Pacific Journal of Mathematics

A REMARK ON THE LATTICE OF IDEALS OF A PRÜFER DOMAIN

DANIEL D. ANDERSON

Vol. 57, No. 2 February 1975

PACIFIC JOURNAL OF MATHEMATICS
Vol. 57, No. 2, 1975

A REMARK ON THE LATTICE OF IDEALS OF A PRÜFER DOMAIN

D. D. ANDERSON

For a ring R we will use $L(R)$ to denote the lattice of ideals of R. It is known that for a Dedekind domain D, there exists a PID D' such that $L(D)$ and $L(D')$ are isomorphic. In this note we show that for a Prüfer domain D, there exists a Bézout domain D' such that $L(D)$ and $L(D')$ are isomorphic.

We use the Krull-Kaplansky-Jaffard-Ohm theorem which states that any lattice-ordered abelian group is the group of divisibility of a Bézout domain.

Let D be a Prüfer domain and let S be the set of nonzero finitely generated (i.e., invertible) ideals of D. Then $(S, \supseteqq)$ is a partially ordered cancellation monoid under multiplication; moreover, $\supseteqq$ is actually a lattice order. Let $(S^*, \leqq)$ be the group of quotients of S with $\leqq$ the partial order induced by $\supseteqq$. Then $(S^*, \leqq)$ is lattice ordered and $S^*_+ = \{s \in S^* \mid s \geqq 0\} = S$. By the Krull-Kaplansky-Jaffard-Ohm theorem [2], S^* is the group of divisibility of a Bézout domain D', more precisely, there exists a field L and a demivaluation $w: L \to S^* \cup \{\infty\}$ such that $D' = \{x \in L \mid w(x) \geqq 0\}$ and D' is a Bézout domain. We proceed to show that $L(D)$ and $L(D')$ are isomorphic.

THEOREM. *Given a Prüfer domain D, there exists a Bézout domain D' such that $L(D)$ is isomorphic to $L(D')$.*

Proof. We define a mapping $v: L(D) \to \mathscr{P}(S \cup \{\infty\})$ by $v(J) = \{K \in S \mid K \subseteqq J\} \cup \{\infty\}$. We then define a map $\theta: L(D) \to L(D')$ by $\theta(J) = w^{-1}(v(J))$ where w is the demivaluation previously defined. θ is clearly well-defined and preserves order. For an ideal N in D' we consider the subset $w^{-1}(N)$ of S. The set $F = \bigcup \{K \in L(D) \mid K \in w^{-1}(N)\}$ is an ideal of D and $\theta(F) = N$; thus θ is onto. To show that θ is one-to-one and that its inverse preserves order, it is sufficient to show that $\theta(J) \subseteqq \theta(K)$ implies $J \subseteqq K$. Now $0 \neq j \in J$ implies $jD \subseteqq J$ so $jD \in v(J)$. Let $x \in L$ such that $w(x) = jD$. Then $x \in \theta(J) \subseteqq \theta(K)$. Now $x \in \theta(K)$ implies $w(x) \in w(K)$ so $u(x) = jD \subseteqq K$. Thus θ is a lattice isomorphism.

This theorem raises the following question. Given an integral domain D, does there exist an integral domain D' such that $L(D)$ and $L(D')$ are isomorphic and such that every invertible ideal in D'

is principal? More generally, given a commutative ring R, does there exist a commutative ring R' such that $L(R)$ and $L(R')$ are isomorphic and every principal element in $L(R')$ [1] is a truly principal (cyclic) ideal?

REFERENCES

1. R. P. Dilworth, *Abstract commutative ideal theory*, Pacific J. Math., **12** (1962), 481-498.

2. R. Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, New York, 1972.

3. E. W. Johnson and J. P. Lediaev, *Representable distributive Noether lattices*, Pacific J. Math., **28** (1969), 561-564.

Received October 10, 1974.

UNIVERSITY OF IOWA

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

The Pacific Journal of Mathematics expects the author's institution to pay page charges, and reserves the right to delay publication for nonpayment of charges in case of financial emergency.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $72.00 a year (6 Vols., 12 issues). Special rate: $36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 57, No. 2 February, 1975