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For a ring R we will use $L(R)$ to denote the lattice of ideals of R. It is known that for a Dedekind domain D, there exists a PID D' such that $L(D)$ and $L(D')$ are isomorphic. In this note we show that for a Prüfer domain D, there exists a Bézout domain D' such that $L(D)$ and $L(D')$ are isomorphic.

We use the Krull-Kaplansky-Jaffard-Ohm theorem which states that any lattice-ordered abelian group is the group of divisibility of a Bézout domain.

Let D be a Prüfer domain and let S be the set of nonzero finitely generated (i.e., invertible) ideals of D. Then $(S, \supseteqq)$ is a partially ordered cancellation monoid under multiplication; moreover, $\supseteqq$ is actually a lattice order. Let $(S^*, \leqq)$ be the group of quotients of S with $\leqq$ the partial order induced by $\supseteqq$. Then $(S^*, \leqq)$ is lattice ordered and $S^*_+ = \{s \in S^* \mid s \geqq 0\} = S$. By the Krull-Kaplansky-Jaffard-Ohm theorem [2], S^* is the group of divisibility of a Bézout domain D', more precisely, there exists a field L and a demivaluation $w: L \rightarrow S^* \cup \{\infty\}$ such that $D' = \{x \in L \mid w(x) \geqq 0\}$ and D' is a Bézout domain. We proceed to show that $L(D)$ and $L(D')$ are isomorphic.

THEOREM. *Given a Prüfer domain D, there exists a Bézout domain D' such that $L(D)$ is isomorphic to $L(D')$.*

Proof. We define a mapping $v: L(D) \rightarrow \mathscr{P}(S \cup \{\infty\})$ by $v(J) = \{K \in S \mid K \subseteqq J\} \cup \{\infty\}$. We then define a map $\theta: L(D) \rightarrow L(D')$ by $\theta(J) = w^{-1}(v(J))$ where w is the demivaluation previously defined. θ is clearly well-defined and preserves order. For an ideal N in D' we consider the subset $w^{-1}(N)$ of S. The set $F = \bigcup \{K \in L(D) \mid K \in w^{-1}(N)\}$ is an ideal of D and $\theta(F) = N$; thus θ is onto. To show that θ is one-to-one and that its inverse preserves order, it is sufficient to show that $\theta(J) \subseteqq \theta(K)$ implies $J \subseteqq K$. Now $0 \neq j \in J$ implies $jD \subseteqq J$ so $jD \in v(J)$. Let $x \in L$ such that $w(x) = jD$. Then $x \in \theta(J) \subseteqq \theta(K)$. Now $x \in \theta(K)$ implies $w(x) \in w(K)$ so $u(x) = jD \subseteqq K$. Thus θ is a lattice isomorphism.

This theorem raises the following question. Given an integral domain D, does there exist an integral domain D' such that $L(D)$ and $L(D')$ are isomorphic and such that every invertible ideal in D'

is principal? More generally, given a commutative ring R, does there exist a commutative ring R' such that $L(R)$ and $L(R')$ are isomorphic and every principal element in $L(R')$ [1] is a truly principal (cyclic) ideal?

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