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**NONLINEAR HOLOMORPHIC SEMIGROUPS**

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## NONLINEAR HOLOMORPHIC SEMIGROUPS

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**Conditions are given on a nonlinear operator  $A$  in a Banach space  $X$  under which the semigroup,  $S(t)$ , generated by  $-A$  has the property that  $S(t)x$  is analytic in  $t$  for  $|\arg t| < \theta$  for each fixed  $x \in \text{cl}(D(A))$ . Analyticity in  $t$  of solutions of  $u' + Tu = Fu$  where  $-T$  generates a linear holomorphic semigroup in  $X$  and  $F$  maps  $D(T^\alpha)$  analytically into  $X$  for some  $\alpha < 1$  is also established. These results are applied to establish analyticity in  $t$  of solutions to  $\partial u / \partial t + Lu + \beta(u) = 0$  where  $\beta: R \rightarrow R$  is real analytic, monotone increasing and  $\beta(0) = 0$ , and  $L$  is a second order elliptic operator.**

1. **Introduction.** Hille and Yosida proved that if  $A$  is a densely defined linear operator on a Banach space  $X$  such that, for  $\lambda > 0$ ,  $I + \lambda A$  is an isomorphism from  $D(A)$  onto  $X$  and  $(I + \lambda A)^{-1}$  is a contraction, then  $-A$  generates a strongly continuous semigroup  $\{S(t): t \geq 0\}$  of contractions on  $X$ . If  $X$  is a complex Banach space and the above conditions hold for  $|\arg \lambda| < \theta$ , instead of just for  $\lambda > 0$ , then  $S(t)$  has an analytic extension in  $t$  to the sector  $|\arg t| < \theta$ . These holomorphic semigroups have a smoothing property, namely  $S(t)$  maps  $X$  into  $D(A)$  for  $t \neq 0$  so that  $u(t) = S(t)x$  is a solution to  $u'(t) + Au(t) = 0$ ,  $u(0) = x$  for any initial data  $x \in X$ . For the linear theory of semigroups see Yosida [24], Kato [12], and Hille-Phillips [11].

A number of authors (see Kōmura [15, 16], Kato [13, 14], Crandall and Pazy [6], Brezis [2], Crandall and Liggett [5], and the references listed there) have generalized the theory of semigroups to nonlinear operators. They have shown that if  $A \subset X \times X$  is a (multivalued) nonlinear operator such that, for sufficiently small  $\lambda > 0$ ,  $(I + \lambda A)^{-1}$  is a contraction and the range of  $(I + \lambda A)$  contains  $\text{cl}(D(A))$ , the closure of the domain of  $A$ , then  $-A$  generates a strongly continuous semigroup  $\{S(t): t \geq 0\}$  on  $\text{cl}(D(A))$ . In the case when  $X$  is a Hilbert space, Kōmura [16] has given conditions under which  $S(t)$  extends analytically to a sector  $|\arg t| < \theta$ . Brezis [2] has shown that if  $A = \partial\varphi$  is the subdifferential of a lower semicontinuous, convex functional on a Hilbert space then the semigroup  $\{S(t)\}$  generated by  $-A$  has a regularizing property similar to the linear case, namely  $S(t)$  maps  $\text{cl}(D(A))$  into  $D(A)$  for  $t > 0$ .

In this paper (§ 2) we give an extension of Kōmura's result to the case where  $X$  is a Banach space by establishing conditions under which  $S(t)$  extends analytically to  $|\arg t| < \theta$ . These conditions also imply  $S(t)$  maps  $\text{cl}(D(A))$  into  $D(A)$  for  $t \neq 0$ ; in other words,  $S(t)$

has a smoothing action.

In § 3 we establish local analyticity in  $t$  of solutions,  $u(t)$ , of equations of the form  $du/dt + Tu = Fu$  where  $-T$  is the generator of a linear analytic semigroup in a Banach space  $X$  and  $F$  maps  $D(T^\alpha)$  analytically into  $X$  for some  $\alpha < 1$ . We use the integral equation approach developed by Sobolevskii [23], and Fujita and Kato [9]. In § 4 we give applications to semilinear parabolic equations.

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**2. A class of holomorphic nonlinear semigroups.** In the following  $X$  is a complex Banach space. Let  $C \subset X$ , and  $\Sigma_\theta = \{z \in C: |\arg z| < \theta, z \neq 0\}$  be an open sector in the complex plane. A holomorphic semigroup on  $C$  is a function  $S$  on  $\Sigma_\theta \cup \{0\}$  such that  $S(z)$  maps  $C$  into  $C$  for each  $z \in \Sigma_\theta \cup \{0\}$ ;  $S(z+w) = S(z)S(w)$  for  $z, w \in \Sigma_\theta \cup \{0\}$ ; and, for  $x \in C$ ,  $S(z)x$  is a holomorphic function of  $z \in \Sigma_\theta$  with  $S(z)x \rightarrow S(0)x = x$  as  $z \rightarrow 0$  and  $z \in \Sigma_\theta$ . If there is also a real number  $\omega$  such

$$(2.1) \quad \|S(z)x - S(z)y\| \leq e^{\omega|z|} \|x - y\|,$$

$x, y \in C, z \in \Sigma_\theta$ , we will write  $S \in \mathcal{H}_{\omega, \theta}(C)$ . Note that we do not require  $S(z)$  to be holomorphic for fixed  $z$  as did Kōmura [16]. Kōmura noted that a contraction mapping which is holomorphic on all of a complex Banach space must be the translate of a linear operator (a consequence of Liouville's theorem). Hence we wish to avoid the hypothesis that  $S(z)$  be a holomorphic map.

The generator,  $A$ , of a nonlinear semigroup is, in general, a "multivalued" operator which is regarded as a subset of  $X \times X$ . For such operators we use the notation and definitions of Crandall and Liggett [5, page 266].

**THEOREM 2.1.** *Let  $A \subset X \times X$ ,  $\omega, \theta, \varepsilon$  be real numbers such that  $e^{i\varphi}A + \omega I$  is accretive for  $|\varphi| < \theta$  and  $R(I + \lambda A) \supset \text{cl}(D(A))$  for  $|\arg \lambda| < \theta$  and  $|\lambda| < \varepsilon$ . Let  $J_\lambda = (I + \lambda A)^{-1}$  and suppose, for  $x \in D(A)$  and  $n$  a positive integer, the map  $\lambda \mapsto J_\lambda^n x$  is a holomorphic function of  $\lambda$  for  $|\arg \lambda| < \theta, |\lambda| < \min(\varepsilon, |\omega|^{-1})$ . Then*

$$(2.2) \quad \lim_{n \rightarrow \infty} J_{z/n}^n x \equiv S(z)x$$

*exists for  $x \in \text{cl}(D(A))$  and  $z \in \Sigma_\theta$  and  $S \in \mathcal{H}_{\omega, \theta}(\text{cl}(D(A)))$ . If, in addition,  $A$  is a closed subset of  $X \times X$  then for each  $x \in \text{cl}(D(A))$  and  $z \in \Sigma_\theta$ , we have  $S(z)x \in D(A)$  and  $-(d/dz)S(z)x \in AS(z)x$ .*

*Proof.* Let  $K_{\alpha, \varphi} = (I + \alpha e^{i\varphi} A)^{-1}$  be the resolvent of  $e^{i\varphi} A$ . For  $|\varphi| < \theta$ , the operator  $e^{i\varphi} A$  satisfies the hypotheses of Theorem 1 of Crandall and Liggett [5], so  $\lim K_{t/n, \varphi}^n x \equiv T_\varphi(t)x$  exists for  $x \in \text{cl}(D(A))$ ,  $t \geq 0$ , and  $\{T_\varphi(t): t \geq 0\}$  is a (strongly continuous) semigroup with each  $T_\varphi(t)$  Lipschitz with constant  $e^{\omega t}$ . Since  $J_\lambda = K_{|\lambda|, \arg \lambda}$ , it follows that the limit (2.2) exists,  $S(z)x = T_{\arg z}(|z|)x$ , and  $S(z)$  satisfies (2.1) for  $x, y \in \text{cl}(D(A))$ .

Now let  $x \in D(A)$ . Applying the inequalities (ii) and (iii) on p. 268 of [5] to  $e^{i\varphi} A$ , we get  $\|K_{t/n, \varphi}^n x - x\| \leq t(1 - tn^{-1}|\omega|)^{-n} |e^{i\varphi} Ax|$ ,  $t \geq 0$ ,  $t|\omega| < n$ . Substituting  $t = |z|$ ,  $\varphi = \arg z$ , and using  $J_z = K_{|z|, \arg z}$ , and the fact that  $(1 - a/n)^{-n} \leq e^{|a|}$ ,  $a \in \mathbb{R}$ , we obtain  $\|J_{z/n}^n x - x\| \leq |z| e^{|z||\omega|} |Ax|$ ,  $|\arg z| < \theta$ ,  $|z\omega| < n$ . Thus when  $z$  is restricted to lie in a bounded subset of  $\Sigma_\theta$ , the sequence  $\{J_{z/n}^n x\}$  is a uniformly bounded sequence of holomorphic functions of  $z$  which converge pointwise to  $S(z)x$ . It follows (see [11], p. 104) that  $S(z)x$  is holomorphic in  $z$  and  $\|S(z)x - x\| \leq |z| e^{|z||\omega|} |Ax|$ . In particular,  $S(z)x \rightarrow x$  as  $z \rightarrow 0$ .

Now let  $x \in \text{cl}(D(A))$  and choose  $\{x_n\} \subset D(A)$  with  $x_n \rightarrow x$ . Then  $\{S(z)x_n\}$  is a sequence of functions holomorphic on  $\Sigma_\theta$  and continuous at  $z = 0$ . If  $z$  is restricted to lie in a bounded subset of  $\Sigma_\theta \cup \{0\}$  then the  $S(z)$  are Lipschitz with constant independent of  $z$  and, hence,  $\{S(z)x_n\}$  converges uniformly to  $S(z)x$ . Thus  $S(z)x$  is holomorphic on  $\Sigma_\theta$  and continuous at  $z = 0$ .

In order to show the semigroup property, let  $w \in \Sigma_\theta$  be fixed and  $\varphi = \arg w$ . If  $\{T_\varphi(t): t \geq 0\}$  is the semigroup generated by  $-e^{i\varphi} A$  then  $S(te^{i\varphi}) = T_\varphi(t)$ ,  $t \geq 0$ . By Crandall and Liggett,  $T_\varphi(t)$  is a semigroup for real  $t$ , so  $S(te^{i\varphi} + \tau e^{i\varphi}) = S(te^{i\varphi})S(\tau e^{i\varphi})$ . Thus  $S(z + w) = S(z)S(w)$  for  $z = tw$ ,  $t \geq 0$ . If  $x \in \text{cl}(D(A))$  then  $S(z + w)x$  and  $S(z)S(w)x$  are holomorphic functions of  $z \in \Sigma_\theta$  which agree on the ray  $z = tw$ ,  $t \geq 0$ . By the identity theorem for holomorphic functions  $S(z + w)x = S(z)S(w)x$  for all  $z$ .

In the real case (see [5]) a strong solution to the Cauchy problem

$$(2.3) \quad 0 \in du/dt + Au, \quad 0 \leq t \leq T, \quad u(0) = x,$$

is a function  $u: [0, T] \rightarrow X$  so that (i)  $u$  is continuous, (ii)  $u$  is the indefinite integral of a function which is strongly integrable on compact subsets of  $(0, T)$ , (iii)  $u(0) = x$  and (iv)  $u'(t) \in -Au(t)$  for a.e.  $t$  in  $(0, T)$ .

Crandall and Liggett, and Miyadera [20] have shown the following result. Let  $B$  be closed in  $X \times X$ ,  $B + \omega I$  accretive for some real number  $\omega$ ,  $R(I + tB) \supset \text{cl}(D(B))$  for sufficiently small  $t > 0$ , and for  $x \in \text{cl}(D(B))$  let  $T(t)x = \lim (I + (t/n)B)^{-n}x$  be the semigroup generated by  $-B$ . Then if  $x \in \text{cl}(D(B))$  and  $T(t)x$  is strongly differentiable at

$t_0 > 0$ , with  $y = (d/dt)T(t_0)x$ , then  $[T(t_0)x, -y] \in B$ . Then using the fact that for  $x \in D(B)$ ,  $S(t)x$  is Lipschitz continuous on bounded sets of  $t$ , they are able to conclude that if  $S(t)x$  is differentiable a.e. then  $u = S(t)x$  is a strong solution of (2.3).

In our case, since we have shown that  $S(z)x$  is a holomorphic function for  $x \in \text{cl}(D(A))$ , it is immediate that  $S(z)x$  can be recovered as the indefinite integral of an analytic function along a ray.

To finish the details of the proof, let  $A$  be closed,  $x \in \text{cl}(D(A))$ ,  $z \in \Sigma_\theta$  with  $\varphi = \arg z$ , and  $\{T_\varphi(t); t \geq 0\}$  be the semigroup generated by  $-e^{i\varphi}A$  so that  $S(te^{i\varphi}) = T_\varphi(t)$ ,  $t \geq 0$ . If  $x \in \text{cl}(D(A))$  then  $u(z) = S(z)x$  is holomorphic for  $z \in \Sigma_\theta$  which implies that  $v(t) = T_\varphi(t)x$  is differentiable for  $t > 0$  and  $v'(t) = e^{i\varphi}u'(te^{i\varphi})$ .

Since  $-e^{i\varphi}A$  is closed, it follows from the above results of Crandall and Liggett that  $-v'(t) \in e^{i\varphi}Av(t)$ . Hence  $-u'(te^{i\varphi}) \in Au(te^{i\varphi})$ , and together with the comment on holomorphy of  $S(t)x$  for  $x \in \text{cl}(D(A))$ , we have established a strong solution to the Cauchy problem for  $x \in \text{cl}(D(A))$ .

REMARK. We will show in an example that  $J_\lambda$  may not be defined on an open set, so that  $J_\lambda$  is certainly not a holomorphic map in general. However in case  $J_\lambda$  is a holomorphic map, then the hypothesis  $J_\lambda^n x$  is a holomorphic function of  $\lambda$  for all  $n$  is satisfied. We may argue as follows. First since  $J_\lambda$  is locally Lipschitz, both Kōmura [16] and Neuberger [21] have established that  $J_\lambda x$  is holomorphic in  $\lambda$  when  $J_\lambda$  is a holomorphic map. Next let  $g(\lambda_1, \lambda_2, \dots, \lambda_n) = J_{\lambda_1} \cdot J_{\lambda_2} \cdot J_{\lambda_3} \cdots J_{\lambda_n} x$ . Then for fixed  $\lambda_2, \lambda_3, \dots, \lambda_n$ ,  $g$  is holomorphic in  $\lambda_1$ . If  $\lambda_1, \lambda_3, \dots, \lambda_n$  are fixed, then  $J_{\lambda_2} \cdot J_{\lambda_3} \cdots J_{\lambda_n}$  is holomorphic in  $\lambda_2$  and therefore when composed with the holomorphic map  $J_{\lambda_1}$ ,  $g$  is holomorphic in  $\lambda_2$  and so forth. Hence, as is well known [11], p. 107,  $g(\lambda, \lambda, \lambda, \dots)$  is a holomorphic function of  $\lambda$ .

EXAMPLE. Let  $\beta: K \rightarrow C$  be continuous where  $K$  is the closure of an open, convex set  $U \subset C$ . Suppose  $0 \in K$ ,  $\beta(0) = 0$  and  $\beta$  is analytic on  $U$ . Assume there is  $\theta > 0$  such that  $|\arg \beta'(z)| \leq \pi/2 - \theta$ ,  $z \in U$ . Finally suppose there is  $\varepsilon < 0$  such that for  $|\arg \lambda| < \theta$ ,  $|\lambda| < \varepsilon$ , one has  $(I + \lambda\beta)(K) \supset K$  and  $(I + \lambda\beta)(U) \supset U$ . Here  $I(z) = z$  is the identity map on  $C$ .

Let  $X = L^p(\Omega; C)$  where  $\Omega$  is any measure space and  $1 \leq p \leq \infty$ . Let  $D(A) = \{u \in X: u(x) \in K \text{ a.e. and } \beta(u) \in X\}$ , where  $\beta(u)$  is the composition of  $\beta$  and  $u$ . Let  $Au = \beta(u)$  for  $u \in D(A)$ . We shall show that  $A$  satisfies the hypotheses of Theorem 2.1 with  $\omega = 0$  and  $\theta, \varepsilon$  as above.

The hypothesis  $|\arg \beta'(z)| \leq \pi/2 - \theta$ ,  $z \in U$ , implies  $e^{t\varphi}\beta$  is accretive for  $|\varphi| < \theta$ . In particular  $I + \lambda\beta$  is one-to-one and  $(I + \lambda\beta)^{-1}$  is a contraction for  $|\arg \lambda| < \theta$ . Let  $S = \{\lambda \in \mathbb{C}; |\arg \lambda| < \theta, |\lambda| < \varepsilon\}$ . The assumption that  $(I + \lambda\beta)(K) \supset K$ ,  $\lambda \in S$ , implies the function  $j(w, \lambda) = (I + \lambda\beta)^{-1}(w)$  is well defined for  $w \in K$ ,  $\lambda \in S$ . It is a contraction in  $w$  for fixed  $\lambda$ . Since  $\beta$  is analytic on  $U$  and  $(I + \lambda\beta)(U) \supset U$ , the implicit function theorem implies  $j: U \times S \rightarrow U$  is analytic. Since  $\beta(0) = 0$  we have  $j(0, \lambda) = 0$ . Since  $j(\cdot, \lambda)$  is a contraction we have  $|j(w, \lambda)| \leq |w|$ .

Let  $j^1(w, \lambda) = j(w, \lambda)$ ,  $w \in K$ ,  $\lambda \in S$  and  $j^n(w, \lambda) = j(j^{n-1}(w, \lambda), \lambda)$ ,  $w \in K$ ,  $\lambda \in S$ ,  $n \geq 2$ . Since  $j(w, \lambda)$  is a contraction in  $w$ , it follows that  $j^n(w, \lambda)$  is a contraction in  $w$  for fixed  $\lambda$ . Since  $j: UXS \rightarrow U$  is analytic, it follows that  $j^n: UXS \rightarrow U$  is analytic. We claim that  $j^n(w, \lambda)$  is analytic in  $\lambda$  for fixed  $w$ , even if  $w \in K$ . To see this, choose a sequence  $\{w_m\} \subset U$  with  $w_m \rightarrow w$ . Then  $\{j^n(w_m, \lambda)\}$  is a sequence of functions each analytic in  $\lambda$  and  $j^n(w_m, \lambda) \rightarrow j^n(w, \lambda)$  uniformly in  $\lambda$  since  $j^n(w, \lambda)$  is a contraction in  $w$ . It follows that  $j^n(w, \lambda)$  is analytic in  $\lambda$ . Finally we note that  $|j^n(w, \lambda)| \leq |w|$  since  $|j(w, \lambda)| \leq |w|$ .

Now consider the operator  $A$ . We have  $v = (I + \lambda A)u$  if and only if  $v(x) = (I + \lambda\beta)(u(x))$  a.e. If  $|\arg \lambda| < \theta$  then  $I + \lambda\beta$  is  $1 - 1$  so  $v = (I + \lambda A)u$  is equivalent to  $u(x) = (I + \lambda\beta)^{-1}(v(x))$  a.e. In particular  $I + \lambda A$  is  $1 - 1$  and  $J_\lambda \equiv (I + \lambda A)^{-1}$  is contraction. It follows that  $e^{t\varphi}A$  is accretive for  $|\varphi| < \theta$ .

To show  $\text{cl}(D(A)) \subset R(I + \lambda A)$ , note that  $\text{cl}(D(A)) \subset F$  where  $F = \{v \in X; v(x) \in K \text{ a.e.}\}$ . The assumption  $K \subset (I + \lambda\beta)(K)$ ,  $\lambda \in S$  and the definition of  $j$  implies that  $F \subset R(I + \lambda A)$  for  $\lambda \in S$ , and  $J_\lambda v(x) = j(v(x), \lambda)$ ,  $v \in F$ ,  $\lambda \in S$ .

To show  $J_\lambda^n v$  is analytic in  $\lambda$  for fixed  $v \in F$ , note that  $J_\lambda^n v(x) = j^n(v(x), \lambda)$ . It follows from  $|j^n(v(x), \lambda)| \leq |v(x)|$  and the Cauchy integral formula that

$$(2.4) \quad |j_\lambda^n(v(x), \lambda)| \leq |v(x)| \text{dist}(\lambda, \partial S)$$

$$(2.5) \quad |j_{\lambda\lambda}^n(v(x), \lambda)| \leq |v(x)| [\text{dist}(\lambda, \partial S)]^2$$

where  $j_\lambda^n = \partial j^n / \partial \lambda$ ,  $j_{\lambda\lambda}^n = \partial^2 j^n / \partial \lambda^2$ . In the case  $1 \leq p < \infty$  in order to show  $J_\lambda^n v$  is analytic in  $\lambda$  it suffices to show weak analyticity, i.e.  $(d/d\lambda) \int_\Omega j^n(v(x), \lambda) \overline{w(x)} dx = \int_\Omega j_\lambda^n(v(x), \lambda) \overline{w(x)} dx$  for all  $w \in L^q(\Omega)$ ,  $p^{-1} + q^{-1} = 1$ . This is true because  $j^n(v(x), \lambda)$  is analytic in  $\lambda$  for fixed  $x$ , and the estimate (2.4) implies that differentiation under the integral sign is valid. In the case  $p = \infty$  we must show  $r(\mu) \rightarrow 0$  as  $\mu \rightarrow 0$  where  $r(\mu) = \|[j^n(v, \lambda + \mu) - j^n(v, \lambda)]\mu^{-1} - j_\lambda^n(v, \lambda)\|_\infty$ . Note that (2.4) implies that  $j_\lambda^n(v, \lambda)$  is in  $L^\infty(\Omega)$  for each  $\lambda$ . Since  $j^n(v, \lambda + \mu) -$

$j^n(v, \lambda) = \int_{\lambda}^{\lambda+\mu} j_{\lambda}^n(v, \eta) d\eta$  a computation shows that

$$r(\mu) \leq \sup \{ |j_{\lambda}^n(v(x), \lambda + t\mu) - j_{\lambda}^n(v(x), \lambda)| : x \in \Omega, 0 \leq t \leq 1 \} \\ \leq |\mu| \sup \{ |j_{\lambda}^n(v(x), \lambda + t\mu)| : x \in \Omega, 0 \leq t \leq 1 \}.$$

Using (2.5) we get  $r(\mu) \leq 4|\mu| \|v\|_{\infty} [\text{dist}(\lambda, \partial S)]^2$  for  $|\mu| < 2^{-1} \text{dist}(\lambda, \partial S)$ . Thus  $r(\mu) \rightarrow 0$ .

A special case of this example is  $\beta(z) = z^2, z \in K \equiv \{z \in \mathbb{C} : |\arg z| \leq \pi/4\} \cup \{0\}$ . We have  $|\arg \beta'(z)| \leq \pi/4, x \in K$ , so we can take  $\theta = \pi/4$ . Note that  $(I + \lambda\beta)^{-1}w = [-1 + (1 + 4\lambda w)^{1/2}]/2\lambda$  for  $w \in R(I + \lambda\beta)$ ,  $|\arg \lambda| < \pi/4$ . A simple geometric argument shows that if  $|\arg \lambda| < \pi/4$  and  $|\arg w| \leq \pi/4$  (resp.  $|\arg w| < \pi/4$ ) then  $|\arg (I + \lambda\beta)^{-1}w| \leq \pi/4$  (resp.  $< \pi/4$ ). Thus  $K \subset (I + \lambda\beta)(K)$  and  $U \subset (I + \lambda\beta)(U)$ ,  $|\arg \lambda| < \pi/4$ , where  $U$  is the interior of  $K$ .

To obtain an "unbounded generator" version of the above example, let  $X = l^2$ ,  $D(A) = \{x \in l^2 : Ax \in l^2, |\arg x_i| \leq \pi/4\}$ . Let  $\Sigma_{\theta} = \{\lambda \in \mathbb{C} : |\arg \lambda| < \pi/4\}$  and let  $A(x_1, x_2, x_3, \dots) = (x_1^2, 2x_2^2, 3x_3^2, \dots)$ . The hypotheses of Theorem 1 are easy to verify in this case.

Our results include some, but not all, of the linear theory of holomorphic semigroups. If  $A$  is an  $m$ -sectorial operator in a Hilbert space with vertex zero (so that its numerical range is a subset of a sector  $|\arg \varphi| \leq \pi/2 - \theta, \theta < \pi/2$ ), then  $A$  satisfies the hypotheses of Theorem 2.1.

**3. A perturbation theorem.** In this section we consider the equation  $du/dt + Tu(t) = Fu(t), t \geq 0, u(0) = x$ , where  $T$  is a linear operator in a complex Banach space  $X$  and  $F$  is a function with domain and range in  $X$ . Equations of this type have been studied by Sobolevskii [23], Fujita and Kato [9], Friedman [8], Henry [10] and others. We establish analyticity in  $t$  of solutions  $u(t)$  of this equation under suitable conditions on  $T$  and  $F$ . In particular, we assume that

The resolvent of  $T$  exists for  $\text{Re } \lambda \leq 0$  and there exists  
(3.1) a constant  $C$  such that  $\|(\lambda - T)^{-1}\| \leq C(1 + |\lambda|)^{-1}$ ,  
 $\text{Re } \lambda \leq 0$ .

Using the Neumann series representation for the resolvent [12, pp. 37, 173] it is not hard to show that there exists  $C_1, \omega > 0$  such that the resolvent of  $T$  exists and satisfies  $\|(\lambda - T)^{-1}\| \leq C_1 |\lambda|^{-1}$  for  $|\arg \lambda - \pi| < (\pi/2) + \omega$ . This is a well known ([12, p. 488], [8, p. 101]) condition for  $-T$  to generate a holomorphic semigroup  $\{U(t) : |\arg t| < \omega\}$ . The map  $t \rightarrow U(t)$  is a bounded holomorphic map from  $\{t : |\arg t| < \theta, t \neq 0\}$  into  $B(X)$  for any  $\theta < \omega$ .

The assumption (3.1) implies that  $T$  has fractional powers,  $T^\gamma$ , for  $\gamma \in \mathbf{R}$  (see [24, 8, 18]). For  $\gamma \leq 0$ ,  $T^\gamma \in B(X)$ . For  $\gamma \geq 0$ ,  $T^\gamma$  is a closed operator in  $X$  with domain,  $X_\gamma \equiv D(T^\gamma)$ , dense in  $X$ . For all  $\gamma$ ,  $T^\gamma$  is invertible with  $(T^\gamma)^{-1} = T^{-\gamma}$ ; see [8, pp. 158-159]. For  $\gamma > 0$ , we define  $\|x\|_\gamma = \|T^\gamma x\|$ ,  $x \in X_\gamma$  (cf. [10, p. 29]). The fact that  $(T^\gamma)^{-1} \in B(X)$  implies  $\|\cdot\|_\gamma$  is a norm on  $X_\gamma$  which is equivalent to the graph norm,  $\|x\|_\gamma$ , of  $T^\gamma$ , since  $\|x\|_\gamma \equiv \|T^\gamma x\| + \|x\| \leq (1 + \|T^{-\gamma}\|) \|T^\gamma x\|$ .  $X_\gamma$  is a Banach space with the norm  $\|\cdot\|_\gamma$  since  $T^\gamma$  is a closed operator. In § 4 we shall need the following imbedding theorem for domains of fractional powers.

If  $Y$  is a Banach space with  $D(T) \subset Y \subset X$  and  $0 \leq \beta < 1$  and there exists  $C$  such that  $\|u\|_Y \leq C \|Tu\|_X^\beta \|u\|_X^{1-\beta}$ ,  $u \in D(T)$ , then  $D(T^\alpha)$  is continuously imbedded in  $Y$  for  $\beta < \alpha \leq 1$ .

(See Sobolevskii [23, p. 22], Friedman [8, p. 177], and Henry [10, p. 29].)

We shall also need the following facts which relate the semigroup to the fractional powers. For all  $\gamma \geq 0$ ,  $U(t)$  maps  $X$  into  $D(T^\gamma)$  and, for  $\theta < \omega$  there exists a constant  $M_\gamma$  such that

$$(3.3) \quad \|T^\gamma U(t)\| \leq M_\gamma |t|^{-\gamma}, \quad |\arg t| < \theta.$$

(See [8, pp. 105-106, 158-160] where this is proved for real  $t$ . The same argument works for complex  $t$ .)

For  $0 < \gamma \leq 1$ ,  $\theta < \omega$  one has

$$(3.4) \quad \|U(t)x - x\| \leq M_{1-\gamma} \gamma^{-1} |t|^\gamma \|T^\gamma x\|, \quad |\arg t| < \theta, x \in X_\gamma.$$

(To prove this, note that  $(d/ds)U(s)x = -TU(s)x = -T^{1-\gamma}U(s)T^\gamma x$ . Thus  $U(t)x - x = -\int_0^t T^{1-\gamma}U(s)T^\gamma x ds$ . Using (3.3) to estimate  $\|T^{1-\gamma}U(s)\|$ , one obtains (3.4). This proof is due to Henry [10].)

Let  $1 < p \leq \infty$ ,  $0 \leq \gamma < 1 - p^{-1}$ ,  $0 < \varepsilon < \tau$ . Then there exists a constant  $M$  such that if  $u: [0, \tau] \rightarrow X$  is differentiable,  $u(t) \in D(T)$ ,  $0 \leq t \leq \tau$ , and  $u'(t) + Tu(t) = f(t)$ ,  $0 \leq t \leq \tau$ , with  $f \in L^p(0, \tau; X)$  then

$$(3.5) \quad \|T^\gamma u(t)\| \leq M \left[ \|u(0)\| + \left( \int_0^\tau \|f(s)\|^p ds \right)^{1/p} \right], \quad \varepsilon \leq t \leq \tau.$$

To prove (3.5), first note that

$$u(t) = U(t)u(0) + \int_0^t U(t-s)f(s)ds$$

(see [12, p. 486]). By (3.3) we have  $\|T^\gamma U(t)u(0)\| \leq M_\gamma \varepsilon^{-\gamma} \|u(0)\|$ ,

$\varepsilon \leq t \leq \tau$ , and  $\int_0^t \|T^\gamma U(t-s)f(s)\| ds \leq M_\gamma \int_0^t \|f(s)\| (t-s)^{-\gamma} ds \leq M_\gamma \left(\int_0^t \|f(s)\|^p ds\right)^{1/p} \left(\int_0^t (t-s)^{-\gamma q}\right)^{1/q} \leq \text{const.} \left(\int_0^\tau \|f(s)\|^p ds\right)^{1/p}$ ,  $0 \leq t \leq \tau$ ,  $p^{-1} + q^{-1} = 1$ . Note that  $\gamma q < 1$  since  $\gamma < q^{-1} = 1 - p^{-1}$ . This proves (3.5).

**THEOREM 3.1.** Assume  $T$  satisfies (3.1),  $0 \leq \alpha < 1$ ,  $\theta < \omega$ , and  $F$  is a function whose domain,  $D(F)$ , is an open subset of  $X_\alpha$  and  $F: D(F) \rightarrow X$  is Frechet analytic (as a map from  $X_\alpha$  to  $X$ ). Then for each  $x \in D(F)$  there exists  $r > 0$  and a unique function  $u$  mapping  $W_r = \{t \in \mathbb{C}: |\arg t| < \theta, 0 < |t| < r\}$  analytically into  $X_1 = D(T)$  such that for each  $t \in W_r$ ,  $u(t) \in D(F)$  and  $u'(t) + Tu(t) = Fu(t)$ , and  $\|u(t) - x\|_\alpha \rightarrow 0$  as  $t \rightarrow 0$ .

Let  $U \subset D(F) \cap X_\gamma$  for some  $\gamma > \alpha$  and suppose there exists  $\delta > 0$  and  $K$  such that if  $x \in U$  and  $\|y - x\|_\alpha < \delta$  then  $y \in D(F)$  and  $\|Fy\| < K$ . Suppose also that  $U$  is bounded in  $X_\gamma$ . Then the value of  $r$  can be chosen independently of  $x \in U$ .

If, in addition,  $F$  maps  $D(F) \cap X_{s+\alpha}$  analytically into  $X_s$  for  $0 \leq s \leq n$ , then  $u$  is analytic from  $W_r$  to  $X_{n+1}$ .

*Proof.* The differential equation  $du/dt + Au = Fu$  is transformed into the integral equation (3.7) below. This method was introduced by Sobolevskii [23] and Fujita and Kato [9] and is now standard. We use methods similar to Henry [10], and therefore we are as brief as possible.

Choose  $\delta > 0$  and  $K$  so that  $\|y - x\|_\alpha < \delta$  implies  $y \in D(F)$  and  $\|Fy\| \leq K$ . Using the Cauchy integral formula, one has

$$(3.6) \quad \|Fy_1 - Fy_2\| \leq 4K\delta^{-1} \|y_1 - y_2\|_\alpha,$$

if  $\|y_i - x\|_\alpha \leq \delta/2$ ,  $i = 1, 2$ . Let  $S_r$  be the set of all analytic functions  $u: W_r \rightarrow X_\alpha$  such that  $\|u(t) - x\|_\alpha \leq \delta/2$ ,  $t \in W_r$  and  $\|u(t) - x\|_\alpha \rightarrow 0$  as  $t \rightarrow 0$ .  $S_r$  is a complete metric space if we define  $d(u, v) = \sup \times \{\|u(t) - v(t)\|_\alpha: t \in W_r\}$ ,  $u, v \in S_r$ .

For  $u \in S_r$  put

$$(3.7) \quad Gu(t) = U(t)x + \int_0^t U(t-s)Fu(s)ds, \quad t \in W_r,$$

where the integral is taken over the line segment  $\{s = \lambda t, 0 \leq \lambda \leq 1\}$  joining 0 to  $t$ . We shall show  $G$  is a strict contraction from  $S_r$  into  $S_r$  if  $r$  is chosen small enough.

First consider the integral on the right of (3.7); we denote its value by  $v(t)$ . Putting  $s = \lambda t$ ,  $0 \leq \lambda \leq 1$ , we get  $v(t) = t \int_0^1 g(t, \lambda) d\lambda$  where  $g(t, \lambda) = U(t - t\lambda)f(t\lambda)$ , where  $f(t) = Fu(t)$ . Using (3.3) one

sees that there is a constant  $C$  such that  $\|g(t, \lambda)\|_\alpha \leq C|t|^{-\alpha}(1-\lambda)^{-\alpha}$ ,  $t \in W_r$ ,  $0 < \lambda < 1$ . Thus the integral in (3.7) is absolutely convergent in  $X_\alpha$  and  $\|v(t)\|_\alpha \leq C_1|t|^{1-\alpha}$ ,  $t \in W_r$ . In particular,  $\|v(t)\|_\alpha \rightarrow 0$  as  $t \rightarrow 0$ , and we can make  $\|v(t)\|_\alpha \leq \delta/4$ ,  $t \in W_r$ , by choosing  $r$  sufficiently small.

Since  $\|U(t)x - x\|_\alpha = \|U(t)T^\alpha x - T^\alpha x\|$  approaches 0 as  $t \rightarrow 0$ , we can make  $\|U(t)x - x\|_\alpha < \delta/4$  by making  $r$  small. If  $x \in X_r$  for some  $\gamma > \alpha$ , then the size of  $r$  necessary to make  $\|U(t)x - x\|_\alpha < \delta/4$  is determined by  $\|x\|_\gamma$ . This is because (3.4) implies

$$\|U(t)x - x\|_\alpha \leq \text{const.} |t|^{r-\alpha} \|T^{r-\alpha} T^\alpha x\| \leq \text{const.} |t|^{r-\alpha} \|x\|_\gamma.$$

Combining these results, one has  $\|Gu(t) - x\|_\alpha \rightarrow 0$  as  $t \rightarrow 0$ , and  $\|Gu(t) - x\|_\alpha \leq \delta/2$ ,  $t \in W_r$  for  $r$  small.

Since  $U(t)x$  is analytic in  $t$ , it remains to show the integral  $v(t)$  is analytic in  $t$  with values in  $X_\alpha$ . For fixed  $\lambda \in (0, 1)$ ,  $g(t, \lambda)$  is an analytic function of  $t$  with values in  $X_\alpha$  and

$$g_t(t, \lambda) = -(1-\lambda)TU(t-t\lambda)f(t\lambda) + U(t-t\lambda)f'(t\lambda)\lambda,$$

where  $g_t = \partial g / \partial t$ . The function  $f$  is bounded by  $K$ , so by the Cauchy integral formula  $\|f'(t)\| \leq K|t|^{-1} \csc(\theta - |\arg t|)$ . Using this and (3.3), one sees that  $\|g_t(t, \lambda)\|_\alpha$  is bounded by  $\text{const.}(1-\lambda)^{-\alpha}$  for  $t$  in a compact subset of  $W_r$ . Thus the difference quotients  $\|[g(t, \lambda) - g(s, \lambda)]/(t-s)\|_\alpha$  are similarly bounded. Using the dominated convergence theorem, it follows that  $v: W_r \rightarrow X_\alpha$  is analytic. Therefore  $Gu: W_r \rightarrow X_\alpha$  is analytic.

We have shown  $G$  maps  $S_r$  into  $S_r$  for  $r$  small. To show  $G$  is a contraction, we use (3.3) and (3.6) to get

$$\begin{aligned} \|Gu(t) - Gv(t)\|_\alpha &\leq M_\alpha \int_0^t |t-s|^{-\alpha} \|Fu(s) - Fv(s)\| ds \\ &\leq \text{const.} |t|^{1-\alpha} \sup \|u(s) - v(s)\|_\alpha, \end{aligned}$$

$t \in W_r$ ,  $u, v \in S_r$ . By making  $r$  sufficiently small we can make  $G$  a strict contraction. By the fixed point theorem for strict contractions on a complete metric space, there is a unique  $u \in S_r$  such that  $Gu = u$ . In order to show  $u$  satisfies the differential equation  $u'(t) + Tu(t) = Fu(t)$  we will use a known result (see Kato [12], Theorem 1.27, p. 491) on solutions to inhomogeneous equations for holomorphic semigroups. In order to apply this theorem it is necessary to make two changes of variable. Fix  $t \in W_r$  and define  $v(\lambda) = u(\lambda t) = U(\lambda t)x + \int_0^{\lambda t} U(\lambda t-s) \times Fu(s)ds$ . Putting  $s = \sigma t$ ,  $0 \leq \sigma \leq \lambda$ , we get  $v(\lambda) = V(\lambda)x + \int_0^\lambda V(\lambda-\sigma)f(\sigma)d\sigma$  where  $V(\lambda) = U(\lambda t)$  is the (holomorphic) semigroup generated by  $-tT$ , and  $f(\sigma) = tFu(\sigma t)$  is continuous on  $[0, r/|t|)$  and analytic on  $(0, r/|t|)$  with values in  $X$ . Fixing  $\tau < 1$ , it is not hard to show  $v(\lambda + \tau) =$

$V(\lambda)v(\tau) + \int_0^\lambda V(\lambda - \rho)f(\rho + \tau)d\rho$ ,  $0 \leq \lambda < r/|t| - \tau$ . The function  $\rho \mapsto f(\rho + \tau)$  is Hölder continuous on  $[0, r/|t| - \tau)$ . By the above mentioned theorem in [12], it follows that  $v(s) \in D(T)$ ,  $\tau < s < r/|t|$ , with  $v'(s) + tTv(s) = f(s)$ . Putting  $s = 1$  shows  $u(t) \in D(T)$  and  $u'(t) + Tu(t) = Fu(t)$ . So far we know  $u: W_r \rightarrow X_\alpha$  is analytic. If we rewrite the equation as  $u = T^{-1}(Fu - u')$  it follows that  $u: W_r \rightarrow X_1$  is analytic. The solution of  $u' + Tu = Fu$ ,  $u(0) = x$  is unique because any  $u$  satisfying the conclusions of the theorem must also satisfy  $Gu = u$ .

Suppose  $F$  is analytic from  $U \cap X_{s+\alpha}$  to  $X_s$ ,  $0 \leq s \leq n$ . If  $u$  is analytic from  $W_r$  to  $X_{s+\alpha}$  for such an  $s$ , then the equation  $u = T^{-1}(Fu - u')$  shows  $u: W_r \rightarrow X_{s+1}$  is analytic. Repeating this argument shows that  $u: W_r \rightarrow X_{n+1}$  is analytic.

**4. Semilinear parabolic equations.** In this section the results of § 3 are applied to the mixed problem  $\partial u / \partial t + Lu + \beta(u) = 0$ ,  $(x, t) \in \Omega \times [0, \infty)$ ;  $u(x, 0) = \varphi(x)$ ,  $x \in \Omega$ ;  $u(x, t) = 0$ ,  $(x, t) \in \partial\Omega \times [0, \infty)$ ; where  $L$  is a second order elliptic operator of the form  $Lu = -\sum_{i,j} \partial_j [a_{ij} \partial_i u] + \sum_i \partial_i [a_i u] + au$ . Here  $\partial_j = \partial / \partial x_j$  and sums are from 1 to  $n$ .  $\Omega$  is the closure of a bounded, open subset of  $\mathbf{R}^n$ , and  $\Omega$  has smooth boundary  $\partial\Omega$ . The  $a_{ij}$ ,  $a_i$ ,  $a$  are real valued functions on  $\Omega$  with  $a_{ij} = a_{ji}$ ;  $a_{ij}$ ,  $a_i \in C^1(\Omega)$ ,  $a \in C(\Omega)$  and there exists  $\mu > 0$  such that  $\sum_{i,j} a_{ij} \xi_i \xi_j \geq \mu |\xi|^2$ ,  $\xi \in \mathbf{R}^n$ ,  $x \in \Omega$ .  $\beta$  is an analytic function whose domain,  $D(\beta)$ , is an open subset of the complex plane containing the real axis;  $\beta$  maps the real line into itself; for  $t$  real,  $\beta(t)$  is an increasing function of  $t$ , and  $\beta(0) = 0$ .

Equations of this type have been studied by Brezis, Crandall and Pazy [3], Brezis and Strauss [4], Da Prato [7], Konishi [17], Ōuchi [22], and Brezis [2]. Our main result is that the solution of the mixed problem above is an analytic function of  $t > 0$ ; see Theorem 4.4 below. This result is similar to those of Ōuchi, but he only considers the case where  $\beta$  is a polynomial.

$W^{k,p}(\Omega; \mathbf{R})$  (resp.  $W^{k,p}(\Omega; \mathbf{C})$ ) is the Sobolev space of real-valued (resp. complex-valued) functions whose derivatives up to order  $k$  lie in  $L^p(\Omega; \mathbf{R})$  (resp.  $L^p(\Omega; \mathbf{C})$ ). We write  $W^{k,p}(\Omega)$  if it is clear from the context whether  $\mathbf{R}$  or  $\mathbf{C}$  is intended. The norm in  $W^{k,p}(\Omega)$  (resp.  $L^p(\Omega)$ ) is denoted by  $\| \cdot \|_{k,p}$  (resp.  $\| \cdot \|_p$ ).  $W_0^{k,p}(\Omega)$  is the closure of  $C_0^\infty(\Omega^0)$  in the space  $W^{k,p}(\Omega)$ . Here  $\Omega^0$  is the interior of  $\Omega$ . If  $u$  is a function, then  $\beta(u) = \beta \circ u$  is the composition of  $\beta$  and  $u$ .

For  $1 < p < \infty$ , let  $D(T_p) = W^{2,p}(\Omega; \mathbf{C}) \cap W_0^{1,p}(\Omega)$  and, for  $p = 1$ , let  $D(T_1) = \{u \in W^{1,1}(\Omega; \mathbf{C}); Lu \in L^1(\Omega)\}$ , where  $Lu$  is understood in the sense of distributions. Let  $T_p u = Lu$  for  $u \in D(T_p)$ ,  $1 \leq p < \infty$ . For  $1 \leq p < \infty$ , let  $D(A_p) = \{u \in L^p(\Omega; \mathbf{R}); u \in D(T_p), \beta(u) \in L^p(\Omega)\}$ , and  $A_p u = T_p u + \beta(u)$ ,  $u \in D(A_p)$ .

**PROPOSITION 4.1.** *If  $1 < p < \infty$  and  $k \in \mathbf{R}$  is sufficiently large, then  $X = L^p(\Omega, C)$  and  $T = T_p + kI$  satisfy (3.1) and there exists a constant  $C_p$  such that  $\|u\|_{2,p} \leq C_p \|Tu\|_p, u \in D(T)$ . If  $0 < \alpha \leq 1$  and  $p^{-1} - 2\alpha n^{-1} < q^{-1}$  then  $X_\alpha \equiv D(T^\alpha)$  is continuously imbedded in  $L^q(\Omega)$  (or  $C(\Omega)$  if  $q = \infty$ ;  $q = \infty$  corresponds to  $n/2p < \alpha \leq 1$ ).*

*Let  $D(F) = \{u \in X_\alpha: u(x) \in D(\beta), x \in \Omega\}$  and  $Fu = ku - \beta(u), u \in D(F)$ . If  $n/2p < \alpha < 1$  then  $D(F) \subset C(\Omega)$  and  $\beta(u) \in C(\Omega)$  for each  $u \in D(F)$ . Furthermore  $X, T, \alpha$  and  $F$  satisfy the hypotheses of Theorem 3.1. Let  $R > 0$  and  $\Delta$  be a compact subset of  $D(\beta)$  and  $U = \{u \in W^{2,p}(\Omega; C): \|u\|_{2,p} < R; u(x) \in \Delta, x \in \Omega\}$ . Then  $U$  also satisfies the hypotheses of Theorem 3.1.*

*Proof.* The assertions in the first sentence are well known, see Sobolevskii [23, p. 54] and Friedman [8, p. 101]. If  $p^{-1} - 2\alpha n^{-1} < q^{-1}$  then it follows from Friedman [8, Theorems 10.1, 11.1] that  $W^{2,p}(\Omega) \subset L^q(\Omega)$  (or  $W^{2,p}(\Omega) \subset C(\Omega)$  if  $q = \infty$ ) and there is  $\mu < \alpha$  and  $C$  such that  $\|u\|_q \leq C \|u\|_{2,p}^\mu \|u\|_p^{1-\mu}, u \in W^{2,p}(\Omega)$ . Thus  $\|u\|_q \leq C \|Tu\|_p^\mu \|u\|_p^{1-\mu}, u \in D(T)$ . Thus  $X_\alpha \subset L^q(\Omega)$  follows from (3.2).

Now let  $n/2p < \alpha < 1$ . The fact that  $D(F) \subset C(\Omega)$  follows from the first part of the proposition, and  $\beta(u) \in C(\Omega), u \in D(F)$  follows from the fact that  $\beta$  is continuous. To show that  $D(F)$  is open in  $X_\alpha$ , let  $u \in D(F)$ . Then  $u(\Omega) \equiv \{u(x): x \in \Omega\}$  is compact and contained in  $D(\beta)$  which is open. Thus, the distance,  $\delta$ , from  $u(\Omega)$  to  $C \setminus D(\beta)$  is greater than 0. It follows that  $v(x) \in D(\beta)$  if  $\|v - u\|_\infty < \delta$ . Since  $X_\alpha \subset C(\Omega)$  one has  $\|v - u\|_\infty < \delta$  if the  $X_\alpha$  norm of  $v - u$  is sufficiently small. Thus  $D(F)$  is open in  $X_\alpha$ .

To show  $F: D(F) \rightarrow X$  is analytic, it suffices to show  $\|F(u+h) - F(u) - (kh - \beta'(u)h)\|_p \leq \varepsilon(h) \|T^\alpha h\|$  where  $\varepsilon(h) \rightarrow 0$  as  $\|T^\alpha h\| \rightarrow 0$ . In view of the imbeddings  $X_\alpha \subset C(\Omega) \subset X$ , it suffices to show  $\|\beta(u+h) - \beta(u) - \beta'(u)h\|_\infty \leq \varepsilon(h) \|h\|_\infty$  where  $\varepsilon(h) \rightarrow 0$  as  $\|h\|_\infty \rightarrow 0$ . By writing  $\beta(\eta + \xi) - \beta(\eta)$  as the integral of  $\beta'$ , one can show  $|\beta(\eta + \xi) - \beta(\eta) - \beta'(\eta)\xi| \leq \varepsilon(|\xi|) |\xi|, \eta \in u(\Omega)$ , where  $\varepsilon(|\xi|) \rightarrow 0$  as  $|\xi| \rightarrow 0$  and  $\varepsilon(|\xi|)$  is independent of  $\eta \in u(\Omega)$ . Replacing  $\eta$  by  $u(x)$  and  $\xi$  by  $h(x)$  and taking the supremum over  $\Omega$ , one obtains the desired result.

Note that  $U$  is a bounded subset of  $D(T) = X_1$ . Since  $\Delta \subset D(\beta)$  is compact, there exists  $\rho > 0$  such that  $\Delta_1 = \{z + \zeta: z \in \Delta, |\zeta| \leq \rho\} \subset D(\beta)$ . Using an argument similar to the proof that  $D(F)$  is open in  $X_\alpha$ , one can find a  $\delta > 0$  such that if  $u \in U$  and the  $X_\alpha$  norm of  $v - u$  is less than  $\delta$  then  $v(x) \in \Delta_1, x \in \Omega$ , and hence,  $v \in D(F)$ . One has  $\|Fv\| \leq K$  since  $\beta$  is bounded on  $\Delta_1$ .

**PROPOSITION 4.2.** *If  $k \in \mathbf{R}$  is sufficiently large, then  $(I + \lambda(A_p + k))^{-1}$  exists and is a contraction in the norm of  $L^p(\Omega)$  and the range of  $I + \lambda(A_p + k)$  is  $L^p(\Omega; \mathbf{R})$  for  $1 \leq p < \infty, \lambda > 0$ . Furthermore*

$\|\beta(u)\|_p \leq \|(A_p + k)u\|_p$ ,  $\|(T_p + k)u\| \leq 2\|(A_p + k)u\|_p$ ,  $u \in D(A_p)$ . If  $\gamma: \mathbf{R} \rightarrow \mathbf{R}$  is increasing and continuous with  $\gamma(0) = 0$ ,  $p^{-1} + q^{-1} = 1$ ,  $u \in D(T_p) \cap L^p(\Omega; \mathbf{R})$  and  $\gamma(u) \in L^q(\Omega)$  then  $\int_{\Omega} (T_p u + ku)\gamma(u)dx \geq 0$ .

*Proof.* Most of the assertions follow from the results of Brezis and Strauss [4], so we are quite brief and only indicate how to apply their results. Let  $k$  be such that  $a(x) + k \geq 0$  and  $a(x) + \sum_j \partial_j a_j(x) + k \geq 0$ ,  $x \in \Omega$ . Then the operator  $L + k$  satisfies the hypotheses of Theorem 8 of [4]. Thus  $T_1 + k$  (when restricted to  $D(T_1) \cap L^1(\Omega; \mathbf{R})$ ) satisfies Proposition 7 of [4], and Lemma 3\* of [4] can be applied to  $(I + \lambda(T_1 + k))^{-1}$ . It follows that the range of  $I + \lambda(A_1 + k)$  is  $L^1(\Omega; \mathbf{R})$ ,  $(I + \lambda(A_1 + k))^{-1}$  exists and it is a contraction with respect to any norm  $\|\cdot\|_p$ ,  $1 \leq p < \infty$ . In particular,  $(I + \lambda(A_1 + k))^{-1}$  maps  $L^p(\Omega; \mathbf{R})$  into  $D(A_1) \cap L^p(\Omega)$ . Since  $A_1$  is an extension of  $A_p$ ,  $(I + \lambda(A_p + k))^{-1}$  exists and is a contraction in the norm  $\|\cdot\|_p$ ,  $1 \leq p < \infty$ . We still need to show that the range of  $I + \lambda(A_p + k)$  is  $L^p(\Omega)$ . Note that the linear operator  $\lambda(T_1 + k)$  and the monotone function  $u \rightarrow u + \lambda\beta(u)$  satisfy the hypotheses of Theorem 1 of [4]. Let  $f \in L^p(\Omega; \mathbf{R})$  and  $u = (I + \lambda(A_1 + k))^{-1}f$ . As noted above Lemma 3\* of [4] implies  $u \in L^p(\Omega) \cap D(A_1)$ , and Proposition 4 of [4] implies  $u + \lambda\beta(u) \in L^p(\Omega)$ , and, hence,  $\beta(u)$  and  $T_1 u$  belong to  $L^p(\Omega)$ . Using regularity theorems [1] for linear elliptic operators we conclude  $u \in W^{2,p}(\Omega)$ , and, hence,  $u \in D(A_p)$ . Thus, the range of  $I + \lambda(A_p + k)$  is  $L^p(\Omega)$ .

To prove the last part of the proposition, note that  $T_1 + k$  satisfies the hypotheses of Theorem 1 of [4]. Let  $u \in D(A_p)$  and  $f = (A_p + k)u$ . By Proposition 4 of [4] we have  $\|\beta(u)\|_p \leq \|(A_p + k)u\|_p$  and, hence,  $\|(T_p + k)u\|_p \leq 2\|(A_p + k)u\|_p$ . Using Lemma 2 of [4] we get  $\int_{\Omega} (T_p u + ku)\gamma(u)dx \geq 0$ .

**PROPOSITION 4.3.** *Let  $k$  be such that Propositions 4.1 and 4.2 are true.*

(1) *If  $\varphi \in L^1(\Omega; \mathbf{R})$  then  $\lim_{(n \rightarrow \infty)} (I + (t/n)A_1)^{-n} \varphi \equiv u(t) \equiv S(t)\varphi$  exists in  $L^1(\Omega)$  for all  $t \geq 0$ . If  $\varphi \in L^p(\Omega; \mathbf{R})$  for some  $p$ ,  $1 \leq p < \infty$ , then this limit exists in  $L^p(\Omega)$ ,  $u: [0, \infty) \rightarrow L^p(\Omega)$  is continuous and  $S(t): L^p(\Omega) \rightarrow L^p(\Omega)$  is Lipschitz with constant  $e^{kt}$ . In particular,  $\|u(t)\|_p \leq e^{kt} \|\varphi\|_p$ .*

(2) *If  $1 < p < \infty$  and  $\varphi \in D(A_p)$  then  $u(t) \in D(A_p)$ ,  $t \geq 0$ ,  $u: [0, \infty) \rightarrow L^p(\Omega)$  is absolutely continuous, the right derivative,  $D_+ u(t)$  exists and is equal to  $-A_p u(t)$  for all  $t \geq 0$ , and  $\|A_p u(t)\|_p \leq e^{kt} \|A_p \varphi\|_p$ .*

(3) *If  $n/2p < \alpha < 1$  and  $u(t_0) \in D((T_p + k)^{\alpha}) \cap L^p(\Omega; \mathbf{R})$  for some  $t_0 \geq 0$ , then  $u: (t_0, \infty) \rightarrow W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$  is analytic.*

*Proof.* The first part of Proposition 4.2 says that  $A_p + kI$  is

$m$ -accretive as defined by Kato [14, p. 138]. The assertions in part (1) are a direct application of the results of Crandall and Liggett [5, Theorem 1]. The fact that  $\beta(0) = 0$  implies  $A_1\psi = 0$  for  $\psi = 0$ . Thus  $S(t)\psi = 0$  if  $\psi = 0$ . This fact combined with the fact that  $S(t)$  has Lipschitz constant  $e^{kt}$  proves  $\|u(t)\|_p \leq e^{kt} \|\varphi\|_p$ .

If  $1 < p < \infty$  then  $L^p(\Omega)$  and its dual are uniformly convex and, if  $\varphi \in D(A_p)$ , the results of Kato [14, Theorems 7.1, 7.5 and first line of last paragraph of p. 147] imply  $u$  has the properties in (2). (Note that the solution constructed by Kato in [14, Theorem 7.1, 7.5] coincides with  $u(t)$  by virtue of [5, Theorem 2].)

To prove (3), let  $n/2p < \alpha < 1$  and  $u(t_0) \in D((T_p + k)^\alpha) \cap L^p(\Omega; \mathbf{R})$ . By Proposition 4.1 and Theorem 3.1 there exists  $r > 0$  and a continuous function  $v: [t_0, t_0 + r) \rightarrow L^p(\Omega)$  such that  $v: (t_0, t_0 + r) \rightarrow W^{2,p}(\Omega)$  is analytic,  $v_t + (T_p + k)v = kv - \beta(v)$ ,  $t_0 < t < t_0 + r$ , and  $v(t_0) = u(t_0)$ . Since  $v$  satisfies Definition 2.2 of [5] for being a strong solution of  $v_t + A_p v = 0$ ,  $v(t_0) = u(t_0)$ , it follows from Theorem 2 of [5] that  $v = u$  on  $[t_0, t_0 + r)$ . In particular,  $u(t) \in D(A_p)$  for  $t_0 < t < t_0 + r$ . By part (2),  $u(t) \in D(A_p)$ ,  $t_0 < t < \infty$ , and  $\|A_p u(t)\|_p$  is bounded for  $t$  in any interval of the form  $t_1 \leq t \leq t_2$  where  $t_0 < t_1 < t_2 < \infty$ . By Propositions 4.1 and 4.2,  $\|T_p u(t)\|_p$ ,  $\|u(t)\|_{2,p}$  and  $\|u(t)\|_\infty$  are also bounded for  $t_1 \leq t \leq t_2$ . Therefore  $\mathcal{A} = \{u(t)(x): x \in \Omega, t_1 \leq t \leq t_2\}$  is a bounded subset of  $\mathbf{R}$ . Again using Proposition 4.1 and Theorem 3.1, one sees that there exists  $r > 0$  such that for any  $t_3 \in [t_1, t_2]$  there is a continuous function  $v: [t_3, t_3 + r) \rightarrow L^p(\Omega)$  such that  $v: (t_3, t_3 + r) \rightarrow W^{2,p}(\Omega)$  is analytic  $v_t + A_p v(t) = 0$ ,  $t_3 < t < t_3 + r$ , and  $v(t_3) = u(t_3)$ . As above, it follows from Theorem 2 of [5] that  $u = v$  on  $[t_3, t_3 + r)$ . Since  $r$  is independent of  $t_3 \in [t_1, t_2]$ , it follows that  $u: (t_1, t_2) \rightarrow W^{2,p}(\Omega)$  is analytic. Since  $t_1, t_2$  are arbitrary, it follows that  $u: (t_0, \infty) \rightarrow W^{2,p}(\Omega)$  is analytic.

**THEOREM 4.4.** *Let  $\varphi \in W^{2,p}(\Omega; \mathbf{R}) \cap W_0^{1,p}(\Omega)$  and  $\beta(\varphi) \in L^p(\Omega)$ , i.e.  $\varphi \in D(A_p)$ , for some  $p$ ,  $1 < p < \infty$ . Then there exists a differentiable function  $u: [0, \infty) \rightarrow L^p(\Omega; \mathbf{R})$  such that  $u: (0, \infty) \rightarrow W^{2,q}(\Omega; \mathbf{R}) \cap W_0^{1,q}(\Omega)$  is analytic for all  $q$ ,  $1 \leq q < \infty$ ,  $u_t + Lu + \beta(u) = 0$ ,  $0 \leq t < \infty$ , and  $u(0) = \varphi$ . In fact  $u(t) = S(t)\varphi$  is constructed from  $\varphi$  by Proposition 4.3.*

The proof of this theorem uses the a priori inequality in the following lemma. The authors wish to thank Professor H. Brezis for many helpful suggestions regarding this inequality.

**LEMMA 4.5.** *Let  $k$  be such that Propositions 4.1 and 4.2 are true. Let  $1 < p \leq q < \infty$ ,  $0 \leq \alpha < 1 - q^{-1}$ ,  $0 < \varepsilon < \tau$ . Then there is an increasing function  $l: (0, \infty) \rightarrow (0, \infty)$  such that if  $\varphi \in W^{2,\tau}(\Omega; \mathbf{R}) \cap$*

$W_0^{1,r}(\Omega) = D(T_r) = D(A_r)$  for some  $r \geq q$ ,  $r > n/2$  then  $\|(T_q + k)^a u(t)\|_q \leq l(\|A_p \varphi\|_p + \|\varphi\|_p)$ ,  $\varepsilon \leq t \leq \tau$ , where  $u(t) = S(t)\varphi$  is obtained from  $\varphi$  by Proposition 4.3.

*Proof of Lemma 4.5.* It follows from Proposition 4.3 that  $u: (0, \infty) \rightarrow W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$  is analytic,  $u: [0, \infty) \rightarrow L^r(\Omega)$  is differentiable,  $\|A_r u(t)\|_r$  is bounded for  $t$  lying in any bounded interval and  $u_t + (T_r + k)u = ku - \beta(u)$  holds for all  $t \geq 0$ . From Proposition 4.1 and 4.2 it follows that  $\|\beta(u(t))\|_r$ ,  $\|T_r u(t)\|_r$  and  $\|u(t)\|_{2,r}$  are bounded for  $t$  lying in any bounded interval. According to Proposition 4.1, the map  $u \rightarrow \beta(u)$  is analytic from (an open subset of)  $W^{2,r}(\Omega; C)$  to  $L^r(\Omega)$ . Thus  $t \rightarrow \beta(u(t))$  is an analytic function from  $(0, \infty)$  to  $L^r(\Omega)$  and bounded for  $t$  lying in any bounded interval.

For  $1 < \rho \leq r$  we may apply inequality (3.5) with  $X = L^\rho(\Omega)$  and  $T = T_\rho + k$  to obtain

$$\|(T_\rho + k)^\mu u(t)\|_\rho \leq C \left[ \|u(\sigma)\|_\rho + \left( \int_\sigma^\tau \|ku - \beta(u)\|_\rho^\rho dt \right)^{1/\rho} \right],$$

$\sigma + \varepsilon/2 \leq t \leq \tau$ ,  $0 \leq \mu < 1 - \rho^{-1}$ . Using Minkowski's inequality on the integral and estimating  $\|u(t)\|_\rho$  in terms of  $\|u(\sigma)\|_\rho$  (by Proposition 4.3) one obtains

$$(4.1) \quad \|(T_\rho + k)^\mu u(t)\|_\rho \leq C \left[ \|u(\sigma)\|_\rho + \left( \int_\sigma^\tau \|\beta(u)\|_\rho^\rho dt \right)^{1/\rho} \right],$$

$\sigma + \varepsilon/2 \leq t \leq \tau$ ,  $0 \leq \mu < 1 - \rho^{-1}$ . Applying Proposition 4.1 to the left side, one obtains

$$(4.2) \quad \|u(t)\|_\rho \leq C \left[ \|u(\sigma)\|_\rho + \left( \int_\sigma^\tau \|\beta(u)\|_\rho^\rho dt \right)^{1/\rho} \right],$$

$\sigma + \varepsilon/2 \leq t \leq \tau$ ,  $\rho^{-1} \geq s^{-1} > \rho^{-1} - 2\mu n^{-1} > \rho^{-1} - 2n^{-1}(1 - \rho^{-1})$ . This is equivalent to  $\rho \leq s < \rho [1 - 2n^{-1}(\rho - 1)]^{-1}$  if  $1 - 2n^{-1}(\rho - 1) \geq 0$ , and to  $\rho \leq s \leq \infty$  if  $1 - 2n^{-1}(\rho - 1) < 0$ .

We now show that there is an increasing function  $l: (0, \infty) \rightarrow (0, \infty)$  such that

$$(4.3) \quad \|u(t)\|_q + \int_{\sigma+\varepsilon}^\tau \|\beta(u)\|_q^\rho dt \leq l(\|u(\sigma)\|_\rho + \int_\sigma^\tau \|\beta(u)\|_\rho^\rho dt),$$

$\sigma + \varepsilon \leq t \leq \tau$ . Let  $\gamma(\xi) = |\beta(\xi)|^{q-2} \beta(\xi)$ ,  $\xi \in \mathbf{R}$ . Multiplying the equation  $\beta(u) = -u_t - (T_q + k)u + ku$  by  $\gamma(u)$ , integrating over  $\Omega$ , and using Proposition 4.2  $ku\gamma(u) \leq C|u|^q + 2^{-1}|\beta(u)|^q$ , one obtains  $\|\beta(u)\|_q^\rho \leq -2 \int u_t \gamma(u) dx + C\|u\|_q^\rho$ ,  $0 \leq t < \infty$ . Let  $\zeta: \mathbf{R} \rightarrow \mathbf{R}$  be smooth,  $0 \leq \zeta \leq 1$ ,  $\zeta = 0$  on  $(-\infty, \sigma + \varepsilon/2]$ , and  $\zeta = 1$  on  $[\sigma + \varepsilon, \infty)$ . Multiplying the above inequality by  $\zeta$  and integrating from  $\sigma$  to  $\tau$ , one obtains

$$(4.4) \quad \int_{\sigma+\varepsilon}^{\tau} \|\beta(u)\|_q^q dt \leq -2 \int_{\sigma}^{\tau} \zeta(t) \int u_t \gamma(u) dx dt + C \int_{\sigma+\varepsilon/2}^{\tau} \|u\|_q^q dt.$$

Let  $\Gamma(\eta) = \int_0^{\eta} \gamma(\xi) d\xi$ ,  $\eta \in \mathbf{R}$ . Then  $\Gamma' = \gamma$ ,  $\Gamma(0) = 0$ ,  $\Gamma \geq 0$ . Since  $\Gamma$  is convex, we have  $\Gamma(0) - \Gamma(\eta) \geq \gamma(\eta)(0 - \eta)$  i.e.  $\Gamma(\eta) \leq \gamma(\eta)\eta$ . Using the same argument that was used in the proof of Proposition 4.1, one can show that the map  $G: u \rightarrow \Gamma(u)$  is Fréchet differentiable from  $W^{2,r}(\Omega; \mathbf{R})$  to  $L^r(\Omega)$ , and its differential is given by  $DG(u)v = \gamma(u)v$ . Therefore the map  $t \rightarrow \Gamma(u(t))$  is differentiable from  $(0, \infty)$  to  $L^r(\Omega)$  and its derivative is  $\gamma(u(t))u_t(t)$ . Thus  $\int \gamma(u)u_t dx = (d/dt) \int \Gamma(u) dx$ . If we integrate the first term on the right of (4.4) by parts, we get  $\int_{\sigma}^{\tau} \zeta'(t) \int \Gamma(u) dx dt - \int \Gamma(u(\tau)) dx$  (since  $\zeta(\tau) = 1$ ,  $\zeta(\sigma) = 0$ ). Using the fact that  $\Gamma \geq 0$  and  $\Gamma(\eta) \leq |\beta(\eta)|^{q-2} \beta(\eta)\eta$ , one sees that the preceding integrals are dominated by  $C \int_{\sigma+\varepsilon/2}^{\tau} \int |\beta(u)|^{q-1} |u| dx dt$ . Applying Hölders inequality, one sees that this integral is dominated by  $C \int_{\sigma+\varepsilon/2}^{\tau} \|\beta(u)\|_{(q-1)a}^{q-1} \|u\|_b dt$ , where  $a^{-1} + b^{-1} = 1$ . Using  $xy \leq a^{-1}x^a + b^{-1}y^b$ , one sees that this is dominated by  $C \int_{\sigma+\varepsilon/2}^{\tau} \|\beta(u)\|_{(q-1)a}^{(q-1)a} dt + C \int_{\sigma+\varepsilon/2}^{\tau} \|u\|_b^b dt$ . Let  $p$  be fixed and choose  $b$  so that (4.2) holds with  $\rho$  replaced by  $p$ , i.e.  $0 \leq p^{-1} - b^{-1} < \min\{2n^{-1}(1 - p^{-1}), p^{-1}\}$ . Then choose  $q$  so that  $(q-1)a = p$ , i.e.  $q = p(1 + p^{-1} - b^{-1})$ . This implies  $p \leq q < \min\{p+1, p+2n^{-1}(p-1)\}$ . Then the integrals above are dominated by  $l(\|u(\sigma)\|_p + \int_{\sigma}^{\tau} \|\beta(u)\|_p^p dt)$  where  $l: (0, \infty) \rightarrow (0, \infty)$  is increasing. Putting this together with (4.4) gives

$$(4.5) \quad \int_{\sigma+\varepsilon}^{\tau} \|\beta(u)\|_q^q dt \leq l(\|u(\sigma)\|_p + \int_{\sigma}^{\tau} \|\beta(u)\|_p^p dt) + C \int_{\sigma+\varepsilon/2}^{\tau} \|u\|_q^q dt.$$

We restrict  $q$  so that (4.2) holds with  $s$  replaced by  $q$  and  $\rho$  replaced by  $p$ . Then the second term on the right of (4.5) can be estimated by the first term and we obtain the desired inequality (4.3) for  $p \leq q < \min\{p+1, p+2n^{-1}(p-1), p[1+2n^{-1}(p-1)]^{-1}\}$ . However, we may now proceed to argue inductively on  $p$  and  $q$  to obtain (4.3) for all  $p, q$ ,  $1 < p \leq q < \infty$ .

To finish the proof of the lemma, note that Proposition 4.3 implies  $\|(A_p + k)u(t)\|_p \leq C(\|A_p \varphi\|_p + \|\varphi\|_p)$ ,  $0 \leq t \leq \tau$ . Combining this with Proposition 4.2, one obtains  $\|\varphi\|_p + \int_0^{\tau} \|\beta(u)\|_p^p dt \leq l(\|A_p \varphi\|_p + \|\varphi\|_p)$ . Combining this with (4.3), one obtains  $\|u(t)\|_q + \left(\int_{\varepsilon/2}^{\tau} \|\beta(u)\|_q^q dt\right)^{1/q} \leq l(\|A_p \varphi\|_p + \|\varphi\|_p)$ ,  $\varepsilon/2 \leq t \leq \tau$ . Using (4.1) with  $\rho$  replaced by  $q$  and  $\mu$  replaced by  $\alpha$ , one obtains the inequality in the lemma.

*Proof of Theorem 4.4* Since  $\Omega$  is bounded it suffices to prove the theorem for all  $q$  sufficiently large. We choose  $q$  so large than

$n/2q < \alpha < 1 - q^{-1}$ , and then pick  $\alpha$  so that  $n/2q < \alpha < 1 - q^{-1}$ . For such  $q$  and  $\alpha$  we can apply Proposition 4.3 (part (3)) and Lemma 4.5.

There exists a sequence  $\{\varphi_n\} \subset W^{2,q}(\Omega; \mathbf{R}) \cap W_0^{1,q}(\Omega)$  such that  $\varphi_n \rightarrow \varphi$  and  $A_p \varphi_n \rightarrow \varphi$  in  $L^p(\Omega)$ . (For example, we can take  $\varphi_n = (A_p + k + 1)^{-1} \psi_n = (A_q + k + 1)^{-1} \psi_n$  where  $\{\psi_n\}$  is a sequence in  $L^q(\Omega)$  with  $\psi_n \rightarrow (A_p + k + 1)\varphi$  in  $L^p(\Omega)$  and  $k$  is chosen so that Proposition 4.2 holds.) Let  $u(t) = S(t)\varphi$  and  $u_n = S(t)\varphi_n$  be constructed from  $\varphi$  and  $\varphi_n$  by Proposition 4.3. Since the  $S(t)$  are Lipschitz maps,  $u_n(t)$  converges to  $u(t)$  in  $L^p(\Omega)$ . By Lemma 4.5,  $\{(T_q + k)^\alpha u_n(t)\}$  is a bounded sequence in  $L^q(\Omega)$ , for fixed  $t > 0$ . Since  $L^q(\Omega)$  is reflexive, there is a subsequence  $\{u_{n_j}(t)\}$  such that  $\{u_{n_j}(t)\}$  and  $\{(T_q + k)^\alpha u_{n_j}(t)\}$  converge weakly in  $L^q(\Omega)$ , say  $u_{n_j}(t) \rightharpoonup v$  and  $(T_q + k)^\alpha u_{n_j}(t) \rightharpoonup w$  weakly in  $L^q(\Omega)$ . It follows that  $\{(u_{n_j}(t), (T_q + k)^\alpha u_{n_j}(t))\}$  converges weakly to  $(v, w)$  in  $L^q(\Omega) \times L^q(\Omega)$ . Since the graph of  $(T_q + k)^\alpha$  is closed (and, hence weakly closed),  $v \in D((T_q + k)^\alpha)$ . However, we must have  $u(t) = v$ , since  $(u_{n_j}(t), \psi) \rightarrow (u(t), \psi)$  and  $(u_{n_j}(t), \psi) \rightarrow (v, \psi)$  for every test function  $\psi$ . It follows that  $u(t) \in D((T_q + k)^\alpha)$ . From part (3) of Proposition 4.3 it follows that  $u: (t, \infty) \rightarrow W^{2,q}(\Omega)$  is analytic. Since  $t > 0$  is arbitrary, this proves the theorem.

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