \). This sequence has the property that for \( J_\ell \) and \( U_{\ell}(n) \) either for all \( k \neq j \) sufficiently large, depending on \( n \), \( y_k(t) - y_j(t) \in U_{\ell}(n) \) for all \( t \in J_\ell \) or else for all \( k \neq j \) sufficiently large, depending on \( n \), there is some \( t \in J_\ell \) such that \( y_k(t) - y_j(t) \in U_{\ell}(n) \).

We now repeat the entire process described in the preceding paragraph consecutively on the intervals \( J_{\ell+1}, J_{\ell+2}, \ldots \) and then take the diagonal subsequence again denoted by \( \{ y_k \} \). This sequence has the property that for \( J_\ell \) and \( U_{\ell}(n) \) either \( y_k(t) - y_j(t) \in U_{\ell}(n) \) for all \( t \in J_\ell \) and \( k \neq j \) sufficiently large depending on \( \ell \) and \( n \) or else there is a \( t \in J_\ell \) depending on \( k, j \) with \( y_k(t) - y_j(t) \in U_{\ell}(n) \), for \( k \neq j \) sufficiently large depending on \( \ell \) and \( n \).

We will now show by contradiction that for all but countably many values of \( x \in S \) the sequence \( \{ y_k(x) \} \) is either convergent or eventually monotone. For \( x \in S \) such that \( \{ y_k(x) \} \) is neither convergent nor eventually monotone let \( \{ F_{x_i} \} \) be the subsequence of \( \{ J_\ell \} \) consisting of the intervals which contain \( x \). There must be a smallest positive integer \( n_{x_i} \), such that \( y_k(t) - y_j(t) \in U_i(n_{x_i}) \) for all \( k \neq j \) sufficiently large, depending on \( i \), for some \( t \in F_{x_i} \) or else \( \{ y_k \} \) would be Cauchy on \( F_{x_i} \) and hence would be convergent at each point in \( F_{x_i} \). In particular, \( \{ y_k(x) \} \) would be convergent which contradicts the choice.
of $x$. If $\lim_{t \to +\infty} n_{x(t)} = +\infty$ then there is a subsequence $\{n_{x(t)}(a)\}$ of $\{n_{x(t)}\}$ such that $\lim_{n \to +\infty} n_{x(t)}(a) = +\infty$ and by the definition of $n_{x(t)}(a)$ and the nestedness of $\{U_i(n)\}$ we have $y_k(t) - y_j(t) \in U_i(n_{x(t)}(a) - 1)$ for all $k \neq j$ sufficiently large, depending on $\alpha$, and all $t \in F_{x(t)}$. Thus $\{y_k(x)\}$ is Cauchy and hence convergent which is contrary to the choice of $x$ so $\lim_{t \to +\infty} n_{x(t)} = c_x < +\infty$. Let $d_x > c_x$ be an upper bound for the set $\{n_{x(t)}\}$.

If there are uncountably many values of $x \in S$ at which $\{y_k(x)\}$ is neither convergent nor eventually monotone then there is some fixed positive integer $d$ so that $d_x \leq d$ holds for uncountably many $x \in S$ at which $\{y_k(x)\}$ is neither convergent nor eventually monotone. Denote this uncountable set of $x$'s by $A$. We now have $x \in A$ and $k \neq j$ sufficiently large, depending on $i$, implies $y_k(t) - y_j(t) \notin U_i(d)$ for some $t \in F_{x(t)}$.

Choose $N > N(d)$ and $u(1) \in A \cap S^o$ and $F_{u(1)}(1) \in \{F_{u(1)}(1)\}$ such that $(S - F_{u(1)}(1)) \cap A$ is uncountable. Choose $u(2) \in (S - F_{u(1)}(1)) \cap (A \cap S^o)$ and $F_{u(2)}(2) \in \{F_{u(2)}(2)\}$ with $F_{u(1)}(1) \cap F_{u(2)}(2) = \emptyset$ and

$$(S - (F_{u(1)}(1) \cup F_{u(2)}(2))) \cap A$$

is uncountable. Continuing in this manner we get $\{u(1), u(2), \ldots, u(2N + 1)\}$ in $A \cap S^o$ and $\{F_{u(1)}(1), F_{u(2)}(2), \ldots, F_{u(2N + 1)}(2N + 1)\}$ which are mutually disjoint. By renaming the points $u(i)$ we may assume $u(1) < u(2) < \cdots < u(2N + 1)$. So choose $k \neq j$, $k, j > H(d)$, sufficiently large that for each odd positive integer $\alpha, 1 \leq \alpha \leq 2N + 1$, $y_k(x(\alpha)) - y_j(x(\alpha)) \notin U_{x(\alpha)}(d)$ for some $x(\alpha) \in F_{u(\alpha)}(\alpha)$ and for each positive even integer $\alpha, 2 \leq \alpha \leq 2N$, $y_k(t_{\alpha}) - y_j(t_{\alpha}) < \theta$ holds for some $t_{\alpha} \in F_{u(\alpha)}(\alpha)$ and $y_k(t_{\alpha}) - y_j(t_{\alpha}) > \theta$ holds for some $t_{\alpha} \in F_{u(\alpha)}(\alpha)$. Now consider the partition $P_0 = \{\beta_1, \beta_2, \ldots, \beta_n\}$ where $\beta_0 = x(\alpha)$. $\beta_0$ is odd; $\beta_0$ is omitted from $P_0$ if $\alpha$ is even and $y_k(x(\alpha - 1)) - y_j(x(\alpha - 1)) < \theta$ and $y_k(x(\alpha + 1)) - y_j(x(\alpha + 1)) > \theta$ or the opposite inequalities hold; $\beta_0$ is taken to be $t_{\alpha}$ if $y_k(x(\alpha - 1)) - y_j(x(\alpha - 1)) > \theta$ and $y_k(x(\alpha + 1)) - y_j(x(\alpha + 1)) < \theta$. Then the partition $P_0$ is such that $(y_k - y_j, P_0)$ is a proper pair and $y_k(x(\alpha)) - y_j(x(\alpha)) \notin U_{x(\alpha)}(d)$ for $\alpha$ odd, $x(\alpha) \in P_0$, and there are $N + 1$ such $x(\alpha)$. This is contrary to the hypothesis of the theorem.

We conclude that the conclusion of theorem holds for all but countably many values of $x$. By choosing a monotone subsequence of $\{y_k(x)\}$ for each such $x$ and diagonalizing, the subsequence, again denoted by $\{y_k\}$, is either eventually monotone or convergent for each $x$ in $S$.

**Note.** If one wishes to consider sequences $\{y_k\}, y_k \in \prod_{e \in S} E_e$ where
COROLLARY 2.3. Let \( B \) be a reflexive ordered Banach space with normal positive cone \( K \) and \( S \) be a nonempty subset of \( R \). Let \( \{y_k\}, y_k: S \to B \) be such that for each \( s \in S \), \( \{y_k(s)\} \) is an eventually comparable norm bounded sequence. If there are nonnegative integers \( N(n) \) and \( H(n) \) such that for all \( k, j \geq H(n) \) \( (y_k - y_j, P) \) is a proper pair then \( P \) contains at most \( N(n) \) points \( x \) such that \( y_k(x) - y_j(x) \in U_x(n) \) then \( \{y_k\} \) contains a subsequence \( \{h_k\} \) which converges at each point of \( S \).

Proof. It follows from Theorem 2.2 that there is a subsequence which at each point \( s \) of \( S \) is either eventually monotone or else is convergent. By [2, Proposition 3.7, p. 93] it follows that this subsequence converges at every point of \( S \).

DEFINITION 2.4. Let \( S \) be a nonempty set of real numbers and \( \{y_k\} \) be a sequence of functions, \( y_k: S \to R \). We say that the sequence \( \{y_k\} \) is equioscillatory if for each \( s \in S \) there exists a neighborhood basis of \( 0 \) of radii \( \varepsilon(n, s) \) and for each positive integer \( n \) there exist positive integers \( N(n) \) and \( H(n) \) such that if \( k, j \geq H(n) \) and \( (y_k - y_j, P) \) is a proper pair then \( P \) contains no more than \( N(n) \) points \( x \) for which \( |y_k(x) - y_j(x)| > \varepsilon(n, x) \).

COROLLARY 2.5. Let \( S \) be a nonempty subset of real numbers and \( \{y_k\} \) be a sequence of functions, \( y_k: S \to R \). The sequence \( \{y_k\} \) has a subsequence which is pointwise convergent if and only if it has a subsequence which is pointwise bounded and equioscillatory.

Proof. The sufficiency follows from Theorem 2.2. The necessity is trivial since if \( N(n) = 0 \) in Definition 2.4 we see that this is equivalent to saying that \( \{y_k\} \) is pointwise Cauchy.

3. Applications. In this section we examine some examples which serve to illustrate the results obtained in §2.

EXAMPLE 1. Let \( H \) be a complex Hilbert space and \( E \) be the ordered locally convex space, over the reals, of continuous linear Hermitian operators on \( H \) with the strong operator topology. Let the order for \( E \) be determined by \( A \geq \theta \Leftrightarrow (Ax, x) \geq 0 \) for all \( x \in H \) and \( A \geq D \Leftrightarrow A - D \geq \theta \) for \( A, D \in E \). Let \( A_k(t) \) be a sequence of
functions from the real interval $I$ into $E$ which satisfies the hypotheses of Theorem 2.2. It is known that monotone sequences in $E$ which are topologically bounded are convergent in the strong operator topology. Thus it follows from Theorem 2.2 that if $\{A_k(t)\}$ is topologically bounded for each $t \in I$ then there is a subsequence $\{D_j(t)\}$ of $\{A_k\}$ such that $\{D_j(t)\}$ is convergent in the strong operator topology on $E$ for every $t \in I$.

**Example 2.** If in Example 1 we take $H$ to be the $d$-dimensional complex Hilbert space $C^d$ and $B$ to be the $d \times d$ Hermitian matrices with the usual operator norm then $B$ is a reflexive Banach with a normal positive cone. Thus a sequence $\{A_k\}$ of functions from a real interval $I$ into $B$ which satisfies the hypotheses of Corollary 2.3 must contain a subsequence which converges in norm for every $t \in I$.

Consider the sequence of linear differential equations

$$y' = A_k(t)y + f_k(t), \quad y(t_k) = y_k$$

where $y \in R^d$, $A_k(t)$ is a $d \times d$ matrix and $f_k(t)$ a $d \times 1$ matrix each with continuous real entries for $t \in I$. Assume that $A_k(t)$ can be partitioned independent of $k$ and $t$ into square submatrices, possibly $1 \times 1$ such that each of the sequences of square submatrices satisfies the hypotheses of Corollary 2.3. Assume also that $f_k(t)$ can be partitioned independent of $k$ and $t$ into square submatrices, necessarily $1 \times 1$, such that each sequence of square submatrices satisfies the hypotheses of Corollary 2.3. Then the sequence $\{A_k\}$, $A_k : I \to B$ must contain a subsequence, $\{A_{k_j}\}$, which converges in the operator norm on the $d \times d$ matrices for each $t \in I$ and hence converges in $R$ in each entry for each $t \in I$. Let us denote this limit by $A_0(t)$. Also, $\{f_{k_j}\}$ must contain a subsequence which converges in $R$ in each entry for each $t \in I$ to a function we will denote by $f_0(t)$. If $t_k \to t_0$ and $y_k \to y_0$ as $k \to +\infty$ where $t_k \in I$ for $k = 0, 1, \ldots$ then it follows that the sequence of solutions of $(3.1)_k$ contains a subsequence which converges at every point of $I$ to a function $y$ which is a solution of $(3.1)_0$ almost everywhere on $I$.

**References**


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