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NONLINEAR HOLOMORPHIC SEMIGROUPS

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Conditions are given on a nonlinear operator A in a Banach space X under which the semigroup, $S(t)$, generated by $-A$ has the property that $S(t)x$ is analytic in t for $|\arg t| < \theta$ for each fixed $x \in \mathrm{cl}(D(A))$. Analyticity in t of solutions of $u' + Tu = Fu$ where $-T$ generates a linear holomorphic semigroup in X and F maps $D(T^\alpha)$ analytically into X for some $\alpha < 1$ is also established. These results are applied to establish analyticity in t of solutions to $\partial u/\partial t + Lu + \beta(u) = 0$ where $\beta: R \to R$ is real analytic, monotone increasing and $\beta(0) = 0$, and L is a second order elliptic operator.

1. Introduction. Hille and Yosida proved that if A is a densely defined linear operator on a Banach space X such that, for $\lambda > 0$, $I + \lambda A$ is an isomorphism from $D(A)$ onto X and $(I + \lambda A)^{-1}$ is a contraction, then $-A$ generates a strongly continuous semigroup $\{S(t): t \geqq 0\}$ of contractions on X. If X is a complex Banach space and the above conditions hold for $|\arg \lambda| < \theta$, instead of just for $\lambda > 0$, then $S(t)$ has an analytic extension in t to the sector $|\arg t| < \theta$. These holomorphic semigroups have a smoothing property, namely $S(t)$ maps X into $D(A)$ for $t \neq 0$ so that $u(t) = S(t)x$ is a solution to $u'(t) + Au(t) = 0$, $u(0) = x$ for *any* initial data $x \in X$. For the linear theory of semigroups see Yosida [24], Kato [12], and Hille-Phillips [11].

A number of authors (see Kōmura [15, 16], Kato [13, 14], Crandall and Pazy [6], Brezis [2], Crandall and Liggett [5], and the references listed there) have generalized the theory of semigroups to nonlinear operators. They have shown that if $A \subset X \times X$ is a (multivalued) nonlinear operator such that, for sufficiently small $\lambda > 0$, $(I + \lambda A)^{-1}$ is a contraction and the range of $(I + \lambda A)$ contains $\mathrm{cl}(D(A))$, the closure of the domain of A, then $-A$ generates a strongly continuous semigroup $\{S(t): t \geqq 0\}$ on $\mathrm{cl}(D(A))$. In the case when X is a Hilbert space, Kōmura [16] has given conditions under which $S(t)$ extends analytically to a sector $|\arg t| < \theta$. Brezis [2] has shown that if $A = \partial\varphi$ is the subdifferential of a lower semicontinuous, convex functional on a Hilbert space then the semigroup $\{S(t)\}$ generated by $-A$ has a regularizing property similar to the linear case, namely $S(t)$ maps $\mathrm{cl}(D(A))$ into $D(A)$ for $t > 0$.

In this paper (§ 2) we give an extension of Kōmura's result to the case where X is a Banach space by establishing conditions under which $S(t)$ extends analytically to $|\arg t| < \theta$. These conditions also imply $S(t)$ maps $\mathrm{cl}(D(A))$ into $D(A)$ for $t \neq \theta$; in other words, $S(t)$

has a smoothing action.

In § 3 we establish local analyticity in t of solutions, $u(t)$, of equations of the form $du/dt + Tu = Fu$ where $-T$ is the generator of a linear analytic semigroup in a Banach space X and F maps $D(T^\alpha)$ analytically into X for some $\alpha < 1$. We use the integral equation approach developed by Sobolevskii [23], and Fujita and Kato [9]. In § 4 we give applications to semilinear parabolic equations.

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2. A class of holomorphic nonlinear semigroups. In the following X is a complex Banach space. Let $C \subset X$, and $\Sigma_\theta = \{z \in C: |\arg z| < \theta, z \neq 0\}$ be an open sector in the complex plane. A holomorphic semigroup on C is a function S on $\Sigma_\theta \cup \{0\}$ such that $S(z)$ maps C into C for each $z \in \Sigma_\theta \cup \{0\}$; $S(z + w) = S(z)S(w)$ for $z, w \in \Sigma_\theta \cup \{0\}$; and, for $x \in C$, $S(z)x$ is a holomorphic function of $z \in \Sigma_\theta$ with $S(z)x \to S(0)x = x$ as $z \to 0$ and $z \in \Sigma_\theta$. If there is also a real number ω such

$$\|S(z)x - S(z)y\| \leqq e^{\omega|z|}\|x - y\|, \tag{2.1}$$

$x, y \in C$, $z \in \Sigma_\theta$, we will write $S \in \mathscr{H}_{\omega,\theta}(C)$. Note that we do not require $S(z)$ to be holomorphic for fixed z as did Kōmura [16]. Kōmura noted that a contraction mapping which is holomorphic on all of a complex Banach space must be the translate of a linear operator (a consequence of Liouville's theorem). Hence we wish to avoid the hypothesis that $S(z)$ be a holomorphic map.

The generator, A, of a nonlinear semigroup is, in general, a "multivalued" operator which is regarded as a subset of $X \times X$. For such operators we use the notation and definitions of Crandall and Liggett [5, page 266].

THEOREM 2.1. *Let $A \subset X \times X$, ω, θ, ε be real numbers such that $e^{i\varphi}A + \omega I$ is accretive for $|\varphi| < \theta$ and $R(I + \lambda A) \supset \mathrm{cl}\,(D(A))$ for $|\arg \lambda| < \theta$ and $|\lambda| < \varepsilon$. Let $J_\lambda = (I + \lambda A)^{-1}$ and suppose, for $x \in D(A)$ and n a positive integer, the map $\lambda \mapsto J_\lambda^n x$ is a holomorphic function of λ for $|\arg \lambda| < \theta$, $|\lambda| < \min(\varepsilon, |\omega|^{-1})$. Then*

$$\lim_{n\to\infty} J_{z/n}^n x \equiv S(z)x \tag{2.2}$$

exists for $x \in \mathrm{cl}\,(D(A))$ and $z \in \Sigma_\theta$ and $S \in \mathscr{H}_{\omega,\theta}(\mathrm{cl}\,(D(A)))$. If, in addition, A is a closed subset of $X \times X$ then for each $x \in \mathrm{cl}\,(D(A))$ and $z \in \Sigma_\theta$, we have $S(z)x \in D(A)$ and $-(d/dz)S(z)x \in AS(z)x$.

Proof. Let $K_{\alpha,\varphi} = (I + \alpha e^{i\varphi}A)^{-1}$ be the resolvent of $e^{i\varphi}A$. For $|\varphi| < \theta$, the operator $e^{i\varphi}A$ satisfies the hypotheses of Theorem 1 of Crandall and Liggett [5], so $\lim K^n_{t/n,\varphi}x \equiv T_\varphi(t)x$ exists for $x \in \mathrm{cl}\,(D(A))$, $t \geqq 0$, and $\{T_\varphi(t): t \geqq 0\}$ is a (strongly continuous) semigroup with each $T_\varphi(t)$ Lipschitz with constant $e^{\omega t}$. Since $J_\lambda = K_{|\lambda|,\arg\lambda}$, it follows that the limit (2.2) exists, $S(z)x = T_{\arg z}(|z|)x$, and $S(z)$ satisfies (2.1) for $x, y \in \mathrm{cl}\,(D(A))$.

Now let $x \in D(A)$. Applying the inequalities (ii) and (iii) on p. 268 of [5] to $e^{i\varphi}A$, we get $\|K^n_{t/n,\varphi}x - x\| \leqq t(1 - tn^{-1}|\omega|)^{-n}|e^{i\varphi}Ax|$, $t \geqq 0$, $t|\omega| < n$. Substituting $t = |z|$, $\varphi = \arg z$, and using $J_z = K_{|z|,\arg z}$, and the fact that $(1 - a/n)^{-n} \leqq e^{|a|}$, $a \in R$, we obtain $\|J^n_{z/n}x - x\| \leqq |z|\,e^{|z||\omega|}|Ax|$, $|\arg z| < \theta$, $|z\omega| < n$. Thus when z is restricted to lie in a bounded subset of Σ_θ, the sequence $\{J^n_{z/n}x\}$ is a uniformly bounded sequence of holomorphic functions of z which converge pointwise to $S(z)x$. It follows (see [11], p. 104) that $S(z)x$ is holomorphic in z and $\|S(z)x - x\| \leqq |z|\,e^{|z||\omega|}|Ax|$. In particular, $S(z)x \to x$ as $z \to 0$.

Now let $x \in \mathrm{cl}\,(D(A))$ and choose $\{x_n\} \subset D(A)$ with $x_n \to x$. Then $\{S(z)x_n\}$ is a sequence of functions holomorphic on Σ_θ and continuous at $z = 0$. If z is restricted to lie in a bounded subset of $\Sigma_\theta \cup \{0\}$ then the $S(z)$ are Lipschitz with constant independent of z and, hence, $\{S(z)x_n\}$ converges uniformly to $S(z)x$. Thus $S(z)x$ is holomorphic on Σ_θ and continuous at $z = 0$.

In order to show the semigroup property, let $w \in \Sigma_\theta$ be fixed and $\varphi = \arg w$. If $\{T_\varphi(t): t \geqq 0\}$ is the semigroup generated by $-e^{i\varphi}A$ then $S(te^{i\varphi}) = T_\varphi(t)$, $t \geqq 0$. By Crandall and Liggett, $T_\varphi(t)$ is a semigroup for real t, so $S(te^{i\varphi} + \tau e^{i\varphi}) = S(te^{i\varphi})S(\tau e^{i\varphi})$. Thus $S(z + w) = S(z)S(w)$ for $z = tw$, $t \geqq 0$. If $x \in \mathrm{cl}\,(D(A))$ then $S(z + w)x$ and $S(z)S(w)x$ are holomorphic functions of $z \in \Sigma_\theta$ which agree on the ray $z = tw$, $t \geqq 0$. By the identity theorem for holomorphic functions $S(z + w)x = S(z)S(w)x$ for all z.

In the real case (see [5]) a strong solution to the Cauchy problem

$$(2.3) \qquad 0 \in du/dt + Au\,, \quad 0 \leqq t \leqq T\,, \quad u(0) = x\,,$$

is a function u: $[0, T] \to X$ so that (i) u is continuous, (ii) u is the indefinite integral of a function which is strongly integrable on compact subsets of $(0, T)$, (iii) $u(0) = x$ and (iv) $u'(t) \in -Au(t)$ for a.e. t in $(0, T)$.

Crandall and Liggett, and Miyadera [20] have shown the following result. Let B be closed in $X \times X$, $B + \omega I$ accretive for some real number ω, $R(I + tB) \supset \mathrm{cl}\,(D(B))$ for sufficiently small $t > 0$, and for $x \in \mathrm{cl}\,(D(B))$ let $T(t)x = \lim\,(I + (t/n)B)^{-n}x$ be the semigroup generated by $-B$. Then if $x \in \mathrm{cl}\,(D(B))$ and $T(t)x$ is strongly differentiable at

$t_0 > 0$, with $y = (d/dt)T(t_0)x$, then $[T(t_0)x, -y] \in B$. Then using the fact that for $x \in D(B)$, $S(t)x$ is Lipschitz continuous on bounded sets of t, they are able to conclude that if $S(t)x$ is differentiable a.e. then $u = S(t)x$ is a strong solution of (2.3).

In our case, since we have shown that $S(z)x$ is a holomorphic function for $x \in \mathrm{cl}\,(D(A))$, it is immediate that $S(z)x$ can be recovered as the indefinite integral of an analytic function along a ray.

To finish the details of the proof, let A be closed, $x \in \mathrm{cl}\,(D(A))$, $z \in \Sigma_\theta$ with $\varphi = \arg z$, and $\{T_\varphi(t); t \geqq 0\}$ be the semigroup generated by $-e^{i\varphi}A$ so that $S(te^{i\varphi}) = T_\varphi(t)$, $t \geqq 0$. If $x \in \mathrm{cl}\,(D(A))$ then $u(z) = S(z)x$ is holomorphic for $z \in \Sigma_\theta$ which implies that $v(t) = T_\varphi(t)x$ is differentiable for $t > 0$ and $v'(t) = e^{i\varphi}u'(te^{i\varphi})$.

Since $-e^{i\varphi}A$ is closed, it follows from the above results of Crandall and Liggett that $-v'(t) \in e^{i\varphi}Av(t)$. Hence $-u'(te^{i\varphi}) \in Au(te^{i\varphi})$, and together with the comment on holomorphy of $S(t)x$ for $x \in \mathrm{cl}\,(D(A))$, we have established a strong solution to the Cauchy problem for $x \in \mathrm{cl}\,(D(A))$.

REMARK. We will show in an example that J_λ may not be defined on an open set, so that J_λ is certainly not a holomorphic map in general. However in case J_λ is a holomorphic map, then the hypothesis $J_\lambda^n x$ is a holomorphic function of λ for all n is satisfied. We may argue as follows. First since J_λ is locally Lipschitz, both Kōmura [16] and Neuberger [21] have established that $J_\lambda x$ is holomorphic in λ when J_λ is a holomorphic map. Next let $g(\lambda_1, \lambda_2, \cdots, \lambda_n) = J_{\lambda_1} \cdot J_{\lambda_2} \cdot J_{\lambda_3} \cdots J_{\lambda_n} x$. Then for fixed $\lambda_2, \lambda_3, \cdots, \lambda_n$, g is holomorphic in λ_1. If $\lambda_1, \lambda_3, \cdots, \lambda_n$ are fixed, then $J_{\lambda_2} \cdot J_{\lambda_3} \cdots J_{\lambda_n}$ is holomorphic in λ_2 and therefore when composed with the holomorphic map J_{λ_1}, g is holomorphic in λ_2 and so forth. Hence, as is well known [11], p. 107, $g(\lambda, \lambda, \lambda, \cdots)$ is a holomorphic function of λ.

EXAMPLE. Let $\beta: K \to \boldsymbol{C}$ be continuous where K is the closure of an open, convex set $U \subset \boldsymbol{C}$. Suppose $0 \in K$, $\beta(0) = 0$ and β is analytic on U. Assume there is $\theta > 0$ such that $|\arg \beta'(z)| \leqq \pi/2 - \theta$, $z \in U$. Finally suppose there is $\varepsilon < 0$ such that for $|\arg \lambda| < \theta$, $|\lambda| < \varepsilon$, one has $(I + \lambda\beta)(K) \supset K$ and $(I + \lambda\beta)(U) \supset U$. Here $I(z) = z$ is the identity map on $\boldsymbol{C}$.

Let $X = L^p(\Omega; \boldsymbol{C})$ where Ω is any measure space and $1 \leqq p \leqq \infty$. Let $D(A) = \{u \in X: u(x) \in K$ a.e. and $\beta(u) \in X\}$, where $\beta(u)$ is the composition of β and u. Let $Au = \beta(u)$ for $u \in D(A)$. We shall show that A satisfies the hypotheses of Theorem 2.1 with $\omega = 0$ and θ, ε as above.

The hypothesis $|\arg \beta'(z)| \leqq \pi/2 - \theta$, $z \in U$, implies $e^{i\varphi}\beta$ is accretive for $|\varphi| < \theta$. In particular $I + \lambda\beta$ is one-to-one and $(I + \lambda\beta)^{-1}$ is a contraction for $|\arg \lambda| < \theta$. Let $S = \{\lambda \in C: |\arg \lambda| < \theta, |\lambda| < \varepsilon\}$. The assumption that $(I + \lambda\beta)(K) \supset K$, $\lambda \in S$, implies the function $j(w, \lambda) = (I + \lambda\beta)^{-1}(w)$ is well defined for $w \in K$, $\lambda \in S$. It is a contraction in w for fixed λ. Since β is analytic on U and $(I + \lambda\beta)(U) \supset U$, the implicit function theorem implies $j: U \times S \to U$ is analytic. Since $\beta(0) = 0$ we have $j(0, \lambda) = 0$. Since $j(\cdot, \lambda)$ is a contraction we have $|j(w, \lambda)| \leqq |w|$.

Let $j^1(w, \lambda) = j(w, \lambda)$, $w \in K$, $\lambda \in S$ and $j^n(w, \lambda) = j(j^{n-1}(w, \lambda), \lambda)$, $w \in K$, $\lambda \in S$, $n \geqq 2$. Since $j(w, \lambda)$ is a contraction in w, it follows that $j^n(w, \lambda)$ is a contraction in w for fixed λ. Since j: $UXS \to U$ is analytic, it follows that j^n: $UXS \to U$ is analytic. We claim that $j^n(w, \lambda)$ is analytic in λ for fixed w, even if $w \in K$. To see this, choose a sequence $\{w_m\} \subset U$ with $w_m \to w$. Then $\{j^n(w_m, \lambda)\}$ is a sequence of functions each analytic in λ and $j^n(w_m, \lambda) \to j^n(w, \lambda)$ uniformly in λ since $j^n(w, \lambda)$ is a contraction in w. It follows that $j^n(w, \lambda)$ is analytic in λ. Finally we note that $|j^n(w, \lambda)| \leqq |w|$ since $|j(w, \lambda)| \leqq |w|$.

Now consider the operator A. We have $v = (I + \lambda A)u$ if and only if $v(x) = (I + \lambda\beta)(u(x))$ a.e. If $|\arg \lambda| < \theta$ then $I + \lambda\beta$ is $1 - 1$ so $v = (I + \lambda A)u$ is equivalent to $u(x) = (I + \lambda\beta)^{-1}(v(x))$ a.e. In particular $I + \lambda A$ is $1 - 1$ and $J_\lambda \equiv (I + \lambda A)^{-1}$ is contraction. It follows that $e^{i\varphi}A$ is accretive for $|\varphi| < \theta$.

To show $\mathrm{cl}(D(A)) \subset R(I + \lambda A)$, note that $\mathrm{cl}(D(A)) \subset F$ where $F = \{v \in X: v(x) \in K \text{ a.e.}\}$. The assumption $K \subset (I + \lambda\beta)(K)$, $\lambda \in S$ and the definition of j implies that $F \subset R(I + \lambda A)$ for $\lambda \in S$, and $J_\lambda v(x) = j(v(x), \lambda)$, $v \in F$, $\lambda \in S$.

To show $J_\lambda^n v$ is analytic in λ for fixed $v \in F$, note that $J_\lambda^n v(x) = j^n(v(x), \lambda)$. It follows from $|j^n(v(x), \lambda)| \leqq |v(x)|$ and the Cauchy integral formula that

$$(2.4) \qquad |j_\lambda^n(v(x), \lambda)| \leqq |v(x)| \,\mathrm{dist}\,(\lambda, \partial S)$$

$$(2.5) \qquad |j_{\lambda\lambda}^n(v(x), \lambda)| \leqq |v(x)| \,[\mathrm{dist}\,(\lambda, \partial S)]^2$$

where $j_\lambda^n = \partial j^n/\partial\lambda$, $j_{\lambda\lambda}^n = \partial^2 j^n/\partial\lambda^2$. In the case $1 \leqq p < \infty$ in order to show $J_\lambda^n v$ is analytic in λ it suffices to show weak analyticity, i.e. $(d/d\lambda)\int_\Omega j^n(v(x), \lambda)\overline{w(x)}dx = \int_\Omega j_\lambda^n(v(x), \lambda)\overline{w(x)}dx$ for all $w \in L^q(\Omega)$, $p^{-1} + q^{-1} = 1$. This is true because $j^n(v(x), \lambda)$ is analytic in λ for fixed x, and the estimate (2.4) implies that differentiation under the integral sign is valid. In the case $p = \infty$ we must show $r(\mu) \to 0$ as $\mu \to 0$ where $r(\mu) = \|[j^n(v, \lambda + \mu) - j^n(v, \lambda)]\mu^{-1} - j_\lambda^n(v, \lambda)\|_\infty$. Note that (2.4) implies that $j_\lambda^n(v, \lambda)$ is in $L^\infty(\Omega)$ for each λ. Since $j^n(v, \lambda + \mu) -$

$j^n(v, \lambda) = \int_\lambda^{\lambda+\mu} j^n_\lambda(v, \eta)d\eta$ a computation shows that

$$\begin{aligned} r(\mu) &\leqq \sup\{|j^n_\lambda(v(x), \lambda + t\mu) - j^n_\lambda(v(x), \lambda)| : x \in \Omega, 0 \leqq t \leqq 1\} \\ &\leqq |\mu| \sup\{|j^n_{\lambda\lambda}(v(x), \lambda + t\mu)| : x \in \Omega, 0 \leqq t \leqq 1\} . \end{aligned}$$

Using (2.5) we get $r(\mu) \leqq 4|\mu| \, \|v\|_\infty [\mathrm{dist}(\lambda, \partial S)]^2$ for $|\mu| < 2^{-1}\mathrm{dist}(\lambda, \partial S)$. Thus $r(\mu) \to 0$.

A special case of this example is $\beta(z) = z^2, z \in K \equiv \{z \in C: |\arg z| \leqq \pi/4\} \cup \{0\}$. We have $|\arg \beta'(z)| \leqq \pi/4, x \in K$, so we can take $\theta = \pi/4$. Note that $(I + \lambda\beta)^{-1}w = [-1 + (1 + 4\lambda w)^{1/2}]/2\lambda$ for $w \in R(I + \lambda\beta)$, $|\arg\lambda| < \pi/4$. A simple geometric argument shows that if $|\arg\lambda| < \pi/4$ and $|\arg w| \leqq \pi/4$ (resp. $|\arg w| < \pi/4$) then $|\arg(I + \lambda\beta)^{-1}w| \leqq \pi/4$ (resp. $< \pi/4$). Thus $K \subset (I + \lambda\beta)(K)$ and $U \subset (I + \lambda\beta)(U)$, $|\arg\lambda| < \pi/4$, where U is the interior of K.

To obtain an "unbounded generator" version of the above example, let $X = l^2$, $D(A) = \{x \in l^2: Ax \in l^2, |\arg x_i| \leqq \pi/4\}$. Let $\Sigma_\theta = \{\lambda \in C: |\arg\lambda| < \pi/4\}$ and let $A(x_1, x_2, x_3, \cdots) = (x_1^2, 2x_2^2, 3x_3^2, \cdots)$. The hypotheses of Theorem 1 are easy to verify in this case.

Our results include some, but not all, of the linear theory of holomorphic semigroups. If A is an m-sectorial operator in a Hilbert space with vertex zero (so that its numerical range is a subset of a sector $|\arg\varphi| \leqq \pi/2 - \theta, \theta < \pi/2$), then A satisfies the hypotheses of Theorem 2.1.

3. A perturbation theorem. In this section we consider the equation $du/dt + Tu(t) = Fu(t), t \geqq 0, u(0) = x$, where T is a linear operator in a complex Banach space X and F is a function with domain and range in X. Equations of this type have been studied by Sobolevskii [23], Fujita and Kato [9], Friedman [8], Henry [10] and others. We establish analyticity in t of solutions $u(t)$ of this equation under suitable conditions on T and F. In particular, we assume that

(3.1) The resolvent of T exists for $\mathrm{Re}\,\lambda \leqq 0$ and there exists a constant C such that $\|(\lambda - T)^{-1}\| \leqq C(1 + |\lambda|)^{-1}$, $\mathrm{Re}\,\lambda \leqq 0$.

Using the Neumann series representation for the resolvent [12, pp. 37, 173] it is not hard to show that there exists $C_1, \omega > 0$ such that the resolvent of T exists and satisfies $\|(\lambda - T)^{-1}\| \leqq C_1|\lambda|^{-1}$ for $|\arg\lambda - \pi| < (\pi/2) + \omega$. This is a well known ([12, p. 488], [8, p. 101]) condition for $-T$ to generate a holomorphic semigroup $\{U(t): |\arg t| < \omega\}$. The map $t \to U(t)$ is a bounded holomorphic map from $\{t: |\arg t| < \theta, t \neq 0\}$ into $B(X)$ for any $\theta < \omega$.

The assumption (3.1) implies that T has fractional powers, T^γ, for $\gamma \in \boldsymbol{R}$ (see [24, 8, 18]). For $\gamma \leqq 0$, $T^\gamma \in B(X)$. For $\gamma \geqq 0$, T^γ is a closed operator in X with domain, $X_\gamma \equiv D(T^\gamma)$, dense in X. For all γ, T^γ is invertible with $(T^\gamma)^{-1} = T^{-\gamma}$; see [8, pp. 158–159]. For $\gamma > 0$, we define $\| x \|_\gamma = \| T^\gamma x \|$, $x \in X_\gamma$ (cf. [10, p. 29]). The fact that $(T^\gamma)^{-1} \in B(X)$ implies $\| \quad \|_\gamma$ is a norm on X_γ which is equivalent to the graph norm, $||| \quad |||$, of T^γ, since $||| x ||| \equiv \| T^\gamma x \| + \| x \| \leqq (1 + \| T^{-\gamma} \|) \| T^\gamma x \|$. X_γ is a Banach space with the norm $\| \quad \|_\gamma$ since T^γ is a closed operator. In § 4 we shall need the following imbedding theorem for domains of factional powers.

(3.2) If Y is a Banach space with $D(T) \subset Y \subset X$ and $0 \leqq \beta < 1$ and there exists C such that $\| u \|_Y \leqq C \| Tu \|_X^\beta \, \| u \|_X^{1-\beta}$, $u \in D(T)$, then $D(T^\alpha)$ is continuously imbedded in Y for $\beta < \alpha \leqq 1$.

(See Sobolevskii [23, p. 22], Friedman [8, p. 177], and Henry [10, p. 29].)

We shall also need the following facts which relate the semigroup to the fractional powers. For all $\gamma \geqq 0$, $U(T)$ maps X into $D(T^\gamma)$ and, for $\theta < \omega$ there exists a constant M_γ such that

$$\| T^\gamma U(t) \| \leqq M_\gamma |t|^{-\gamma}, \quad |\arg t| < \theta . \tag{3.3}$$

(See [8, pp. 105–106, 158–160] where this is proved for real t. The same argument works for complex t.)

For $0 < \gamma \leqq 1$, $\theta < \omega$ one has

$$\| U(t)x - x \| \leqq M_{1-\gamma} \gamma^{-1} |t|^\gamma \| T^\gamma x \| , \tag{3.4}$$

$|\arg t| < \theta$, $x \in X_\gamma$. (To prove this, note that $(d/ds)U(s)x = -TU(s)x = -T^{1-\gamma}U(s)T^\gamma x$. Thus $U(t)x - x = -\int_0^t T^{1-\gamma}U(s)T^\gamma x\,ds$. Using (3.3) to estimate $\| T^{1-\gamma}U(s) \|$, one obtains (3.4). This proof is due to Henry [10].)

Let $1 < p \leqq \infty$, $0 \leqq \gamma < 1 - p^{-1}$, $0 < \varepsilon < \tau$. Then there exists a constant M such that if $u\colon [0, \tau] \to X$ is differentiable, $u(t) \in D(T)$, $0 \leqq t \leqq \tau$, and $u'(t) + Tu(t) = f(t)$, $0 \leqq t \leqq \tau$, with $f \in L^p(0, \tau; X)$ then

$$\| T^\gamma u(t) \| \leqq M\left[\| u(0) \| + \left(\int_0^\tau \| f(s) \|^p \, ds \right)^{1/p} \right], \tag{3.5}$$

$\varepsilon \leqq t \leqq \tau$. To prove (3.5), first note that

$$u(t) = U(t)u(0) + \int_0^t U(t-s)f(s)ds$$

(see [12, p. 486]). By (3.3) we have $\| T^\gamma U(t)u(0) \| \leqq M_\gamma \varepsilon^{-\gamma} \| u(0) \|$,

$\varepsilon \leqq t \leqq \tau$, and $\int_0^t \| T^\gamma U(t-s)f(s) \| ds \leqq M_\gamma \int_0^t \| f(s) \| (t-s)^{-\gamma} ds \leqq M_\gamma \left(\int_0^t \| f(s) \|^p ds \right)^{1/p} \left(\int_0^t (t-s)^{-\gamma q} \right)^{1/q} \leqq \text{const.} \left(\int_0^\tau \| f(s) \|^p ds \right)^{1/p}$, $0 \leqq t \leqq \tau$, $p^{-1} + q^{-1} = 1$. Note that $\gamma q < 1$ since $\gamma < q^{-1} = 1 - p^{-1}$. This proves (3.5).

THEOREM 3.1. *Assume T satisfies* (3.1), $0 \leqq \alpha < 1$, $\theta < \omega$, *and F is a function whose domain, $D(F)$, is an open subset of X_α and $F\colon D(F) \to X$ is Frechet analytic (as a map from X_α to X). Then for each $x \in D(F)$ there exists $r > 0$ and a unique function u mapping $W_r = \{t \in C\colon |\arg t| < \theta, 0 < |t| < r\}$ analytically into $X_1 = D(T)$ such that for each $t \in W_r$, $u(t) \in D(F)$ and $u'(t) + Tu(t) = Fu(t)$, and $\| u(t) - x \|_\alpha \to 0$ as $t \to 0$.*

Let $U \subset D(F) \cap X_\gamma$ for some $\gamma > \alpha$ and suppose there exists $\delta > 0$ and K such that if $x \in U$ and $\| y - x \|_\alpha < \delta$ then $y \in D(F)$ and $\| Fy \| < K$. Suppose also that U is bounded in X_γ. Then the value of r can be chosen independently of $x \in U$.

If, in addition, F maps $D(F) \cap X_{s+\alpha}$ analytically into X_s for $0 \leqq s \leqq n$, then u is analytic from W_r to X_{n+1}.

Proof. The differential equation $du/dt + Au = Fu$ is transformed into the integral equation (3.7) below. This method was introduced by Sobolevskii [23] and Fujita and Kato [9] and is now standard. We use methods similar to Henry [10], and therefore we are as brief as possible.

Choose $\delta > 0$ and K so that $\| y - x \|_\alpha < \delta$ implies $y \in D(F)$ and $\| Fy \| \leqq K$. Using the Cauchy integral formula, one has

$$\| Fy_1 - Fy_2 \| \leqq 4K\delta^{-1} \| y_1 - y_2 \|_\alpha , \tag{3.6}$$

if $\| y_i - x \|_\alpha \leqq \delta/2$, $i = 1, 2$. Let S_r be the set of all analytic functions $u\colon W_r \to X_\alpha$ such that $\| u(t) - x \|_\alpha \leqq \delta/2$, $t \in W_r$ and $\| u(t) - x \|_\alpha \to 0$ as $t \to 0$. S_r is a complete metric space if we define $d(u, v) = \sup \times \{\| u(t) - v(t) \|_\alpha\colon t \in W_r\}$, $u, v \in S_r$.

For $u \in S_r$ put

$$Gu(t) = U(t)x + \int_0^t U(t-s)Fu(s)ds , \quad t \in W_r , \tag{3.7}$$

where the integral is taken over the line segment $\{s = \lambda t, 0 \leqq \lambda \leqq 1\}$ joining 0 to t. We shall show G is a strict contraction from S_r into S_r if r is chosen small enough.

First consider the integral on the right of (3.7); we denote its value by $v(t)$. Putting $s = \lambda t$, $0 \leqq \lambda \leqq 1$, we get $v(t) = t \int_0^1 g(t, \lambda) d\lambda$ where $g(t, \lambda) = U(t - t\lambda) f(t\lambda)$, where $f(t) = Fu(t)$. Using (3.3) one sees that there is a constant C such that $\|g(t, \lambda)\|_\alpha \leqq C|t|^{-\alpha}(1-\lambda)^{-\alpha}$, $t \in W_r$, $0 < \lambda < 1$. Thus the integral in (3.7) is absolutely convergent in X_α and $\|v(t)\|_\alpha \leqq C_1|t|^{1-\alpha}$, $t \in W_r$. In particular, $\|v(t)\|_\alpha \to 0$ as $t \to 0$, and we can make $\|v(t)\|_\alpha \leqq \delta/4$, $t \in W_r$, by choosing r sufficiently small.

Since $\|U(t)x - x\|_\alpha = \|U(t)T^\alpha x - T^\alpha x\|$ approaches 0 as $t \to 0$, we can make $\|U(t)x - x\|_\alpha < \delta/4$ by making r small. If $x \in X_\gamma$ for some $\gamma > \alpha$, then the size of r necessary to make $\|U(t)x - x\|_\alpha < \delta/4$ is determined by $\|x\|_\gamma$. This is because (3.4) implies

$$\|U(t)x - x\|_\alpha \leqq \text{const.}\, |t|^{\gamma-\alpha}\|T^{\gamma-\alpha}T^\alpha x\| \leqq \text{const.}\, |t|^{\gamma-\alpha}\|x\|_\gamma .$$

Combining these results, one has $\|Gu(t) - x\|_\alpha \to 0$ as $t \to 0$, and $\|Gu(t) - x\|_\alpha \leqq \delta/2$, $t \in W_r$ for r small.

Since $U(t)x$ is analytic in t, it remains to show the integral $v(t)$ is analytic in t with values in X_α. For fixed $\lambda \in (0, 1)$, $g(t, \lambda)$ is an analytic function of t with values in X_α and

$$g_t(t, \lambda) = -(1-\lambda)TU(t - t\lambda)f(t\lambda) + U(t - t\lambda)f'(t\lambda)\lambda ,$$

where $g_t = \partial g/\partial t$. The function f is bounded by K, so by the Cauchy integral formula $\|f'(t)\| \leqq K|t|^{-1}\csc(\theta - |\arg t|)$. Using this and (3.3), one sees that $\|g_t(t, \lambda)\|_\alpha$ is bounded by const. $(1-\lambda)^{-\alpha}$ for t in a compact subset of W_r. Thus the difference quotients $\|[g(t, \lambda) - g(s, \lambda)]/(t - s)\|_\alpha$ are similarly bounded. Using the dominated convergence theorem, it follows that $v\colon W_r \to X_\alpha$ is analytic. Therefore $Gu\colon W_r \to X_\alpha$ is analytic.

We have shown G maps S_r into S_r for r small. To show G is a contraction, we use (3.3) and (3.6) to get

$$\begin{aligned}\|Gu(t) - Gv(t)\|_\alpha &\leqq M_\alpha \int_0^t |t - s|^{-\alpha}\|Fu(s) - Fu(s)\|\, d|s| \\ &\leqq \text{const.}\, |t|^{1-\alpha}\sup\|u(s) - v(s)\|_\alpha ,\end{aligned}$$

$t \in W_r$, $u, v \in S_r$. By making r sufficiently small we can make G a strict contraction. By the fixed point theorem for strict constractions on a complete metric space, there is a unique $u \in S_r$ such that $Gu = u$. In order to show u satisfies the differential equation $u'(t) + Tu(t) = Fu(t)$ we will use a known result (see Kato [12], Theorem 1.27, p. 491) on solutions to inhomogeneous equations for holomorphic semigroups. In order to apply this theorem it is necessary to make two changes of variable. Fix $t \in W_r$ and define $v(\lambda) = u(\lambda t) = U(\lambda t)x + \int_0^{\lambda t} U(\lambda t - s) \times Fu(s)ds$. Putting $s = \sigma t$, $0 \leqq \sigma \leqq \lambda$, we get $v(\lambda) = V(\lambda)x + \int_0^\lambda V(\lambda - \sigma)f(\sigma)d\sigma$ where $V(\lambda) = U(\lambda t)$ is the (holomorphic) semigroup generated by $-tT$, and $f(\sigma) = tFu(\sigma t)$ is continuous on $[0, r/|t|)$ and analytic on $(0, r/|t|)$ with values in X. Fixing $\tau < 1$, it is not hard to show $v(\lambda + \tau) =$

$V(\lambda)v(\tau) + \int_0^\lambda V(\lambda - \rho)f(\rho + \tau)d\rho$, $0 \leqq \lambda < r/|t| - \tau$. The function $\rho \mapsto f(\rho + \tau)$ is Hölder continuous on $[0, r/|t| - \tau)$. By the above mentioned theorem in [12], it follows that $v(s) \in D(T)$, $\tau < s < r/|t|$, with $v'(s) + tTv(s) = f(s)$. Putting $s = 1$ shows $u(t) \in D(T)$ and $u'(t) + Tu(t) = Fu(t)$. So far we know $u: W_r \to X_\alpha$ is analytic. If we rewrite the equation as $u = T^{-1}(Fu - u')$ it follows that $u: W_r \to X_1$ is analytic. The solution of $u' + Tu = Fu$, $u(0) = x$ is unique because any u satisfying the conclusions of the theorem must also satisfy $Gu = u$.

Suppose F is analytic from $U \cap X_{s+\alpha}$ to X_s, $0 \leqq s \leqq n$. If u is analytic from W_r to $X_{s+\alpha}$ for such an s, then the equation $u = T^{-1}(Fu - u')$ shows $u: W_r \to X_{s+1}$ is analytic. Repeating this argument shows that $u: W_r \to X_{n+1}$ is analytic.

4. **Semilinear parabolic equations.** In this section the results of § 3 are applied to the mixed problem $\partial u/\partial t + Lu + \beta(u) = 0$, $(x, t) \in \Omega \times [0, \infty)$; $u(x, 0) = \varphi(x)$, $x \in \Omega$; $u(x, t) = 0$, $(x, t) \in \partial\Omega \times [0, \infty)$; where L is a second order elliptic operator of the form $Lu = -\sum_{i,j} \partial_j[a_{ij}\partial_i u] + \sum_i \partial_i[a_i u] + au$. Here $\partial_j = \partial/\partial x_j$ and sums are from 1 to n. Ω is the *closure* of a bounded, open subset of $\boldsymbol{R}^n$, and Ω has smooth boundary $\partial\Omega$. The a_{ij}, a_i, a are real valued functions on Ω with $a_{ij} = a_{ji}$; a_{ij}, $a_i \in C^1(\Omega)$, $a \in C(\Omega)$ and there exists $\mu > 0$ such that $\sum_{ij} a_{ij}\xi_i\xi_j \geqq \mu|\xi|^2$, $\xi \in \boldsymbol{R}^n$, $x \in \Omega$. β is an analytic function whose domain, $D(\beta)$, is an open subset of the complex plane containing the real axis; β maps the real line into itself; for t real, $\beta(t)$ is an increasing function of t, and $\beta(0) = 0$.

Equations of this type have been studied by Brezis, Crandall and Pazy [3], Brezis and Strauss [4], Da Prato [7], Konishi [17], Ōuchi [22], and Brezis [2]. Our main result is that the solution of the mixed problem above is an analytic function of $t > 0$; see Theorem 4.4 below. This result is similar to those of Ōuchi, but he only considers the case where β is a polynomial.

$W^{k,p}(\Omega; \boldsymbol{R})$ (resp. $W^{k,p}(\Omega; \boldsymbol{C})$) is the Sobolev space of real-valued (resp. complex-valued) functions whose derivatives up to order k lie in $L^p(\Omega; \boldsymbol{R})$ (resp. $L^p(\Omega; \boldsymbol{C})$). We write $W^{k,p}(\Omega)$ if it is clear from the context whether $\boldsymbol{R}$ or $\boldsymbol{C}$ is intended. The norm in $W^{k,p}(\Omega)$ (resp. $L^p(\Omega)$) is denoted by $\|\ \|_{k,p}$ (resp. $\|\ \|_p$). $W_0^{k,p}(\Omega)$ is the closure of $C_0^\infty(\Omega^0)$ in the space $W^{k,p}(\Omega)$. Here Ω^0 is the interior of Ω. If u is a function, then $\beta(u) = \beta \circ u$ is the composition of β and u.

For $1 < p < \infty$, let $D(T_p) = W^{2,p}(\Omega; \boldsymbol{C}) \cap W_0^{1,p}(\Omega)$ and, for $p = 1$, let $D(T_1) = \{u \in W^{1,1}(\Omega; \boldsymbol{C}): Lu \in L^1(\Omega)\}$, where Lu is understood in the sense of distributions. Let $T_p u = Lu$ for $u \in D(T_p)$, $1 \leqq p < \infty$, For $1 \leqq p < \infty$, let $D(A_p) = \{u \in L^p(\Omega; \boldsymbol{R}): u \in D(T_p),\ \beta(u) \in L^p(\Omega)\}$, and $A_p u = T_p u + \beta(u)$, $u \in D(A_p)$.

PROPOSITION 4.1. *If $1 < p < \infty$ and $k \in \boldsymbol{R}$ is sufficiently large, then $X = L^p(\Omega, \boldsymbol{C})$ and $T = T_p + kI$ satisfy* (3.1) *and there exists a constant C_p such that $\| u \|_{2,p} \leqq C_p \| Tu \|_p$, $u \in D(T)$. If $0 < \alpha \leqq 1$ and $p^{-1} - 2\alpha n^{-1} < q^{-1}$ then $X_\alpha \equiv D(T^\alpha)$ is continuously imbedded in $L^q(\Omega)$ (or $C(\Omega)$ if $q = \infty$; $q = \infty$ corresponds to $n/2p < \alpha \leqq 1$).*

Let $D(F) = \{u \in X_\alpha : u(x) \in D(\beta),\ x \in \Omega\}$ and $Fu = ku - \beta(u)$, $u \in D(F)$. If $n/2p < \alpha < 1$ then $D(F) \subset C(\Omega)$ and $\beta(u) \in C(\Omega)$ for each $u \in D(F)$. Furthermore X, T, α and F satisfy the hypotheses of Theorem 3.1. *Let $R > 0$ and Δ be a compact subset of $D(\beta)$ and $U = \{u \in W^{2,p}(\Omega; \boldsymbol{C}) : \| u \|_{2,p} < R;\ u(x) \in \Delta,\ x \in \Omega\}$. Then U also satisfies the hypotheses of Theorem* 3.1.

Proof. The assertions in the first sentence are well known, see Sobolevskii [23, p. 54] and Friedman [8, p. 101]. If $p^{-1} - 2\alpha n^{-1} < q^{-1}$ then it follows from Friedman [8, Theorems 10.1, 11.1] that $W^{2,p}(\Omega) \subset L^q(\Omega)$ (or $W^{2,p}(\Omega) \subset C(\Omega)$ if $q = \infty$) and there is $\mu < \alpha$ and C such that $\| u \|_q \leqq C \| u \|_{2,p}^{\mu} \| u \|_p^{1-\mu}$, $u \in W^{2,p}(\Omega)$. Thus $\| u \|_q \leqq C \| Tu \|_p^{\mu} \| u \|_p^{1-\mu}$, $u \in D(T)$. Thus $X_\alpha \subset L^q(\Omega)$ follows from (3.2).

Now let $n/2p < \alpha < 1$. The fact that $D(F) \subset C(\Omega)$ follows from the first part of the proposition, and $\beta(u) \in C(\Omega)$, $u \in D(F)$ follows from the fact that β is continuous. To show that $D(F)$ is open in X_α, let $u \in D(F)$. Then $u(\Omega) \equiv \{u(x) : x \in \Omega\}$ is compact and contained in $D(\beta)$ which is open. Thus, the distance, δ, from $u(\Omega)$ to $\boldsymbol{C} \backslash D(\beta)$ is greater than 0. It follows that $v(x) \in D(\beta)$ if $\| v - u \|_\infty < \delta$. Since $X_\alpha \subset C(\Omega)$ one has $\| v - u \|_\infty < \delta$ if the X_α norm of $v - u$ is sufficiently small. Thus $D(F)$ is open in X_α.

To show $F: D(F) \to X$ is analytic, it sufficies to show $\| F(u + h) - F(u) - (kh - \beta'(u)h) \|_p \leqq \varepsilon(h) \| T^\alpha h \|$ where $\varepsilon(h) \to 0$ as $\| T^\alpha h \| \to 0$. In view of the imbeddings $X_\alpha \subset C(\Omega) \subset X$, it suffices to show $\| \beta(u+h) - \beta(u) - \beta'(u)h \|_\infty \leqq \varepsilon(h) \| h \|_\infty$ where $\varepsilon(h) \to 0$ as $\| h \|_\infty \to 0$. By writing $\beta(\eta + \xi) - \beta(\eta)$ as the integral of β', one can show $| \beta(\eta + \xi) - \beta(\eta) - \beta'(\eta)\xi | \leqq \varepsilon(| \xi |) | \xi |$, $\eta \in u(\Omega)$, where $\varepsilon(| \xi |) \to 0$ as $| \xi | \to 0$ and $\varepsilon(| \xi |)$ is independent of $\eta \in u(\Omega)$. Replacing η by $u(x)$ and ξ by $h(x)$ and taking the supremum over Ω, one obtains the desired result.

Note that U is a bounded subset of $D(T) = X_1$. Since $\Delta \subset D(\beta)$ is compact, there exists $\rho > 0$ such that $\Delta_1 = \{z + \zeta : z \in \Delta,\ | \zeta | \leqq \rho\} \subset D(\beta)$. Using an argument similar to the proof that $D(F)$ is open in X_α, one can find a $\delta > 0$ such that if $u \in U$ and the X_α norm of $v - u$ is less than δ then $v(x) \in \Delta_1$, $x \in \Omega$, and hence, $v \in D(F)$. One has $\| Fv \| \leqq K$ since β is bounded on Δ_1.

PROPOSITION 4.2. *If $k \in \boldsymbol{R}$ is sufficiently large, then $(I + \lambda(A_p + k))^{-1}$ exists and is a contraction in the norm of $L^p(\Omega)$ and the range of $I + \lambda(A_p + k)$ is $L^p(\Omega; \boldsymbol{R})$ for $1 \leqq p < \infty$, $\lambda > 0$. Furthermore*

$\| \beta(u) \|_p \leqq \| (A_p + k)u \|_p$, $\| (T_p + k)u \| \leqq 2 \| (A_p + k)u \|_p$, $u \in D(A_p)$. *If* $\gamma: \boldsymbol{R} \to \boldsymbol{R}$ *is increasing and continuous with* $\gamma(0) = 0$, $p^{-1} + q^{-1} = 1$, $u \in D(T_p) \cap L^p(\Omega; \boldsymbol{R})$ *and* $\gamma(u) \in L^q(\Omega)$ *then* $\int_\Omega (T_p u + ku)\gamma(u)dx \geqq 0$.

Proof. Most of the assertions follow from the results of Brezis and Strauss [4], so we are quite brief and only indicate how to apply their results. Let k be such that $a(x) + k \geqq 0$ and $a(x) + \sum_j \partial_j a_j(x) + k \geqq 0$, $x \in \Omega$. Then the operator $L + k$ satisfies the hypotheses of Theorem 8 of [4]. Thus $T_1 + k$ (when restricted to $D(T_1) \cap L^1(\Omega; \boldsymbol{R})$) satisfies Proposition 7 of [4], and Lemma 3* of [4] can be applied to $(I + \lambda(T_1 + k))^{-1}$. It follows that the range of $I + \lambda(A_1 + k)$ is $L^1(\Omega; \boldsymbol{R})$, $(I + \lambda(A_1 + k))^{-1}$ exists and it is a contraction with respect to any norm $\| \quad \|_p$, $1 \leqq p < \infty$. In particular, $(I + \lambda(A_1 + k))^{-1}$ maps $L^p(\Omega; \boldsymbol{R})$ into $D(A_1) \cap L^p(\Omega)$. Since A_1 is an extension of A_p, $(I + \lambda(A_p + k))^{-1}$ exists and is a contraction in the norm $\| \quad \|_p$, $1 \leqq p < \infty$. We still need to show that the range of $I + \lambda(A_p + k)$ is $L^p(\Omega)$. Note that the linear operator $\lambda(T_1 + k)$ and the monotone function $u \to u + \lambda\beta(u)$ satisfy the hypotheses of Theorem 1 of [4]. Let $f \in L^p(\Omega; \boldsymbol{R})$ and $u = (I + \lambda(A_1 + k))^{-1}f$. As noted above Lemma 3* of [4] implies $u \in L^p(\Omega) \cap D(A_1)$, and Proposition 4 of [4] implies $u + \lambda\beta(u) \in L^p(\Omega)$, and, hence, $\beta(u)$ and $T_1 u$ belong to $L^p(\Omega)$. Using regularity theorems [1] for linear elliptic operators we conclude $u \in W^{2,p}(\Omega)$, and, hence, $u \in D(A_p)$. Thus, the range of $I + \lambda(A_p + k)$ is $L^p(\Omega)$.

To prove the last part of the proposition, note that $T_1 + k$ satisfies the hypotheses of Theorem 1 of [4]. Let $u \in D(A_p)$ and $f = (A_p + k)u$. By Proposition 4 of [4] we have $\| \beta(u) \|_p \leqq \| (A_p + k)u \|_p$ and, hence, $\| (T_p + k)u \|_p \leqq 2 \| (A_p + k)u \|_p$. Using Lemma 2 of [4] we get $\int_\Omega (T_p u + ku)\gamma(u)dx \geqq 0$.

PROPOSITION 4.3. *Let k be such that Propositions* 4.1 *and* 4.2 *are true.*

(1) *If* $\varphi \in L^1(\Omega; \boldsymbol{R})$ *then* $\lim_{(n\to\infty)}(I + (t/n)A_1)^{-n}\varphi \equiv u(t) \equiv S(t)\varphi$ *exists in* $L^1(\Omega)$ *for all* $t \geqq 0$. *If* $\varphi \in L^p(\Omega; \boldsymbol{R})$ *for some* $p, 1 \leqq p < \infty$, *then this limit exists in* $L^p(\Omega)$, $u: [0, \infty) \to L^p(\Omega)$ *is continuous and* $S(t): L^p(\Omega) \to L^p(\Omega)$ *is Lipschitz with constant* e^{kt}. *In particular,* $\| u(t) \|_p \leqq e^{kt} \| \varphi \|_p$.

(2) *If* $1 < p < \infty$ *and* $\varphi \in D(A_p)$ *then* $u(t) \in D(A_p)$, $t \geqq 0$, $u: [0, \infty) \to L^p(\Omega)$ *is absolutely continuous, the right derivative,* $D_r u(t)$ *exists and is equal to* $-A_p u(t)$ *for all* $t \geqq 0$, *and* $\| A_p u(t) \|_p \leqq e^{kt} \| A_p \varphi \|_p$.

(3) *If* $n/2p < \alpha < 1$ *and* $u(t_0) \in D((T_p + k)^\alpha) \cap L^p(\Omega; \boldsymbol{R})$ *for some* $t_0 \geqq 0$, *then* $u: (t_0, \infty) \to W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ *is analytic.*

Proof. The first part of Proposition 4.2 says that $A_p + kI$ is

m-accretive as defined by Kato [14, p. 138]. The assertions in part (1) are a direct application of the results of Crandall and Liggett [5, Theorem 1]. The fact that $\beta(0) = 0$ implies $A_1\psi = 0$ for $\psi = 0$. Thus $S(t)\psi = 0$ if $\psi = 0$. This fact combined with the fact that $S(t)$ has Lipschitz constant e^{kt} proves $\| u(t) \|_p \leqq e^{kt} \| \varphi \|_p$.

If $1 < p < \infty$ then $L^p(\Omega)$ and its dual are uniformly convex and, if $\varphi \in D(A_p)$, the results of Kato [14, Theorems 7.1, 7.5 and first line of last paragraph of p. 147] imply u has the properties in (2). (Note that the solution constructed by Kato in [14, Theorem 7.1, 7.5] coincides with $u(t)$ by virtue of [5, Theorem 2].)

To prove (3), let $n/2p < \alpha < 1$ and $u(t_0) \in D((T_p + k)^\alpha) \cap L^p(\Omega; \boldsymbol{R})$. By Proposition 4.1 and Theorem 3.1 there exists $r > 0$ and a continuous function $v: [t_0, t_0 + r) \to L^p(\Omega)$ such that $v: (t_0, t_0 + r) \to W^{2,p}(\Omega)$ is analytic, $v_t + (T_p + k)v = kv - \beta(v)$, $t_0 < t < t_0 + r$, and $v(t_0) = u(t_0)$. Since v satisfies Definition 2.2 of [5] for being a strong solution of $v_t + A_p v = 0$, $v(t_0) = u(t_0)$, it follows from Theorem 2 of [5] that $v = u$ on $[t_0, t_0 + r)$. In particular, $u(t) \in D(A_p)$ for $t_0 < t < t_0 + r$. By part (2), $u(t) \in D(A_p)$, $t_0 < t < \infty$, and $\| A_p u(t) \|_p$ is bounded for t in any interval of the form $t_1 \leqq t \leqq t_2$ where $t_0 < t_1 < t_2 < \infty$. By Propositions 4.1 and 4.2, $\| T_p u(t) \|_p$, $\| u(t) \|_{2,p}$ and $\| u(t) \|_\infty$ are also bounded for $t_1 \leqq t \leqq t_2$. Therefore $\Delta = \{u(t)(x): x \in \Omega, t_1 \leqq t \leqq t_2\}$ is a bounded subset of $\boldsymbol{R}$. Again using Proposition 4.1 and Theorem 3.1, one sees that there exists $r > 0$ such that for any $t_3 \in [t_1, t_2]$ there is a continuous function $v: [t_3, t_3 + r) \to L^p(\Omega)$ such that $v: (t_3, t_3 + r) \to W^{2,p}(\Omega)$ is analytic $v_t + A_p v(t) = 0$, $t_3 < t < t_3 + r$, and $v(t_3) = u(t_3)$. As above, it follows from Theorem 2 of [5] that $u = v$ on $[t_3, t_3 + r)$. Since r is independent of $t_3 \in [t_1, t_2]$, it follows that $u: (t_1, t_2) \to W^{2,p}(\Omega)$ is analytic. Since t_1, t_2 are arbitrary, it follows that $u: (t_0, \infty) \to W^{2,p}(\Omega)$ is analytic.

THEOREM 4.4. *Let $\varphi \in W^{2,p}(\Omega; \boldsymbol{R}) \cap W_0^{1,p}(\Omega)$ and $\beta(\varphi) \in L^p(\Omega)$,* i.e. *$\varphi \in D(A_p)$, for some p, $1 < p < \infty$. Then there exists a differentiable function $u: [0, \infty) \to L^p(\Omega; \boldsymbol{R})$ such that $u: (0, \infty) \to W^{2,q}(\Omega; \boldsymbol{R}) \cap W_0^{1,q}(\Omega)$ is analytic for all q, $1 \leqq q < \infty$, $u_t + Lu + \beta(u) = 0$, $0 \leqq t < \infty$, and $u(0) = \varphi$. In fact $u(t) = S(t)\varphi$ is constructed from φ by Proposition* 4.3.

The proof of this theorem uses the a priori inequality in the following lemma. The authors wish to thank Professor H. Brezis for many helpful suggestions regarding this inequality.

LEMMA 4.5. *Let k be such that Propositions* 4.1 *and* 4.2 *are true. Let $1 < p \leqq q < \infty$, $0 \leqq \alpha < 1 - q^{-1}$, $0 < \varepsilon < \tau$. Then there is an increasing function $l: (0, \infty) \to (0, \infty)$ such that if $\varphi \in W^{2,r}(\Omega: \boldsymbol{R}) \cap$*

$W_0^{1,r}(\Omega) = D(T_r) = D(A_r)$ *for some* $r \geqq q$, $r > n/2$ *then* $\|(T_q + k)^\alpha u(t)\|_q \leqq l(\|A_p\varphi\|_p + \|\varphi\|_p)$, $\varepsilon \leqq t \leqq \tau$, *where* $u(t) = S(t)\varphi$ *is obtained from* φ *by Proposition* 4.3.

Proof of Lemma 4.5. It follows from Proposition 4.3 that $u: (0, \infty) \to W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$ is analytic, $u: [0, \infty) \to L^r(\Omega)$ is differentiable, $\|A_r u(t)\|_r$ is bounded for t lying in any bounded interval and $u_t + (T_r + k)u = ku - \beta(u)$ holds for all $t \geqq 0$. From Proposition 4.1 and 4.2 it follows that $\|\beta(u(t))\|_r$, $\|T_r u(t)\|_r$ and $\|u(t)\|_{2,r}$, are bounded for t lying in any bounded interval. According to Proposition 4.1, the map $u \to \beta(u)$ is analytic from (an open subset of) $W^{2,r}(\Omega; C)$ to $L^r(\Omega)$. Thus $t \to \beta(u(t))$ is an analytic function from $(0, \infty)$ to $L^r(\Omega)$ and bounded for t lying in any bounded interval.

For $1 < \rho \leqq r$ we may apply inequality (3.5) with $X = L^\rho(\Omega)$ and $T = T_\rho + k$ to obtain

$$\|(T_\rho + k)^\mu u(t)\|_\rho \leqq C\left[\|u(\sigma)\|_\rho + \left(\int_\sigma^\tau \|ku - \beta(u)\|_\rho^\rho \, dt\right)^{1/\rho}\right],$$

$\sigma + \varepsilon/2 \leqq t \leqq \tau$, $0 \leqq \mu < 1 - \rho^{-1}$. Using Minkowski's inequality on the integral and estimating $\|u(t)\|_\rho$ in terms of $\|u(\sigma)\|_\rho$ (by Proposition 4.3) one obtains

$$\|(T_\rho + k)^\mu u(t)\|_\rho \leqq C\left[\|u(\sigma)\|_\rho + \left(\int_\sigma^\tau \|\beta(u)\|_\rho^\rho \, dt\right)^{1/\rho}\right], \tag{4.1}$$

$\sigma + \varepsilon/2 \leqq t \leqq \tau$, $0 \leqq \mu < 1 - \rho^{-1}$. Applying Proposition 4.1 to the left side, one obtains

$$\|u(t)\|_s \leqq C\left[\|u(\sigma)\|_\rho + \left(\int_\sigma^\tau \|\beta(u)\|_\rho^\rho \, dt\right)^{1/\rho}\right], \tag{4.2}$$

$\sigma + \varepsilon/2 \leqq t \leqq \tau$, $\rho^{-1} \geqq s^{-1} > \rho^{-1} - 2\mu n^{-1} > \rho^{-1} - 2n^{-1}(1 - \rho^{-1})$. This is equivalent to $\rho \leqq s < \rho\,[1 - 2n^{-1}(\rho - 1)]^{-1}$ if $1 - 2n^{-1}(\rho - 1) \geqq 0$, and to $\rho \leqq s \leqq \infty$ if $1 - 2n^{-1}(\rho - 1) < 0$.

We now show that there is an increasing function $l: (0, \infty) \to (0, \infty)$ such that

$$\|u(t)\|_q + \int_{\sigma+\varepsilon}^\tau \|\beta(u)\|_q^q \, dt \leqq l(\|u(\sigma)\|_p + \int_\sigma^\tau \|\beta(u)\|_p^p \, dt), \tag{4.3}$$

$\sigma + \varepsilon \leqq t \leqq \tau$. Let $\gamma(\xi) = |\beta(\xi)|^{q-2}\beta(\xi)$, $\xi \in \boldsymbol{R}$. Multiplying the equation $\beta(u) = -u_t - (T_q + k)u + ku$ by $\gamma(u)$, integrating over Ω, and using Proposition 4.2 $ku\gamma(u) \leqq C|u|^q + 2^{-1}|\beta(u)|^q$, one obtains $\|\beta(u)\|_q^q \leqq -2\int u_t\gamma(u)dx + C\|u\|_q^q$, $0 \leqq t < \infty$. Let $\zeta: \boldsymbol{R} \to \boldsymbol{R}$ be smooth, $0 \leqq \zeta \leqq 1$, $\zeta = 0$ on $(-\infty, \sigma + \varepsilon/2]$, and $\zeta = 1$ on $[\sigma + \varepsilon, \infty)$. Multiplying the above inequality by ζ and integrating from σ to τ, one obtains

$$(4.4) \qquad \int_{\sigma+\varepsilon}^{\tau} \| \beta(u) \|_q^q \, dt \leqq -2\int_{\sigma}^{\tau} \zeta(t) \int u_t \gamma(u) \, dx \, dt + C\int_{\sigma+\varepsilon/2}^{\tau} \| u \|_q^q \, dt \, .$$

Let $\Gamma(\eta) = \int_0^\eta \gamma(\xi) d\xi$, $\eta \in \boldsymbol{R}$. Then $\Gamma' = \gamma$, $\Gamma(0) = 0$, $\Gamma \geqq 0$. Since Γ is convex, we have $\Gamma(0) - \Gamma(\eta) \geqq \gamma(\eta)(0 - \eta)$ i.e. $\Gamma(\eta) \leqq \gamma(\eta)\eta$. Using the same argument that was used in the proof of Proposition 4.1, one can show that the map $G: u \longrightarrow \Gamma(u)$ is Fréchet differentiable from $W^{2,r}(\Omega; \boldsymbol{R})$ to $L^r(\Omega)$, and its differential is given by $DG(u)v = \gamma(u)v$. Therefore the map $t \longrightarrow \Gamma(u(t))$ is differentiable from $(0, \infty)$ to $L^r(\Omega)$ and its derivative is $\gamma(u(t))u_t(t)$. Thus $\int \gamma(u)u_t dx = (d/dt) \int \Gamma(u)dx$. If we integrate the first term on the right of (4.4) by parts, we get $\int_\sigma^\tau \zeta'(t) \int \Gamma(u) dx \, dt - \int \Gamma(u(\tau)) dx$ (since $\zeta(\tau) = 1$, $\zeta(\sigma) = 0$). Using the fact that $\Gamma \geqq 0$ and $\Gamma(\eta) \leqq | \beta(\eta) |^{q-2}\beta(\eta)\eta$, one sees that the preceding integrals are dominated by $C\int_{\sigma+\varepsilon/2}^{\tau} \int | \beta(u) |^{q-1} | u | \, dxdt$. Applying Hölders inequality, one sees that this integral is dominated by $C \int_{\sigma+\varepsilon/2}^{\tau} \| \beta(u) \|_{(q-1)a}^{q-1} | u \|_b \, dt$, where $a^{-1} + b^{-1} = 1$. Using $xy \leqq a^{-1}x^a + b^{-1}y^b$, one sees that this is dominated by $C\int_{\sigma+\varepsilon/2}^{\tau} \| \beta(u) \|_{(q-1)a}^{(q-1)a} dt + C\int_{\sigma+\varepsilon/2}^{\tau} \| u \|_b^b \, dt$. Let p be fixed and choose b so that (4.2) holds with ρ replaced by p, i.e. $0 \leqq p^{-1} - b^{-1} < \min \{2n^{-1}(1 - p^{-1}), p^{-1}\}$. Then choose q so that $(q - 1)a = p$, i.e. $q = p(1 + p^{-1} - b^{-1})$. This implies $p \leqq q < \min \{p + 1, p + 2n^{-1}(p - 1)\}$. Then the integrals above are dominated by $l(\| u(\sigma) \|_p + \int_\sigma^\tau \| \beta(u) \|_p^p dt)$ where $l: (0, \infty) \longrightarrow (0, \infty)$ is increasing. Putting this together with (4.4) gives

$$(4.5) \quad \int_{\sigma+\varepsilon}^{\tau} \| \beta(u) \|_q^q \, dt \leqq l(\| u(\sigma) \|_p + \int_\sigma^\tau \| \beta(u) \|_p^p \, dt) + C\int_{\sigma+\varepsilon/2}^{\tau} \| u \|_q^q \, dt \, .$$

We restrict q so that (4.2) holds with s replaced by q and ρ replaced by p. Then the second term on the right of (4.5) can be estimated by the first term and we obtain the desired inequality (4.3) for $p \leqq q < \min \{p + 1, p + 2n^{-1}(p - 1), p[1 + 2n^{-1}(p - 1)]^{-1}\}$. However, we may now proceed to argue inductively on p and q to obtain (4.3) for all p, q, $1 < p \leqq q < \infty$.

To finish the proof of the lemma, note that Proposition 4.3 implies $\| (A_p + k)u(t) \|_p \leqq C(\| A_p\varphi \|_p + \| \varphi \|_p)$, $0 \leqq t \leqq \tau$. Combining this with Proposition 4.2, one obtains $\| \varphi \|_p + \int_0^\tau \| \beta(u) \|_p^p dt \leqq l(\| A_p\varphi \|_p + \| \varphi \|_p)$. Combining this with (4.3), one obtains $\| u(t) \|_q + \left(\int_{\varepsilon/2}^{\tau} \| \beta(u) \|_q^q dt\right)^{1/q} \leqq l(\| A_p\varphi \|_p + \| \varphi \|_p)$, $\varepsilon/2 \leqq t \leqq \tau$. Using (4.1) with ρ replaced by q and μ replaced by α, one obtains the inequality in the lemma.

Proof of Theorem 4.4 Since Ω is bounded it suffices to prove the theorem for all q sufficiently large. We choose q so large than

$n/2q < \alpha < 1 - q^{-1}$, and then pick α so that $n/2q < \alpha < 1 - q^{-1}$. For such q and α we can apply Proposition 4.3 (part (3)) and Lemma 4.5.

There exists a sequence $\{\varphi_n\} \subset W^{2,q}(\Omega; \boldsymbol{R}) \cap W_0^{1,q}(\Omega)$ such that $\varphi_n \to \varphi$ and $A_p\varphi_n \to \varphi$ in $L^p(\Omega)$. (For example, we can take $\varphi_n = (A_p + k + 1)^{-1}\psi_n = (A_q + k + 1)^{-1}\psi_n$ where $\{\psi_n\}$ is a sequence in $L^q(\Omega)$ with $\psi_n \to (A_p + k + 1)\varphi$ in $L^p(\Omega)$ and k is chosen so that Proposition 4.2 holds.) Let $u(t) = S(t)\varphi$ and $u_n = S(t)\varphi_n$ be constructed from φ and φ_n by Proposition 4.3. Since the $S(t)$ are Lipschitz maps, $u_n(t)$ converges to $u(t)$ in $L^p(\Omega)$. By Lemma 4.5, $\{(T_q + k)^\alpha u_n(t)\}$ is a bounded sequence in $L^q(\Omega)$, for fixed $t > 0$. Since $L^q(\Omega)$ is reflexive, there is a subsequence $\{u_{n_j}(t)\}$ such that $\{u_{n_j}(t)\}$ and $\{(T_q + k)^\alpha u_{n_j}(t)\}$ converge weakly in $L^q(\Omega)$, say $u_{n_j}(t) \rightharpoonup v$ and $(T_q + k)^\alpha u_{n_j}(t) \rightharpoonup w$ weakly in $L^q(\Omega)$. It follows that $\{(u_{n_j}(t), (T_q + k)^\alpha u_{n_j}(t))\}$ converges weakly to (v, w) in $L^q(\Omega) \times L^q(\Omega)$. Since the graph of $(T_q + k)^\alpha$ is closed (and, hence weakly closed), $v \in D((T_q + k)^\alpha)$. However, we must have $u(t) = v$, since $(u_{n_j}(t), \psi) \to (u(t), \psi)$ and $(u_{n_j}(t), \psi) \to (v, \psi)$ for every test function ψ. It follows that $u(t) \in D((T_q + k)^\alpha)$. From part (3) of Proposition 4.3 it follows that $u\colon (t, \infty) \to W^{2,q}(\Omega)$ is analytic. Since $t > 0$ is arbitrary, this proves the theorem.

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