CHARACTERIZATIONS OF INFINITE-DIMENSIONAL AND NONREFLEXIVE SPACES

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Infinite-dimensional, resp. nonreflexive spaces are characterized in terms of subsets having a finite visibility property without being starshaped.

1. Introduction. A well-known result of Smulian [4] states that every nonreflexive normed linear space contains a decreasing sequence of nonempty closed and bounded convex sets whose intersection is empty. This result was used by V. L. Klee [1] to show that a normed linear space is nonreflexive if, and only if, it contains a decreasing sequence of closed and bounded starshaped sets whose intersection is empty. Also proved by Klee [2] is the following. Theorem [Klee]. Every infinite dimensional normed linear space contains a decreasing sequence of unbounded but linearly bounded closed convex sets whose intersection is empty. Here, a set is called linearly bounded if each straight line intersects it in a bounded set.

In the present paper other characterizations of infinite-dimensional, and of nonreflexive spaces are given which are similar in spirit and not unrelated to those mentioned above. To this end use is made of the notion of finite visibility. A set $S$ is said to have the finite visibility property, f.v.p. for short, if for any finite $F \subseteq S$ there is an $x \in S$ such that the line segment $[x, y]$ is contained in $S$ for all $y$ in $F$. As customary a set $S$ is called starshaped if an $s \in S$ exists such that the above condition is satisfied with $s$ replacing $x$ and $S$ replacing $F$. A well-known theorem of Krasnoselski [3] implies that in a finite dimensional normed linear space $X$ if $S$ is closed and bounded and has f.v.p. then $S$ is starshaped. (In fact, if $\dim X = n$, and card $S \geq n + 1$, then the above mentioned theorem holds if the hypothesis is satisfied for all $F$ with card $F = n + 1$.) A previous version of this paper was mainly concerned with showing that in some Banach spaces a weakly closed bounded set may have f.v.p. without being starshaped. The broader scope of the present paper is due to suggestions made by Professor Klee in a personal communication, in which he conjectured the two theorems of this paper and directed us to relevant passages in some of his works. It is indeed a pleasure to acknowledge his help.

2. Preliminary results.

Lemma 1. A compact subset $S$ of a Hausdorff linear topological
space $X$ is starshaped if it has the finite visibility property.

Proof. For $x \in S$, let $S_x = \{y \in S : [x, y] \subset S\}$, a closed set. The family $\{S_x : x \in S\}$ has the finite intersection property by f.v.p. so $\bigcap S_x \neq \emptyset$ by compactness, and $S$ is starshaped.

**Lemma 2.** Let $E$ be a closed subspace of a normed linear space $X$, $S$ a closed convex linearly bounded set in $E$ and $x$ a point in $X \sim E$. Then $K = \text{co} \{\{x\} \cup S\}$ is closed.

Proof. Let $y \in \bar{K}$, $y \neq x$, and let $F$ be the subspace spanned by $x$ and $S$. Clearly $y \in F$. Thus if $R$ is the ray emanating from $x$, through $y$, i.e. $R = \{z \in X : z = x + \alpha(y - x), \alpha \geq 0\}$, then $R$ is contained in $F$. Moreover, $R$ cannot be parallel to $E$, for if parallel, then with $w \in S$, $R' = \{z \in X : z = w + \alpha(y - x), \alpha \geq 0\}$ is contained in $E$ and by linear boundedness there is a $w' \in R' \sim S$. But then $w'$ and $S$ can be separated by a hyperplane $H \subset E$, relative to $E$. The subspace spanned by $H$ and $x$ clearly determines a closed half space of $F$ which contains $\{\{x\} \cup S\}$ and is disjoint from $y$, leading to a contradiction, since $y \in \bar{K}$. Suppose now that $u$ is the point of intersection of $R$ and $E$. It suffices to show that $u \in S$. If not, then there is an open ball $B$ about $u$ which is disjoint from $S$ and $\text{co} \{\{x\} \cup B\}$ is a neighborhood of $u$ which contains no point of the form $\lambda x + (1 - \lambda)s$ for any $\lambda$, $0 \leq \lambda < 1$ and $s \in S$. This is impossible since $y \in \bar{K}$. Hence $y \in K$ and $K = \bar{K}$ as claimed.

**Lemma 3.** Let $x$ be a normed linear space, $E$ a closed subspace of $X$ and $l$ a line skew to $E$, i.e. $l$ neither intersects $E$ nor is parallel to any line of $E$. Let $\{C_k : k = 1, 2, \cdots\}$ be a decreasing sequence of closed convex subsets of $E$ and $\{p_k : k = 1, 2, \cdots\}$ a sequence on $l$ converging to some $p_\circ$. Let $K_i = \text{co} \{\{p_i\} \cup C_i\}$ for $i \geq 1$ and $K_\circ = \text{co} \{\{p_\circ\} \cup C_\circ\}$.

Then $S = \bigcup \{K_i : i = 0, 1, \cdots\}$ is weakly closed. If, in addition, $C_i$ is linearly bounded then so is $S$.

Proof. To prove that $S$ is weakly closed let $x \in X \sim S$. Then $x \in K_\circ$, which is closed by Lemma 2, and convex. Thus there is a hyperplane $H$ such that $x \in H^+$ and $K_\circ \subset H^-$ where $H^+$ and $H^-$ are open half spaces determined by $H$. Let $n_\circ$ be such that $p_n \in H^-$ whenever $n > n_\circ$. Then, for such $n$, $K_n \subset H^-$ since $\{\{p_n\} \cup C_n\} \subset H^-$. On the other hand, as $\bigcup \{K_i : i \leq n_\circ\}$ is weakly closed there is a weak neighborhood $W$ of $x$ which is disjoint from it. It follows that $W \cap H^+$ is a weak neighborhood of $x$ which is disjoint from $S$. Hence $S$ is weakly closed. To prove linear boundedness observe first that,
as can be readily verified, in finite dimensional spaces boundedness and linear boundedness are equivalent for closed convex sets. If now \( l_i \) is a line in \( X \) let \( L \) be the subspace spanned by \( l \cup l_i \). Then \( L \cap C_i \) is bounded and closed and \( l_i \cap S \) is contained in the compact set

\[
\text{co} \{\{p_i; k = 0, 1, \cdots\} \cup (C_i \cap L)\}
\]

and therefore bounded. Hence \( S \) is linearly bounded, as asserted.

**Lemma 4.** Let \( X \) be a linear space, \( E \) a subspace of \( X \) and \( l \) a line in \( X \) which is skew to \( E \). If \( p, q \in l \), \( p \neq q \), and \( A, B \) are convex subsets of \( E \) then

\[
\text{co} \{(p) \cup A\} \cap \text{co} \{(q) \cup B\} = A \cap B.
\]

**Proof.** Let \( x \in \text{co} \{(p) \cup A\} \cap \text{co} \{(q) \cup B\} \). It suffices to show that \( x \in A \cap B \). If this were not the case then \( x \in [p, a) \cap [q, b) \) for some \( a \in A \) and \( b \in B \), with \( a \neq b \). But then \( a, b, p, q \) would have to be coplanar against the assumption that \( l \) is skew to \( E \).

**Lemma 5.** Let \( X \) be a linear space, \( E \) a subspace of \( X \) and \( l \) a line in \( X \) which is skew to \( E \). Suppose \( p_i; i = 1, 2, \cdots \) is a sequence of distinct points on \( l \). Let \( C_i \subset E \) be convex, \( K_i = \text{co} \{(p_i) \cup C_i\} i = 1, 2, \cdots \) and \( S = \bigcup \{K_i; i = 1, 2, \cdots\} \). Then \( S \) is starshaped if, and only if, \( \bigcap \{C_i; i = 1, 2, \cdots\} \neq \emptyset \) and \( S \) has f.v.p. if, and only if, \( \{C_i; i = 1, 2, \cdots\} \) has the finite intersection property.

**Proof.** If \( l' \) is a line such that \( l' \cap (K_j \sim C_j) \neq \emptyset \) then \( \text{card} (l' \cap K) \leq 1 \) for any \( i \neq j \). Indeed, if for some \( i \neq j \) \( l' \cap K_i \) contains two or more points then \( l' \) is contained in \( L_i \), the linear span of \( K_i \); but then \( l' \cap (K_j \sim C_j) = \emptyset \) since \( L_i \cap K_j \subset C_j \) by the preceding lemma. Hence \( [u, p_i] \), with \( u \in K_j \sim C_j \) and \( i \neq j \), is not contained in \( S \) as \( \text{card} ([u, p_i] \cap S) \leq \aleph_0 \). Thus \( \bigcup \{[u, p_m]; S: m \in M\} \), where \( M \) is a set of two or more positive integers, implies that \( u \) and for it to have f.v.p. \( \{C_i; i = 1, 2, \cdots\} \) has to have the finite intersection property.

For the converse note that \( u \in \bigcap \{C_i; i = 1, 2, \cdots\} \) implies \( S_u = S \) and if \( F \subset S \) is finite then, for \( N \) sufficiently large, \( F \subset \bigcup \{K_i: i = 1, 2, \cdots\} \) and this last set is contained in \( S_u \) for any \( u \in \bigcap \{C_i; i = 1, 2, \cdots, N\} \).

3. Main results.

**Theorem 1.** A normed linear space is infinite-dimensional if,
and only if, it contains a linearly bounded, weakly closed subset $S$ which has the finite visibility property but fails to be starshaped.

Proof. If $X$ contains a set $S$ with the stated properties then by the Krasnoselski theorem [3] $X$ must be infinite-dimensional.

Assume now that $X$ is infinite-dimensional and $E$ is a closed subspace of $X$ of codimension 2. By the theorem of Klee quoted in the introduction, $E$ contains a decreasing sequence $\{C_k: k = 1, 2, \ldots\}$ of nonempty, closed, linearly bounded subsets whose intersection is empty. Let $l$ be a line which is skew to $E$ and $\{p_k: k = 1, 2, \ldots\}$ a sequence of distinct points on $l$ converging to $p_0 \in l$. Let $K_i, i = 0, 1, \ldots$ and $S$ be as in Lemma 3. Then $S$ is weakly closed and linearly bounded by that lemma. By Lemma 4 $S$ has f.v.p. but fails to be starshaped.

Theorem 2. A normed linear space $X$ is nonreflexive if, and only if, it contains a set $S$ which is bounded, weakly closed, has the finite visibility property but fails to be starshaped.

Proof. If $X$ contains a set $S$ with the stated properties then, by Lemma 1, it fails to be reflexive.

Assume now that $X$ is nonreflexive and, as in the construction of the proof of Theorem 1, let $E$ be a closed subspace of $X$ of codimension 2 and $l$ a line skew to $E$. Let $\{p_k\}$ be a sequence of distinct points on $l$ converging to $p_0 \in l$. By the Smulian theorem [3] there exists a decreasing sequence $\{C_k: k = 1, 2, \ldots\}$ of nonempty, closed and bounded convex sets in $E$ whose intersection is empty. Let $K_i, i = 0, 1, \ldots$ and $S$ be defined as in the proof of Theorem 1. Then the arguments used there apply again to the effect that $S$ is weakly closed, bounded, with f.v.p. but not starshaped.

4. An example in $l_1$. The following is an example of a concrete subset of $l_1$ having all the properties of the set $S$ of Theorem 2. Let $S$ consist of all $x = (x_1, x_2, \ldots, x_n, \ldots) \in l_1$ such that

(i) $x_n \geq 0$ for $n = 1, 2, \ldots$;

(ii) $\|x\| = 1$;

(iii) if $x_{2n} \neq 0$ then $x_k = 0$ for $1 \leq k < 2n$.

To show that $S$ has the finite visibility property let $F \subset S$ be finite and $N$ an odd integer which is larger than the index of the first positive coordinate of each member of $F$. If $e_S \in S$ has 1 for its $N$th coordinate then clearly $[u, e_n] \subset S$ for all $u \in F$.

To prove that $S$ is weakly closed let $y = (y_1, y_2, \ldots, y_n, \ldots) \in l_1 \sim S$ and assume, as we may, that $\|y\| = 1$. Since $y \in S$, there must be
positive integers $n, k$ such that $k < 2n$ and $y_k > 0$ and $y_{2n} > 0$. If $u = (u_1, \ldots, u_k, \ldots), v = (v_1, \ldots, v_{2n}, \ldots) \in l_\infty$ are such that $u_k = v_{2n} = 1$ and all other coordinates $= 0$ then

$$W = \{z \in l_1: u(z) > 0 \text{ and } v(z) > 0\}$$

is a weak neighborhood of $y$ which is disjoint from $S$. Since boundedness of $S$ is obvious it remains to show that $S$ is not starshaped. If now $u = (u_1, u_2, \ldots, u_k, \ldots) \in S$ and $u_k \neq 0$ then for $x = (x_1, \ldots, x_n, \ldots) \in S$ with $s_{2k} = 1$ we have $[u, x] \in S$.

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