A REMARK ON THE LATTICE OF IDEALS OF A PRÜFER
DOMAIN

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For a ring $R$ we will use $L(R)$ to denote the lattice of ideals of $R$. It is known that for a Dedekind domain $D$, there exists a PID $D'$ such that $L(D)$ and $L(D')$ are isomorphic. In this note we show that for a Prüfer domain $D$, there exists a Bézout domain $D'$ such that $L(D)$ and $L(D')$ are isomorphic.

We use the Krull-Kaplansky-Jaffard-Ohm theorem which states that any lattice-ordered abelian group is the group of divisibility of a Bézout domain.

Let $D$ be a Prüfer domain and let $S$ be the set of nonzero finitely generated (i.e., invertible) ideals of $D$. Then $(S, \geq)$ is a partially ordered cancellation monoid under multiplication; moreover, $\geq$ is actually a lattice order. Let $(S^*, \leq)$ be the group of quotients of $S$ with $\leq$ the partial order induced by $\geq$. Then $(S^*, \leq)$ is lattice ordered and $S^*_+ = \{s \in S^* | s \geq 0\} = S$. By the Krull-Kaplansky-Jaffard-Ohm theorem [2], $S^*$ is the group of divisibility of a Bézout domain $D'$, more precisely, there exists a field $L$ and a demivaluation $w: L \to S^* \cup \{\infty\}$ such that $D' = \{x \in L | w(x) \geq 0\}$ and $D'$ is a Bézout domain. We proceed to show that $L(D)$ and $L(D')$ are isomorphic.

**Theorem.** Given a Prüfer domain $D$, there exists a Bézout domain $D'$ such that $L(D)$ is isomorphic to $L(D')$.

**Proof.** We define a mapping $v: L(D) \to \mathcal{P}(S \cup \{\infty\})$ by $v(J) = \{K \in S | K \subseteq J\} \cup \{\infty\}$. We then define a map $\theta: L(D) \to L(D')$ by $\theta(J) = w^{-1}(v(J))$ where $w$ is the demivaluation previously defined. $\theta$ is clearly well-defined and preserves order. For an ideal $N$ in $D'$ we consider the subset $w^{-1}(N)$ of $S$. The set $F = \bigcup \{K \in L(D) | K \in w^{-1}(N)\}$ is an ideal of $D$ and $\theta(F) = N$; thus $\theta$ is onto. To show that $\theta$ is one-to-one and that its inverse preserves order, it is sufficient to show that $\theta(J) \subseteq \theta(K)$ implies $J \subseteq K$. Now $0 \neq j \in J$ implies $jD \subseteq J$ so $jD \in v(J)$. Let $x \in L$ such that $w(x) = jD$. Then $x \in \theta(J) \subseteq \theta(K)$. Now $x \in \theta(K)$ implies $w(x) \in w(K)$ so $w(x) = jD \subseteq K$. Thus $\theta$ is a lattice isomorphism.

This theorem raises the following question. Given an integral domain $D$, does there exist an integral domain $D'$ such that $L(D)$ and $L(D')$ are isomorphic and such that every invertible ideal in $D'$
is principal? More generally, given a commutative ring $R$, does there exist a commutative ring $R'$ such that $L(R)$ and $L(R')$ are isomorphic and every principal element in $L(R')$ [1] is a truly principal (cyclic) ideal?

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