

# Pacific Journal of Mathematics

## **A REMARK ON THE LATTICE OF IDEALS OF A PRÜFER DOMAIN**

DANIEL D. ANDERSON

## A REMARK ON THE LATTICE OF IDEALS OF A PRÜFER DOMAIN

D. D. ANDERSON

**For a ring  $R$  we will use  $L(R)$  to denote the lattice of ideals of  $R$ . It is known that for a Dedekind domain  $D$ , there exists a PID  $D'$  such that  $L(D)$  and  $L(D')$  are isomorphic. In this note we show that for a Prüfer domain  $D$ , there exists a Bézout domain  $D'$  such that  $L(D)$  and  $L(D')$  are isomorphic.**

We use the Krull-Kaplansky-Jaffard-Ohm theorem which states that any lattice-ordered abelian group is the group of divisibility of a Bézout domain.

Let  $D$  be a Prüfer domain and let  $S$  be the set of nonzero finitely generated (i.e., invertible) ideals of  $D$ . Then  $(S, \supseteq)$  is a partially ordered cancellation monoid under multiplication; moreover,  $\supseteq$  is actually a lattice order. Let  $(S^*, \leq)$  be the group of quotients of  $S$  with  $\leq$  the partial order induced by  $\supseteq$ . Then  $(S^*, \leq)$  is lattice ordered and  $S_+^* = \{s \in S^* \mid s \geq 0\} = S$ . By the Krull-Kaplansky-Jaffard-Ohm theorem [2],  $S^*$  is the group of divisibility of a Bézout domain  $D'$ , more precisely, there exists a field  $L$  and a demivaluation  $w: L \rightarrow S^* \cup \{\infty\}$  such that  $D' = \{x \in L \mid w(x) \geq 0\}$  and  $D'$  is a Bézout domain. We proceed to show that  $L(D)$  and  $L(D')$  are isomorphic.

**THEOREM.** *Given a Prüfer domain  $D$ , there exists a Bézout domain  $D'$  such that  $L(D)$  is isomorphic to  $L(D')$ .*

*Proof.* We define a mapping  $v: L(D) \rightarrow \mathcal{P}(S \cup \{\infty\})$  by  $v(J) = \{K \in S \mid K \subseteq J\} \cup \{\infty\}$ . We then define a map  $\theta: L(D) \rightarrow L(D')$  by  $\theta(J) = w^{-1}(v(J))$  where  $w$  is the demivaluation previously defined.  $\theta$  is clearly well-defined and preserves order. For an ideal  $N$  in  $D'$  we consider the subset  $w^{-1}(N)$  of  $S$ . The set  $F = \bigcup \{K \in L(D) \mid K \in w^{-1}(N)\}$  is an ideal of  $D$  and  $\theta(F) = N$ ; thus  $\theta$  is onto. To show that  $\theta$  is one-to-one and that its inverse preserves order, it is sufficient to show that  $\theta(J) \subseteq \theta(K)$  implies  $J \subseteq K$ . Now  $0 \neq j \in J$  implies  $jD \subseteq J$  so  $jD \in v(J)$ . Let  $x \in L$  such that  $w(x) = jD$ . Then  $x \in \theta(J) \subseteq \theta(K)$ . Now  $x \in \theta(K)$  implies  $w(x) \in w(K)$  so  $w(x) = jD \subseteq K$ . Thus  $\theta$  is a lattice isomorphism.

This theorem raises the following question. Given an integral domain  $D$ , does there exist an integral domain  $D'$  such that  $L(D)$  and  $L(D')$  are isomorphic and such that every invertible ideal in  $D'$

is principal? More generally, given a commutative ring  $R$ , does there exist a commutative ring  $R'$  such that  $L(R)$  and  $L(R')$  are isomorphic and every principal element in  $L(R')$  [1] is a truly principal (cyclic) ideal?

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Norman Larrabee Alling, <i>On Cauchy's theorem for real algebraic curves with boundary</i> .....	315
Daniel D. Anderson, <i>A remark on the lattice of ideals of a Prüfer domain</i> .....	323
Dennis Neal Barr and Peter D. Miletta, <i>A necessary and sufficient condition for uniqueness of solutions to two point boundary value problems</i> .....	325
Ladislav Beran, <i>On solvability of generalized orthomodular lattices</i> .....	331
L. Carlitz, <i>A three-term relation for some sums related to Dedekind sums</i> .....	339
Arthur Herbert Copeland, Jr. and Albert Oscar Shar, <i>Images and pre-images of localization maps</i> .....	349
G. G. Dandapat, John L. Hunsucker and Carl Pomerance, <i>Some new results on odd perfect numbers</i> .....	359
M. Edelstein and L. Keener, <i>Characterizations of infinite-dimensional and nonreflexive spaces</i> .....	365
Francis James Flanigan, <i>On Levi factors of derivation algebras and the radical embedding problem</i> .....	371
Harvey Friedman, <i>Provable equality in primitive recursive arithmetic with and without induction</i> .....	379
Joseph Braucher Fugate and Lee K. Mohler, <i>The fixed point property for tree-like continua with finitely many arc components</i> .....	393
John Norman Ginsburg and Victor Harold Saks, <i>Some applications of ultrafilters in topology</i> .....	403
Arjun K. Gupta, <i>Generalisation of a "square" functional equation</i> .....	419
Thomas Lee Hayden and Frank Jones Massey, <i>Nonlinear holomorphic semigroups</i> .....	423
V. Kannan and Thekkedath Thrivikraman, <i>Lattices of Hausdorff compactifications of a locally compact space</i> .....	441
J. E. Kerlin and Wilfred Dennis Pepe, <i>Norm decreasing homomorphisms between group algebras</i> .....	445
Young K. Kwon, <i>Behavior of <math>\Phi</math>-bounded harmonic functions at the Wiener boundary</i> .....	453
Richard Arthur Levaro, <i>Projective quasi-coherent sheaves of modules</i> .....	457
Chung Lin, <i>Rearranging Fourier transforms on groups</i> .....	463
David Lowell Lovelady, <i>An asymptotic analysis of an odd order linear differential equation</i> ..	475
Jerry Malzan, <i>On groups with a single involution</i> .....	481
J. F. McClendon, <i>Metric families</i> .....	491
Carl Pomerance, <i>On multiply perfect numbers with a special property</i> .....	511
Mohan S. Putcha and Adil Mohamed Yaqub, <i>Polynomial constraints for finiteness of semisimple rings</i> .....	519
Calvin R. Putnam, <i>Hyponormal contractions and strong power convergence</i> .....	531
Douglas Conner Ravenel, <i>Multiplicative operations in <math>BP^*BP</math></i> .....	539
Judith Roitman, <i>Attaining the spread at cardinals which are not strong limits</i> .....	545
Kazuyuki Saitô, <i>Groups of *-automorphisms and invariant maps of von Neumann algebras</i> ....	553
Brian Kirkwood Schmidt, <i>Homotopy invariance of contravariant functors acting on smooth manifolds</i> .....	559
Kenneth Barry Stolarsky, <i>The sum of the distances to N points on a sphere</i> .....	563
Mark Lawrence Teply, <i>Semiprime rings with the singular splitting property</i> .....	575
J. Pelham Thomas, <i>Maximal connected Hausdorff spaces</i> .....	581
Charles Thomas Tucker, II, <i>Concerning <math>\sigma</math>-homomorphisms of Riesz spaces</i> .....	585
Rangachari Venkataraman, <i>Compactness in abelian topological groups</i> .....	591
William Charles Waterhouse, <i>Basically bounded functors and flat sheaves</i> .....	597
David Westreich, <i>Bifurcation of operator equations with unbounded linearized part</i> .....	611
William Robin Zame, <i>Extendibility, boundedness and sequential convergence in spaces of holomorphic functions</i> .....	619