A NECESSARY AND SUFFICIENT CONDITION FOR UNIQUENESS OF SOLUTIONS TO TWO POINT BOUNDARY VALUE PROBLEMS

DENNIS NEAL BARR AND PETER D. MILETTA
A NECESSARY AND SUFFICIENT CONDITION FOR
UNIQUENESS OF SOLUTIONS TO TWO POINT
BOUNDARY VALUE PROBLEMS

DENNIS BARR AND PETER MILETTA

In this paper it is shown that the uniqueness of solutions
to two point boundary value problems in which one end
point is held fixed is equivalent to the existence of a family
of Liapunov functions.

T. Yoshizawa [6] and H. Okamura [5] demonstrated that the
uniqueness of solutions to initial value problems was equivalent
and S. Bernfeld [1] provided necessary and sufficient conditions for
the existence of solutions to initial value problems on \([t_0, \infty)\) with
the use of Liapunov functions. With regard to boundary value
problems J. H. George and W. G. Sutton [2] obtained a sufficient
condition in terms of a Liapunov function for the existence and
uniqueness of solutions to two point boundary value problems. In
this paper we shall employ a variation of the Okamura function to
obtain a generalization of the latter result for a certain class of
two point boundary value problems.

1. Preliminaries. In this section we state the definition of a
Liapunov function and establish a theorem which will be used in
the next section. We shall consider the second order differential
equation:

\[ x'' = f(t, x, x') , \]

where \(f\) is a real-valued function defined and continuous on \([a, b] \times \mathbb{R}^2\). It will be assumed that initial value problems associated with
(1) exist, are unique, and that solutions are defined throughout
\([a, b]\). In particular we shall be concerned with the uniqueness of
solutions to (1) satisfying

\[ x(t_1) = y_1, \quad x(t_2) = y_2 \]

where \(a \leq t_1 < t_2 \leq b\) and \(y_1, y_2 \in \mathbb{R}\). If \(x_0(t)\) is any solution of (1)
satisfying (2) for some points \(t_1\) and \(t_2\), then by setting \(x(t) = y(t) + x_0(t)\) we obtain

\[ y'(t) = F(t, y(t), y'(t)) , \]

where \(F(t, y(t), y'(t)) = f(t, y(t) + x_0(t), y'(t) + x_0'(t)) - f(t, x_0(t), x_0'(t))\).
Thus equation (1) has a unique solution satisfying (2) if and only if 
\( y(t) = 0 \) is the only solution of (3) such that 
\( y(t_1) = y(t_2) = 0 \). Thus we shall restrict our attention to the differential equation

(4) \[ x'' = F(t, x, x') , \]

where \( F(t, 0, 0) = 0 \) and the boundary conditions

(5) \[ x(t_1) = x(t_2) = 0 . \]

The principal tool employed in this paper will be Liapunov functions.

**DEFINITION.** A Liapunov function for (4) is a real valued function 
\( V \) defined on \( D = [a, b] \times S \) where \( S \) is a closed subset of \( \mathbb{R}^2 \) and \((0, 0) \in S\), such that

(6) \[ V(t, 0, x_2) = 0 \]

(7) \[ V(t, x_1, x_2) > 0 \text{ if } x_1 \neq 0 \]

(8) \[ V(t, x_1, x_2) \text{ is nondecreasing along solution curves of (4)} \]

By condition (8) we shall mean that if \( x(t) \) is a solution of (4),
then \( V(t, x(t_1), x'(t_1)) \leq V(t_2, x(t_2), x'(t_2)) \) for all points \( t_1 \) and \( t_2, t_1 < t_2 \), such that \( (t_1, x(t_1), x'(t_1)) \in D \).

If \( V \) is a real valued function satisfying (6) and (7), then the following theorem provides a sufficient condition for \( V \) to be a Liapunov function.

**THEOREM 1.** Suppose \( V \) is continuous and satisfies a Lipschitz condition locally with respect to \( x_1 \) and \( x_2 \) in \( D \), and

\[ V(t, x_1, x_2) = \lim_{h \to 0^+} \frac{1}{h} [V(t + h, x_1 + hx_2, x_2 + hF(t, x_1, x_2)) - V(t, x_1, x_2)] \geq 0 \]

for \( t, x_1, x_2 \) in the interior of \( D \). Then \( V(t, x_1, x_2) \) is nondecreasing along solution curves of (4).

**Proof.** Yoshizawa [6].

The following theorem gives a sufficient condition for the uniqueness of solutions of (4) satisfying (5).

**THEOREM 2.** Suppose \( V(t, x_1, x_2) \) is a Liapunov function for (4) defined on \([a, b] \times \mathbb{R}^2 \). Then for any \( t_1 \) and \( t_2 \), \( a \leq t_1 < t_2 \leq b \), there exists at most one solution to (4) satisfying (5).
Proof. Employing a stronger definition of a Liapunov function George and Sutton [2] have given a proof of this theorem. We include a proof for referral at a later time. Suppose \( y(t) \) is a non-zero solution of (4) satisfying (5). Then there exists a \( t_0 \in (t_1, t_2) \) such that \( y(t_0) \neq 0 \). Thus \( V(t_0, y(t_0), y'(t_0)) > 0 \). However \( y(t_2) = 0 \) implies that \( V(t_2, y(t_2), y'(t_2)) = 0 \). This is a contradiction to the assumption that \( V(t, x_i, x_2) \) is nondecreasing along the solution curves of (4).

Two known conditions insuring the uniqueness of solutions of (4) satisfying (5) are consequences of Theorem 2. Hartman [3, page 433] proved that if \( \{x(t)f + x(t)F(t, x(t), x'(t)) > 0 \) whenever \( x(t) \neq 0 \) and \( x'(t)x''(t) = 0 \), for all solutions \( x(t) \) of (4) then \( x(t) = 0 \) is the only solution of (4) satisfying (5). It was noted by George and Sutton that \( V(t, x_1, x_2) = (x_1)^2 \) satisfied their definition of a Liapunov function and therefore by Theorem 2 the result of Hartman follows. The same choice for \( V(t, x_1, x_2) \) will satisfy our definition. It is also well known that if \( F(t, x_1, x_2) \) is continuous and strictly increasing in \( x_1 \) for fixed \( (t, x_2) \) then \( x(t) = 0 \) is the only solution of (4) satisfying (5). This result also follows from Theorem 2 by choosing

\[
V(t, x_1, x_2) = \int_a^t |F(s, x_1, x_2) - F(s, 0, x_2)| \, ds.
\]

2. Necessary and sufficient condition. In this section we will further restrict our attention to the boundary value problem

\[
\begin{align*}
(9) & \quad x'' = F(t, x, x') \\
(10) & \quad x(a) = 0 \quad x(\gamma) = 0 ,
\end{align*}
\]

where \( F(t, 0, 0) = 0 \) and \( a < \gamma \leq b \). Thus we shall fix \( t_1 = a \). We now proceed to derive a necessary and sufficient condition for the uniqueness of solutions of (9) satisfying (10).

For each \( M > 0 \) let \( D_M \) denote the subset of \([a, b] \times \mathbb{R}^2\) defined by \( D_M = \{(t, \alpha, \beta): 2|\alpha/(t - a)| \leq M, 2|\alpha/(t - b)| \leq M, 2|\beta| \leq M \} \cup \{(a, 0, \beta): 2|\beta| \leq M \} \cup \{(b, 0, \beta): 2|\beta| \leq M \} \). For each \( (t_0, \alpha, \beta) \in D_M, a < t_0 < b \), let \( X^M_{(t_0, \alpha, \beta)} \) denote the set of all continuously differentiable functions \( x(t) \) defined on \([a, b]\) whose second derivative exists and is continuous for all except at most one point of \([a, b]\) and which satisfy \( |x'(t)| \leq M \) for all \( t \in [a, b], \) \( x(a) = x(b) = 0, \) \( x(t_0) = \alpha, \) and \( x'(t_0) = \beta. \) The restrictions on \( (t_0, \alpha, \beta) \) in the definition of \( D_M \) insure that \( X^M_{(t_0, \alpha, \beta)} \) is not empty.

**Lemma 1.** Suppose \( a < t_0 < b. \) There exists a solution to (9)
satisfying \( x(a) = x(b) = 0, x(t_0) = \alpha, \) and \( x'(t_0) = \beta \) if and only if

\[
\inf_{x(t) \in X^M_{(t_0,a,\beta)}} \int_{t_0}^{b} |x''(t) - F(t, x(t), x'(t))| \, dt = 0
\]

for some \( M \).

**Proof.** If there exists a solution to (9) satisfying the above conditions then for some \( M > 0, x(t) \in X^M_{(t_0,a,\beta)} \). For this \( M \), or any larger \( M \), the above infimum will be zero. Conversely suppose the above infimum is zero for some \( M > 0 \). Let \( \{x_k(t)\} \) be a sequence of functions in \( X^M_{(t_0,a,\beta)} \) such that

\[
\lim_{k \to \infty} \int_{a}^{b} |x_k''(t) - F(t, x_k(t), x_k'(t))| \, dt = 0.
\]

Letting \( y_k(t) = \int_{a}^{t} (x_k''(s) - F(s, x_k(s), x_k'(s))) \, ds \) we see by (12) that \( y_k(t) \) converges to zero uniformly on \([a, b]\). Let \( z_k(t) = x_k(t) - \int_{a}^{t} y_k(s) \, ds \). Then \( z'_k(t) = x'_k(t) - y_k(t) = x'_k(a) + \int_{a}^{t} F(s, x_k(s), x_k'(s)) \, ds \) and for \( a \leq t_1 < t_2 \leq b \)

\[
|z'_k(t_1) - z'_k(t_2)| \leq \int_{t_1}^{t_2} \left| F(s, x_k(s), x_k'(s)) \right| \, ds \leq K_M |t_2 - t_1|,
\]

where \( K_M = \max_{(t, x_1, x_2) \in D_M} |F(t, x_1, x_2)| \). Also

\[
|z'_k(t)| \leq M + K_M (b - a).
\]

Therefore, there exists a uniformly convergent subsequence of \( \{z'_k(t)\} \) which we shall again denote by \( \{z'_k(t)\} \). In a similar manner the sequence of functions \( \{z_k(t)\} \) can easily be shown to be equicontinuous and uniformly bounded. Thus we again obtain a subsequence which we denote by \( \{z_k(t)\} \), such that \( \{z_k(t)\} \) and \( \{z'_k(t)\} \) converge uniformly on \([a, b]\). Denote the limit function by \( z(t) \). Since \( z_k(a) = 0 \) for all \( k \), we have that \( z(a) = 0 \). Also \( y_k(t) \) converges to 0 uniformly on \([a, b]\) implies that \( x_k(t) \) converges uniformly to \( z(t) \) and \( x'_k(t) \) converges uniformly to \( z'(t) \) on \([a, b]\).

Thus \( z(b) = 0, z(t_0) = \alpha, \) \( z'(t_0) = \beta, \) and

\[
z''(t) = F(t, z(t), z'(t)).
\]

Thus \( z(t) \) is a solution with the desired properties.

For each \( M > 0 \) we define a real valued function \( V_M \) with domain \( D_M \) by
A NECESSARY AND SUFFICIENT CONDITION 329

\[ \begin{aligned}
V_M(t, x_1, x_2) &= \begin{cases} 
\inf_{\tau(t) \in X^M_{(\tau, x_1, x_2)}} \int_0^b \left| x'(t) - F(t, x(t), x'(t)) \right| dt, & x_1 \neq 0 \\
0, & x_1 = 0
\end{cases} 
\end{aligned} \]

\[ (13) \]

**Theorem 3.** Suppose that there exists at most one solution to (9) satisfying (10) for every \( \gamma \), \( a < \gamma \leq b \). Then for each \( M > 0 \), \( V_M \) satisfies the following conditions:

\[ (14) \]

\[ V_M(t, 0, x_2) = 0 \]

\[ (15) \]

\[ V_M(t, x_1, x_2) > 0 \quad \text{if } x_1 \neq 0 \]

\[ (16) \]

\[ V_M(t, x_1, x_2) \text{ is nondecreasing along solution curves } x(t) \text{ of (9) which satisfy } x(a) = 0 \text{ and } (t, x(t), x'(t)) \in D_M \text{ for all } t \in [a, b]. \]

**Proof.** \( V_M \) clearly satisfies (14). Suppose \( x_1 \neq 0 \) and \( V_M(t, x_1, x_2) = 0 \). Then by Lemma 1 there exists a solution \( x(t) \) of (9) such that \( x(a) = x(b) = 0 \) and \( x(t) = x_1 \). This contradicts the uniqueness assumption. Thus \( V_M \) satisfies condition (15). Let \( x(t) \) be a solution of (9) such that \( (t, x(t), x'(t)) \in D_M \) for all \( t \in [a, b] \) and \( x(a) = 0 \). Let \( a \leq t_1 < t_2 \leq b \). If \( t_1 = a \) or \( t_2 = b \) then it follows trivially from the uniqueness of solutions of (9) satisfying (10), the uniqueness of initial value problems associated with (9), and properties (14) and (15) that \( V_M(t_1, x(t_1), x'(t_1)) \leq V_M(t_2, x(t_2), x'(t_2)) \). Thus assume that \( a < t_1 < t_2 < b \). Again from uniqueness of solutions of (9) satisfying (10) it follows that \( x(t_1) \neq 0 \) and \( x(t_2) \neq 0 \). For each \( y(t) \in X^M_{(t_1, x(t_1), x'(t_1))} \) the function

\[ z(t) = \begin{cases} 
x(t) & a \leq t \leq t_1 \\
y(t) & t_1 < t \leq b
\end{cases} \]

is again an element of \( X^M_{(t_1, x(t_1), x'(t_1))} \). Therefore

\[ V_M(t_1, x(t_1), x'(t_1)) = \inf_{\tau(t) \in X^M_{(\tau, x_1, x_2)}} \int_{t_1}^b \left| x''(t) - F(t, x(t), x'(t)) \right| dt \]

and in a similar manner

\[ V_M(t_2, x(t_2), x'(t_2)) = \inf_{\tau(t) \in X^M_{(\tau, x_1, x_2)}} \int_{t_2}^b \left| x''(t) - F(t, x(t), x'(t)) \right| dt . \]

Let \( \{x_k(t)\} \) be a sequence in \( X^M_{(t_2, x(t_2), x'(t_2))} \) such that

\[ V_M(t_2, x(t_2), x'(t_2)) = \lim_{k \to \infty} \int_{t_2}^b \left| x_k''(t) - F(t, x_k(t), x_k'(t)) \right| dt . \]

Then if
we have
\[
V_M(t, x(t), x'(t)) \leq \lim_{h \to \infty} \int_{t_1}^{b} |y''(t) - F(t, y(t), y'(t))| \, dt
\]
\[
= \lim_{h \to \infty} \int_{t_2}^{b} |x''(t) - F(t, x(t), x'(t))| \, dt
\]
\[
= V(t_2, x(t_2), x'(t_2)).
\]
Thus $V_M$ satisfies (16).

On the other hand if there exists a family of subsets $[a, b] \times S_M = D_M$, $S_M$ closed, of $[a, b] \times \mathbb{R}^2$ such that every solution $x(t)$ of (9) satisfies $(t, x(t), x'(t)) \in D_M$ for some $M$ and a family of real valued functions $V_M$ defined on $D_M$ and satisfying (14), (15), and (16), then the only solution of (9) satisfying (10) for any $\gamma \in (a, b]$ is identically zero on $[a, b]$. The proof of this is exactly the same as the proof of Theorem 2. Note that condition (16) is weaker than condition (8).

Theorem 4. There exists at most one solution of (9) satisfying (10) for all $\gamma \in (a, b]$ if and only if there exists a family of subsets $D_M = [a, b] \times S_M$ of $[a, b] \times \mathbb{R}^2$ such that every solution $x(t)$ of (9) satisfies $(t, x(t), x'(t)) \in D_M$ for some $M$ and a family of real valued functions $V_M$ defined on $D_M$ satisfying condition (14), (15), and (16).

Note that a similar theorem to Theorem 4 can be proved if the right end point is fixed.

References


Received June 24, 1974.

University of Puerto Rico
<table>
<thead>
<tr>
<th>Title</th>
<th>Volume/Issue</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>On Cauchy's theorem for real algebraic curves with boundary</td>
<td>57, No. 2</td>
<td>315</td>
</tr>
<tr>
<td>A remark on the lattice of ideals of a Prüfer domain</td>
<td></td>
<td>323</td>
</tr>
<tr>
<td>A necessary and sufficient condition for uniqueness of solutions to two point boundary value problems</td>
<td></td>
<td>325</td>
</tr>
<tr>
<td>On solvability of generalized orthomodular lattices</td>
<td></td>
<td>331</td>
</tr>
<tr>
<td>A three-term relation for some sums related to Dedekind sums</td>
<td></td>
<td>339</td>
</tr>
<tr>
<td>Images and pre-images of localization maps</td>
<td></td>
<td>349</td>
</tr>
<tr>
<td>Some new results on odd perfect numbers</td>
<td></td>
<td>359</td>
</tr>
<tr>
<td>Characterizations of infinite-dimensional and nonreflexive spaces</td>
<td></td>
<td>365</td>
</tr>
<tr>
<td>On Levi factors of derivation algebras and the radical embedding problem</td>
<td></td>
<td>371</td>
</tr>
<tr>
<td>Provable equality in primitive recursive arithmetic with and without induction</td>
<td></td>
<td>379</td>
</tr>
<tr>
<td>The fixed point property for tree-like continua with finitely many arc components</td>
<td></td>
<td>393</td>
</tr>
<tr>
<td>Some applications of ultrafilters in topology</td>
<td></td>
<td>403</td>
</tr>
<tr>
<td>Generalisation of a “square” functional equation</td>
<td></td>
<td>419</td>
</tr>
<tr>
<td>Nonlinear holomorphic semigroups</td>
<td></td>
<td>423</td>
</tr>
<tr>
<td>The fixed point property for tree-like continua with finitely many arc components</td>
<td></td>
<td>441</td>
</tr>
<tr>
<td>Norm decreasing homomorphisms between group algebras</td>
<td></td>
<td>445</td>
</tr>
<tr>
<td>Behavior of Φ-bounded harmonic functions at the Wiener boundary</td>
<td></td>
<td>453</td>
</tr>
<tr>
<td>Projective quasi-coherent sheaves of modules</td>
<td></td>
<td>457</td>
</tr>
<tr>
<td>Rearranging Fourier transforms on groups</td>
<td></td>
<td>463</td>
</tr>
<tr>
<td>An asymptotic analysis of an odd order linear differential equation</td>
<td></td>
<td>475</td>
</tr>
<tr>
<td>On groups with a single involution</td>
<td></td>
<td>481</td>
</tr>
<tr>
<td>Metric families</td>
<td></td>
<td>491</td>
</tr>
<tr>
<td>On multiply perfect numbers with a special property</td>
<td></td>
<td>511</td>
</tr>
<tr>
<td>Polynomial constraints for finiteness of semisimple rings</td>
<td></td>
<td>519</td>
</tr>
<tr>
<td>Hyponormal contractions and strong power convergence</td>
<td></td>
<td>531</td>
</tr>
<tr>
<td>Multiplicative operations in BP*BP</td>
<td></td>
<td>539</td>
</tr>
<tr>
<td>Attaining the spread at cardinals which are not strong limits</td>
<td></td>
<td>545</td>
</tr>
<tr>
<td>Groups of *-automorphisms and invariant maps of von Neumann algebras</td>
<td></td>
<td>553</td>
</tr>
<tr>
<td>Homotopy invariance of contravariant functors acting on smooth manifolds</td>
<td></td>
<td>559</td>
</tr>
<tr>
<td>The sum of the distances to N points on a sphere</td>
<td></td>
<td>563</td>
</tr>
<tr>
<td>Semiprime rings with the singular splitting property</td>
<td></td>
<td>575</td>
</tr>
<tr>
<td>Maximal connected Hausdorff spaces</td>
<td></td>
<td>581</td>
</tr>
<tr>
<td>Concerning σ-homomorphisms of Riesz spaces</td>
<td></td>
<td>585</td>
</tr>
<tr>
<td>Compactness in abelian topological groups</td>
<td></td>
<td>591</td>
</tr>
<tr>
<td>Basically bounded functors and flat sheaves</td>
<td></td>
<td>597</td>
</tr>
<tr>
<td>Bifurcation of operator equations with unbounded linearized part</td>
<td></td>
<td>611</td>
</tr>
<tr>
<td>Extendibility, boundedness and sequential convergence in spaces of holomorphic functions</td>
<td></td>
<td>619</td>
</tr>
</tbody>
</table>