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**REARRANGING FOURIER TRANSFORMS ON GROUPS**

CHUNG LIN

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Let  $G$  denote an infinite locally compact abelian group and  $X$  its character group. Let  $\theta$  be a suitable Haar measure on  $X$ , and  $1 < p < 2$ . For a  $\theta$ -measurable function  $\phi$  on  $X$ , we define  $\theta_\phi(t) = (\{\chi \in X: |\phi(\chi)| > t\})$  and  $\phi^*(x) = \inf \{t > 0: \theta_\phi(t) \leq x\}$  for  $x > 0$ .  $\phi^*$  is called the nonincreasing rearrangement of  $\phi$ . Note that even though  $\phi$  is defined on  $X$ , the domain of  $\phi^*$  is  $(0, \infty)$ . A nonnegative function  $g$  defined on  $(0, \infty)$  is called admissible if  $g$  is nonincreasing and  $\lim_{x \rightarrow \infty} g(x) = 0$ .

**Theorems:**

1. Let  $G$  be nondiscrete with a compact open subgroup and  $g$  admissible. Then  $g|_N = \hat{f}^*|_N$ , where  $N$  is the set of positive integers, for some  $f \in L^p(G)$  if  $\sum_{k=1}^{\infty} g(k)^p k^{p-2} < \infty$ .

2. Let  $G$  be nondiscrete with no compact open subgroup and  $g$  admissible. Then  $g = \hat{f}^*m$  a.e. for some  $f \in L^p(G)$  if  $\int_0^{\infty} g(x)^p x^{p-2} dx < \infty$ .

3. Let  $G$  be an infinite discrete abelian group which contains  $Z, Z(r^\infty)$  or  $Z(r)^{\aleph_0}$  as a subgroup,  $g$  admissible. Then  $g|_{(0,1)} = \hat{f}^*|_{(0,1)}m$  a.e. for some  $f \in L^p(G)$  if  $\int_0^1 g(x)^p x^{p-2} dx < \infty$ .

**I. Introduction.** As usual the Fourier transform  $\hat{f}$  of a function  $f \in L^1(G)$  is defined on  $X$  such that  $\hat{f}(\chi) = \int_G f\chi d\lambda$ , where  $\lambda$  is a fixed but arbitrary Haar measure on  $G$ . For  $1 < p < 2$ ,  $\hat{f} \in L^{p'}(G)$  and  $p'$  is the conjugate exponent of  $p$ . The set of real numbers,  $n$ -dimensional Euclidean space, the circle group, the integers, the  $r$ -adic integers, the countable product of the group of integers modulo  $r$  and the subgroup of the circle whose elements have order a power of  $r$  are denoted by  $R, R^n, T, Z, A_r, \prod Z(r)$  and  $Z(r^\infty)$ , respectively. Also  $p$  will denote any number such that  $1 < p < 2$ . Let  $m$  be  $1/\sqrt{2\pi}$  Lebesgue measure on  $R$ .

Hardy and Littlewood [1], [2] characterized functions on  $Z$  such that every rearrangement is the Fourier transform of a function in  $L^p(T)$ ,  $2 < p < \infty$ . They also characterized functions on  $Z$  such that some rearrangement is the Fourier transform of a function in  $L^p(T)$ ,  $1 < p < 2$ . Hewitt and Ross [4] generalized these results to arbitrary compact infinite abelian groups. We are interested in the case of LCA (locally compact abelian) groups. Here are our results.

**THEOREM 1.** *Let  $G$  be nondiscrete with a compact open subgroup,*

and  $g$  an admissible function. Then  $g|_N = \hat{f}^*|_N$  for some  $f \in L^p(G)$  if and only if  $\sum_{k=1}^{\infty} g(k)^p k^{p-2} < \infty$ . Moreover, there exists a constant  $A_p$  that depends on  $p$  only such that

$$\left( \sum_{k=1}^{\infty} g(k)^p k^{p-2} \right)^{1/p} \leq A_p \|f\|_p$$

for every such  $f$ .

**THEOREM 2.** Let  $G$  be a nondiscrete LCA group with no compact open subgroup and  $g$  an admissible function. Then  $g = \hat{f}^*$  for some  $f \in L^p(G)$  if and only if  $\int_0^{\infty} g(x)^p x^{p-2} dx < \infty$ . Moreover, there exists  $A_p$  that depends only on  $p$  such that

$$\left( \int_0^{\infty} g(x)^p x^{p-2} dx \right)^{1/p} \leq A_p \|f\|_p$$

for every such  $f$ .

**THEOREM 3.** Let  $G$  be an infinite discrete abelian group containing  $Z$ ,  $Z(r^{\infty})$  or  $Z(r)_*^{\infty}$  as a subgroup and  $g$  an admissible function. Then  $g|_{(0,1)} = \hat{f}^*|_{(0,1)}$  for some  $f \in L^p(G)$  if and only if  $\int_0^1 g(x)^p x^{p-2} dx < \infty$ . Moreover there exists  $A_p$  that depends only on  $p$  such that

$$\left( \int_0^1 g(x)^p x^{p-2} dx \right)^{1/p} \leq A_p \|f\|_p$$

for every such  $f$ .

Theorems 1 and 2 give us a complete solution for all nondiscrete LCA groups. Theorem 3 holds for "almost all" discrete abelian groups, but I am not able to settle the case where  $G$  contains  $\prod_{n=1}^{\infty} Z(r_n)$  as a subgroup, with  $r_n \rightarrow \infty$ .

The forward implications "= $\Rightarrow$ " of all three theorems and the existence of the constants  $A_p$  are due to Hunt [5]; see Stein and Weiss [6], Chapter V, Corollary 3.16.

## II. A few lemmas.

**LEMMA 1.** Let  $G$  be a LCA group and  $H$  an open subgroup of  $G$ . Let  $H^{\perp} = \{\chi \in X: \chi = 1 \text{ on } H\}$ . Then for each  $f_0 \in L^p(H)$ , there exists  $f \in L^p(G)$  such that  $\hat{f}^* = \hat{f}_0^* m$  a.e. (where we use suitable Haar measures on  $X$  and  $X/H^{\perp}$  for the definitions of  $\hat{f}^*$  and  $\hat{f}_0^*$ ).

*Proof.* Let  $f_0 \in L^p(H)$  and define  $f(x) = f_0(x)$  if  $x \in H$  and  $f(x) = 0$  otherwise. Since  $H$  is open,  $f$  is still  $\lambda$ -measurable in  $G$

and  $f \in L^p(G)$ . Choose Haar measure  $\lambda_H$  on  $H$  to be the restriction of  $\lambda$  to  $H$ . Choose  $\theta_{H^\perp}$  to be the normalized Haar measure on  $H^\perp$ , and  $\theta_X$  to be an arbitrary Haar measure on  $X$ . Then a Haar measure  $\theta_1$  on  $X/H^\perp$  exists so that Weil's theorem applies [3; Vol. II, 28.54].  $\hat{f}$  is clearly constant on each coset of  $H^\perp$ . That is,  $\hat{f}(\chi) = \hat{f}_0(\chi H^\perp)$  for all  $\chi \in X$ . A calculation, using Weil's theorem shows that  $\hat{f}^* = \hat{f}_0^* m$  a.e.

For the rest of this paper, we let  $g$  be a fixed admissible function on  $(0, \infty)$ ,  $1 < p < 2$  and  $\int_0^\infty g(x)^p x^{p-2} dx$  is finite.

LEMMA 2. (i)  $\int_0^1 g(ct) dm(t) < \infty$  for all  $c > 0$ .  
 (ii)  $0 \leq \int_0^\infty g(ct) \sin xt dm(t) \leq \int_0^{\pi/x} g(ct) \sin xt dm(t) < \infty$  for all  $x > 0, c > 0$ .

Proof. (i) Since

$$\begin{aligned} \int_0^1 g(ct)^p dm(t) &\leq \int_0^1 g(ct)^p t^{p-2} dm(t) \leq \int_0^\infty g(ct)^p t^{p-2} dm(t) \\ &= \frac{1}{c^{p-1}} \int_0^\infty g(t)^p t^{p-2} dm(t) < \infty, \end{aligned}$$

we see that  $\int_0^1 g(ct)^p dm(t)$  is finite and hence  $\int_0^1 g(ct) dm(t)$  is finite.

(ii) For  $k = 1, 2, \dots$ , let

$$\nu_k = (-1)^{k+1} \int_{(k-1)\pi/x}^{k\pi/x} g(ct) \sin xt dm(t).$$

It is clear that  $\nu_1 \geq \nu_2 \geq \nu_3 \geq \dots \geq 0$  and  $\nu_k \rightarrow 0$ .

It follows that

$$\int_0^\infty g(ct) \sin xt dt = \sum_{k=1}^\infty (-1)^{k+1} \nu_k$$

and hence

$$0 \leq \int_0^\infty g(ct) \sin xt dt \leq \nu_1 = \int_0^{\pi/x} g(ct) \sin xt dm(t) < \infty.$$

This completes the proof of Lemma 2.

Define  $G_c(x) = \int_0^{|x|} g(ct) dm(t)$  for  $x \in R$ . This is well-defined because  $\int_0^1 g(ct) dm(t) < \infty$  by (i) of Lemma 2 and  $g$  is bounded in between 1 and  $|x|$ .

LEMMA 3. (i)  $G_c(x) = o(x^{1/p})$  as  $x \rightarrow 0$  and as  $x \rightarrow \infty$ .

(ii)  $\int_0^\infty G_c(x)^p x^{-2} dm(x) < \infty$  for all  $c > 0$ .

*Proof.* See [7], Vol. I, Ch. I, §9.16.

LEMMA 4. *There exists  $f \in L^p(\mathbf{R})$  such that  $\hat{f}^* = gm$  a.e.*

*Proof.* Define, for  $x \in \mathbf{R}$

$$\varphi(x) = \int_0^\infty g(2t) \sin xt \, dm(t).$$

Then, by part (ii) of Lemma 2  $0 \leq \varphi(x) \leq G_2(\pi/x)$ , for  $x > 0$ , because  $0 \leq \varphi(x) \leq \int_0^{\pi/x} g(2t) \sin xt \, dm(t) \leq \int_0^{\pi/x} g(2t) \, dm(t) = G_2(\pi/x)$ . Since  $G_2$  is an even function, we have that  $|\varphi(x)| \leq G_2(\pi/x)$  for all  $x \in \mathbf{R} \setminus \{0\}$ . Part (ii) of Lemma 3 says that  $G_2(\pi/x) \in L^p(\mathbf{R})$ . It follows then that  $\varphi \in L^p(\mathbf{R})$ . Define, for  $n \in \mathbf{N}$ ,

$$\varphi_n(x) = \int_0^n g(2t) \sin xt \, dm(t) \quad (x \in \mathbf{R}).$$

Let  $x > 0$ . For each  $n$ , choose  $m \in \mathbf{N}$  such that  $|2m\pi/x - n| \leq \pi/x$ . Then

$$\begin{aligned} |\varphi_n(x)| &\leq \int_0^{2m\pi/x} g(2t) \sin xt \, dm(t) + \left| \int_{2m\pi/x}^n g(2t) \sin xt \, dm(t) \right| \\ &\leq \int_0^\infty g(2t) \sin xt \, dm(t) + g\left(\frac{2(2m-1)\pi}{x}\right) \left| \frac{2m\pi}{x} - n \right| \\ &\leq \varphi(x) + g\left(\frac{2\pi}{x}\right) \frac{\pi}{x} \leq \varphi(x) + \int_0^{\pi/x} g(2t) \, dm(t) \\ &= \varphi(x) + G_2\left(\frac{\pi}{x}\right). \end{aligned}$$

This shows that  $|\varphi_n(x)| \leq |\varphi(x)| + |G_2(\pi/x)|$  for all  $x \in \mathbf{R} \setminus \{0\}$ . Since  $\varphi_n(x) \rightarrow \varphi(x)$  pointwise and  $\varphi(x), G_2(\pi/x) \in L^p(\mathbf{R})$ , we must have  $\|\varphi_n - \varphi\|_p \rightarrow 0$  by the dominated convergence theorem. So we can obtain  $\varphi$  by approximating  $\varphi_n$ . Let us compute  $\varphi_n$ :

$$\begin{aligned} 2i\varphi_n(x) &= 2i \int_0^n g(2t) \sin xt \, dm(t) = \int_0^n g(2t) (e^{-ixt} - e^{ixt}) \, dm(t) \\ &= \int_{\mathbf{R}} g(-2t) I_{[-n, 0]}(t) e^{-ixt} \, dm(t) \\ &\quad - \int_{\mathbf{R}} g(2t) I_{[0, n]}(t) e^{-ixt} \, dm(t). \end{aligned}$$

Recall that the Haar measure  $m$  on  $\mathbf{R}$  is chosen so that the inversion theorem holds. We know that  $g(2t)I_{[0, n]}(t)$  and  $g(-2t)I_{[-n, 0]}(t) \in$

$L^1(\mathbb{R})$  and  $\varphi_n \in L^p(\mathbb{R})$ . Hence, by [3; Vol. II, 31.44 (b)], we have

$$2i\varphi(x) = \begin{cases} -g(2x) & \text{if } x \geq 0 \\ g(-2x) & \text{if } x < 0 \end{cases} \quad m \text{ a.e.}$$

Now define  $f = 2i\varphi$  so that  $|\hat{f}(x)| = g(|2x|)$   $m$  a.e. It is then easy to check that  $\hat{f}^* = gm$  a.e., which is what we needed to prove.

**LEMMA 5.** *For each  $n \in \mathbb{N}$ , there exists  $f \in L^p(\mathbb{R}^n)$  such that  $\hat{f}^* = gm$  a.e.*

*Proof.* By Lemma 4, we may assume that  $n > 1$ . Define, for  $k \in \mathbb{N}$ ,

$$\begin{aligned} \varphi(x) &= \int_0^\infty g(2^n t) \sin xt \, dm(t) \\ \varphi_k(x) &= \int_0^k g(2^n t) \sin xt \, dm(t) \\ f(x_1, \dots, x_n) &= 2^n i\varphi(x_1) \frac{\sin x_2}{x_2} \dots \frac{\sin x_n}{x_n} \\ f_k(x_1, \dots, x_n) &= 2^n i\varphi_k(x_1) \frac{\sin x_2}{x_2} \dots \frac{\sin x_n}{x_n} \end{aligned}$$

Let  $m_n = m \times m \times \dots \times m$  on  $\mathbb{R}^n$ ,  $x = (x_1, \dots, x_n)$ . Then

$$\varphi(x), \varphi_k(x), \frac{\sin x}{x} \in L^p(\mathbb{R}) .$$

Therefore

$$\begin{aligned} &\int_{\mathbb{R}^n} |f_k - f|^p \, dm_n \\ &= 2^{np} \int_{\mathbb{R}^n} \left| \varphi_k(x_1) - \varphi(x_1) \right|^p \left| \frac{\sin x_2}{x_2} \right|^p \dots \left| \frac{\sin x_n}{x_n} \right|^p \, dm_n \\ &= 2^{np} \int_{\mathbb{R}} \left| \varphi_k - \varphi \right|^p \, dm \left( \int_{\mathbb{R}} \left| \frac{\sin x}{x} \right|^p \, dm \right)^{n-1} . \end{aligned}$$

As in the proof of Lemma 4, we have  $\|\varphi_k - \varphi\|_p \rightarrow 0$ , and so  $\|f_k - f\|_p \rightarrow 0$  in  $L^p(\mathbb{R}^n)$ . Straight forward calculations show that

$$\hat{f}_k(x_1, \dots, x_n) = \begin{cases} g(-2^n x_1) & \text{if } -k \leq x_1 < 0 \text{ and } x_j \in [-1, 1] \\ & \text{for } 2 \leq j \leq n \\ -g(2^n x_1) & \text{if } 0 \leq x_1 \leq k \text{ and } x_j \in [-1, 1] \\ & \text{for } 2 \leq j \leq n \\ 0 & \text{otherwise} \end{cases}$$

$m_n$  a.e. and hence

$$\hat{f}(x_1, \dots, x_n) = \begin{cases} g(-2^n x_1) & \text{if } x_1 < 0, |x_j| \leq 1, 2 \leq j \leq n \\ -g(2^n x_1) & \text{if } x_1 > 0, |x_j| \leq 1, 2 \leq j \leq n \\ 0 & \text{otherwise} \end{cases}$$

$m_n$  a.e. It follows that

$$m_n\{x \in R^n: |\hat{f}(x)| > t\} = 2^n m\{x_1 > 0: g(2^n x_1) > t\}.$$

This in turn shows that for  $x > 0$

$$\hat{f}^*(x) = \inf \{t > 0: 2^n m\{x_1 > 0: g(2^n x_1) > t\} \leq x\} = g(x)$$

$m$  a.e., which completes the proof of Lemma 5.

III. Proof for the nondiscrete case. Let  $G$  be an infinite LCA group. To prove Theorem 1 and Theorem 2, Lemma 1 and the structure theorem [3, Vol. I, 24.30] shows that we may assume  $G = K \times R^n$ , where  $K$  is a compact abelian group.

*Proof of Theorem 1.* In this  $n = 0$ , so that  $G = K$ . Then there exists  $f_0 \in L^p(K)$ , by [4], such that  $\hat{f}_0^*|_N = g|_N$ .

*Proof of Theorem 2.* In this case  $n > 0$ . By Lemma 5, there exists  $f_0 \in L^p(R^n)$  such that  $\hat{f}_0^* = gm$  a.e. Define  $f(x, y) = f_0(y)$  for  $x \in K$  and  $y \in R^n$ . Let  $m_n = m \times \dots \times m$  be the Haar measure on  $R^n$ ,  $\lambda_K$  be the normalized Haar measure on  $K$  and  $\lambda_{K \times R^n}$  the Haar measure on  $K \times R^n$  so that Weil's theorem holds. It follows that  $f$  is in  $L^p(K \times R^n)$  and  $\|f\|_p = \|f_0\|_p$ . Moreover, for  $\chi_1 \in \hat{K}$ ,  $\chi_2 \in R^n$ , we have

$$f(\chi_1 \chi_2) = \begin{cases} f_0(\chi_2) & \text{if } \chi_1 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Choose  $\theta_{\hat{K} \times R^n}$ ,  $\theta_{\hat{K}}$  and  $\theta_{R^n}$  the Haar measures on  $\hat{K} \times R^n$ ,  $\hat{K}$  and  $R^n$  respectively, so that Planchel's theorem holds. Then Weil's theorem holds for these measures by [3, 31.46(c)]. Clearly  $\theta_{\hat{K}}$  is the discrete measure on  $\hat{K}$ . Then for  $t > 0$

$$\begin{aligned} (\theta_{\hat{K} \times R^n})\hat{f}(t) &= \int_{\hat{K} \times R^n} I_{\{x: |\hat{f}(x)| > t\}} d\theta_{\hat{K} \times R^n} \\ &= \int_{R^n} \int_{\hat{K}} I_{\{x: |\hat{f}(x)| > t\}} d\theta_{\hat{K}} d\theta_{R^n} \\ &= \int_{R^n} I_{\{x: |\hat{f}_0(x)| > t\}} d\theta_{R^n} = (\theta_{R^n})\hat{f}_0(t), \end{aligned}$$

and it follows that for  $x > 0$ ,

$$\begin{aligned} \hat{f}^*(x) &= \inf \{t > 0: (\theta_{\hat{K} \times R^n})_{\hat{f}}(t) \leq x\} = \inf \{t > 0: (\theta_{R^n})_{\hat{f}_0}(t) \leq x\} \\ &= \hat{f}_0^*(x) = g(x)m \text{ a.e.} \end{aligned}$$

Note that Theorem 1 is essentially the theorem in [4].

IV. *Proof of Theorem 3.* For each  $n = 1, 2, \dots$ , let  $r_n$  be an integer  $\geq 2$ . Denote by  $\theta$  the normalized Haar measure on  $X = \prod_{n=1}^{\infty} Z(r_n)$  and  $\lambda$  the usual restriction of Lebesgue measure to  $[0, 1]$ . Define a function  $\varphi: X \rightarrow [0, 1]$  via

$$\varphi(\varepsilon) = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{p_1 p_2 \dots p_n} \quad \varepsilon = (\varepsilon_1, \dots, \dots) \in X.$$

Then  $g$  is measure preserving; in fact, the following is well known.

LEMMA 6. *E is measurable in X if and only if  $\varphi(E)$  is measurable in  $[0, 1]$ , and  $\theta(E) = \lambda(\varphi(E))$ .  $\varphi$  is an onto map and  $\varphi$  is one-to-one on X except for a countable set. Moreover,*

$$\int_x h \circ \varphi d\theta = \int_0^1 h d\lambda$$

for all bounded  $\lambda$  measurable functions  $h$  on  $[0, 1]$ .

LEMMA 7. *Theorem 3 is true if  $G \supset Z$ .*

*Proof.* By Lemma 1, we may assume  $G = Z$ . Define

$$a_0(n) = \frac{1}{2\pi} \int_0^{2\pi} g(t) \sin nt dt \quad \text{for } n \in Z.$$

The values of the integrals involved are finite, by (i) of Lemma 2. Also  $a_0 \in l^p(Z)$  because

$$\begin{aligned} (2\pi)^p \sum_{\substack{n \in Z \\ n \neq 0}} |a_0(n)|^p &= \sum_{n \in Z} \left| \int_0^{2\pi} g(t) \sin nt dt \right|^p \leq \sum_{\substack{n \in Z \\ n \neq 0}} \left| \int_0^{\pi/n} g(t) dt \right|^p \\ &= \sum_{\substack{n \in Z \\ n \neq 0}} G_1 \left( \frac{\pi}{n} \right)^p \leq \int_R G_1^p \left( \frac{\pi}{x} \right) dx = \pi \int_R G_1^p(y) y^{-2} dy. \end{aligned}$$

The last integral is finite by (ii) of Lemma 3. Similarly, if we define

$$b_0(n) = \frac{1}{2\pi} \int_0^{2\pi} g(t) \cos nt dt \quad \text{for } n \in Z$$

then  $b_0 \in l^p(Z)$ . So if we set  $c(n) = b_0(n) - ia_0(n) = 1/2\pi \int_0^{2\pi} g(t) e^{-int} dt$  for  $n \in Z$ , then  $c \in l^p(Z)$  and  $\hat{c}(t) = g(t)$  a.e. [3, 31.44, (b)]. Since  $g$  is nonincreasing in  $[0, 2\pi]$ , we then have  $\hat{c}^* = g\theta$  a.e.



LEMMA 8. *Theorem 3 is true is  $G \supset \Pi^*Z(r)$ , where  $r \in N, r \geq 2$ .*

*Proof.* We may assume that  $G = \Pi^*Z(r)$ , by Lemma 1.

Let  $X = Z(r)^{\aleph_0}$ , the character group of  $G$ . Define  $\varphi(\varepsilon) = \sum_{n=1}^{\infty} \varepsilon_n/r_n$  for  $\varepsilon = (\varepsilon_n) \in X$ , and note that Lemma 6 applies to  $\varphi$ . For a real number  $t$ , denote  $[t]$  by the greatest integer which is not greater than  $t$ . For  $m \in N$ , define

$$\chi_m(t) = e^{i2\pi[r^m t]/r}$$

for  $t \in [0, 1]$ . Then  $\chi_m \circ \varphi(\varepsilon) = e^{i(2\pi/r)\varepsilon_m}$  where  $\varepsilon \in X$  and  $\varepsilon_m$  is the  $m$ th component of  $\varepsilon$ . It follows that  $G$  is isomorphic to the group of finite products of elements in  $\{\chi_m \circ \varphi\}_{m=1}^{\infty}$ . In this proof we write  $I_{m,\nu}$  for the characteristic function of the interval  $[\nu/r^m, (\nu + 1)/r^m]$

$$\chi_m(t) = \sum_{u=1}^{r^m-1} \left( \sum_{j=0}^{r-1} w^j I_{m, (u-1)r+j}(t) \right)$$

for  $\theta$  a.e.  $t$ , where  $w = e^{i(2\pi/r)}$ . And hence

$$\chi_{m_1}^{l_1}(t) \cdots \chi_{m_k}^{l_k}(t) = \sum_{u=1}^{r^{m_1-1}} a_u \left( \sum_{j=0}^{r-1} w^{l_1 j} I_{m_1, (u-1)r+j}(t) \right)$$

where  $a_u^r = 1$  for all  $u = 1, \dots, r^{m_1-1}$ ;  $m_1 > m_2 > \dots > m_k$  and  $0 \leq l_1, l_2, \dots, l_k \leq r - 1, l_1 > 0$ .

Define a function  $f$  on  $G$  via

$$f(\chi_{m_1}^{l_1} \circ \varphi, \dots, \chi_{m_k}^{l_k} \circ \varphi) = \int_X g \circ \varphi(\varepsilon) \chi_{m_1}^{l_1} \circ \varphi(\varepsilon) \cdots \chi_{m_k}^{l_k} \circ \varphi(\varepsilon) d\varepsilon.$$

Define, for  $u = 1, 2, \dots, r^{m_1-1}$  and  $j = 0, \dots, r - 1$ ,

$$k_{(u-1)r+j} = \int I_{m_1, (u-1)r+j}(t) g(t) dt, \quad b_{(u-1)r+j} = a_u w^{j l_1}.$$

Then  $\{k_0, k_1, \dots, k_{r^{m_1-1}}\}$  is a positive nonincreasing sequence, and

$$\left| \sum_{l=0}^s b_l \right| \leq r \quad \text{for all } s = 0, 1, 2, \dots, r^{m_1} - 1$$

In fact,

$$\sum_{j=0}^{r-1} b_{(u-1)r+j} = \sum_{j=0}^{r-1} a_u w^{j l_1} = a_u \sum_{j=0}^{r-1} w^{j l_1} = 0.$$

It follows that

$$|f(\chi_{m_1}^{l_1} \circ \varphi, \dots, \chi_{m_k}^{l_k} \circ \varphi)| = \left| \int_0^1 g(t) \chi_{m_1}^{l_1}(t), \dots, \chi_{m_k}^{l_k}(t) dt \right|$$

$$\begin{aligned} &= \sum_{u=1}^{r^{m_1-1}} \left( \sum_{j=0}^{r-1} \alpha_u w^{j l_1} \int_{I_{m_1, (u-1)r+j}} g(t) dt \right) = \left| \sum_{l=0}^{r^{m_1-1}} b_l k_l \right| \\ &\leq k_0 \max_{0 \leq s \leq r^{m_1-1}} \left| \sum_{l=0}^s b_l \right| \leq k_0 r = r \int_0^{1/r^{m_1}} g(t) dt = r G\left(\frac{1}{r^{m_1}}\right). \end{aligned}$$

Writing  $\Sigma'$  for a sum over all  $(m_1, \dots, m_k, l_1, \dots, l_k)$  satisfying  $k \in N, m_1 > m_2 > \dots > m_k \geq 0, 0 < l_1 \leq r - 1, 0 \leq l_j \leq r - 1$  for  $j = 2, \dots, k$ , we obtain

$$\begin{aligned} \|f\|_p^p &= \Sigma' |f(\chi_{m_1}^{l_1} \varphi, \dots, \chi_{m_k}^{l_k} \varphi)|^p \leq \Sigma' r^p G^p\left(\frac{1}{r^{m_1}}\right) \\ &\leq \sum_{m_1=0}^{\infty} r^{m_1} r^p G^p\left(\frac{1}{r^{m_1}}\right) = r^{p+1} \sum_{m_1=0}^{\infty} r^{m_1-1} G^p\left(\frac{1}{r^{m_1}}\right) \\ &\leq r^{p+1} \sum_{m_1=0}^{\infty} (r^{m_1} - r^{m_1-1}) G^p\left(\frac{1}{r^{m_1}}\right) \leq r^{p+1} \int_0^{\infty} G^p\left(\frac{1}{x}\right) dx < \infty. \end{aligned}$$

So  $f \in L^p(G)$  and hence  $\hat{f} = g \circ \varphi$ . It follows that  $\hat{f}^* = g I_{[0,1]}$  *m a.e.*

LEMMA 9. *Theorem 3 is true if  $G$  contains  $Z(r^\infty)$ , ( $r \geq 2$ ).*

*Proof.* We may assume that  $G = Z(r^\infty)$  by Lemma 1. Let  $\Delta_r$  be the group of  $r$ -adic integers; then  $Z(r^\infty)$  is a discrete group with  $Z(r^\infty)^\wedge = \Delta_r$ . Define

$$\varphi(\varepsilon) = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{r^n} \varepsilon = (\varepsilon_n) \in \Delta_r.$$

As in Lemma 6,  $\varphi$  is a measure preserving map from  $\Delta_r$  onto  $[0, 1]$ , and

$$(2) \quad \int_{\Delta_r} h \circ \varphi d^\theta = \int_0^1 h dt$$

for all bounded measurable functions  $h$  on  $[0, 1]$ , where  $\theta$  is the normalized Haar measures on  $\Delta_r$ . We write  $I_{m, s_1, \dots, s_m}$  for the characteristic function of the interval

$$\left[ \frac{r^{m-1}s_1 + r^{m-2}s_2 + \dots + s_m}{r^m}, \frac{r^{m-1}s_1 + r^{m-2}s_2 + \dots + s_m + 1}{r^m} \right].$$

For  $m \in N$ , define

$$(3) \quad \chi_m(t) = \sum_{s_1, \dots, s_m=0}^{r-1} w_m^{s_1 + r s_2 + \dots + r^{m-1} s_m} I_{m, s_1, \dots, s_m}(t)$$

where  $w_m = e^{i(2\pi/r^m)}$ . Then  $\chi_m \circ \varphi(\varepsilon) = w_m^{s_1 + r s_2 + \dots + r^{m-1} s_m} \theta$  *a.e.* where  $(\varepsilon) \in \Delta_r$  and  $\varepsilon_1, \dots, \varepsilon_m$  are the first  $m$  coordinates of  $(\varepsilon)$ . It follows that  $G$  is isomorphic to the group generated by  $\{\chi_m \circ \varphi\}_{m=1}^\infty$ . Define for

$m, l \in N$  and  $(l, r) = 1$

$$f(\chi_m^l) = \int_{\mathcal{A}_r} g \circ \varphi(\varepsilon) \chi_m^l \circ \varphi(\varepsilon) d\theta$$

Then  $f$  is a function on  $G$ , and by (2) and (3),

$$\begin{aligned} f(\chi_m^l) &= \int_0^1 g(t) \chi_m^l(t) dt \\ &= \sum_{s_1, \dots, s_m=0}^{r-1} (w_m^l)^{s_1+r s_2+\dots+r^{m-1} s_m} \int I_{m, s_1, \dots, s_m}(t) g(t) dt . \end{aligned}$$

Let  $k_{r^{m-1} s_1+\dots+s_m} = \int I_{m, s_1, \dots, s_m}(t) g(t) dt$ . Then  $\{k_0, k_1, \dots, k_{r^m-1}\}$  is a positive, nonincreasing sequence. Let  $b_{r^{m-1} s_1+\dots+s_m} = (w_m^l)^{s_1+r s_2+\dots+r^{m-1} s_m}$ . For any  $0 \leq s \leq r^m - 1$ , we write  $s = r^{m-1} s_1 + \dots + s_m$  with  $0 \leq s_1, \dots, s_m < r$ . Then

$$\begin{aligned} \sum_{n=0}^s b_n &= \sum_{n=1}^{r^{m-1} s_1+\dots+s_m} b_n \\ &= \left( \sum_{u=1}^{r^{m-2} s_1+\dots+s_{m-1}} \sum_{h=0}^{r-1} b_{(u-1)r+h} \right) + \left( \sum_{j=0}^{s_m} b_{r^{m-1} s_1+\dots+r s_{m-1}+j} \right) \end{aligned}$$

For each  $u = 1, \dots, r^{m-2} s_1 + \dots + s_{m-1}$ . Choose  $0 \leq u_1, \dots, u_{m-1} < r$  such that  $(u-1)r = r^{m-1} u_1 + \dots + r u_{m-1}$ , and hence

$$\begin{aligned} \sum_{h=0}^{r-1} b_{(u-1)r+h} &= \sum_{h=0}^{r-1} b_{r^{m-1} u_1+\dots+r u_{m-1}+h} \\ &= \sum_{h=0}^{r-1} (w_m^l)^{u_1+r u_2+\dots+r^{m-2} u_{m-1}+r^{m-1} h} \\ &= (w_m^l)^{u_1+r u_2+\dots+r^{m-2} u_{m-1}} \sum_{h=0}^{r-1} (w_m^l) r^{m-1} h \\ &= (w_m^l)^{u_1+r u_2+\dots+r^{m-2} u_{m-1}} \sum_{h=0}^{r-1} (e^{i(2\pi l/r)})^h = 0 . \end{aligned}$$

The last equality holds because  $(l, r) = 1$ . This shows that

$$\left| \sum_{n=0}^s b_n \right| = \left| \sum_{j=0}^{s_m} b_{r^{m-1} s_1+\dots+r s_{m-1}+j} \right| \leq s_m + 1 \leq r$$

and hence

$$\begin{aligned} |f(\chi_m^l)| &= \left| \sum_{n=0}^{r^m-1} b_n k_n \right| \leq k_0 \max_{0 \leq s \leq r^m-1} \left| \sum_{n=0}^s b_n \right| \leq r k_0 \\ &= r \int_0^{1/r^m} g(t) dt = r G_1 \left( \frac{1}{r^m} \right) \end{aligned}$$

for all  $m, l \in N$  and  $(l, r) = 1$ . Denote by  $\Sigma'$  the sum over  $(m, l) \in N$ ,  $(l, r) = 1$  and  $0 \leq l < r^m$ . Then we have

$$\|f\|_p^p = \sum' |f(\chi_m^l)|^p \leq \sum' r^p G_1^p\left(\frac{1}{r^m}\right) \leq \sum_{m=0}^{\infty} r^m r^p G_1^p\left(\frac{1}{r^m}\right).$$

As in Lemma 8, we conclude that  $f \in L^p(G)$  and  $\hat{f}^* = gI_{[0,1]}m$  a.e.

Patching Lemmas 7, 8 and 9 together gives the proof of Theorem 3.

I would like to extend my sincere thanks here to Professor K. A. Ross for his helpful suggestions.

The remaining open question is whether Theorem 3 holds if  $G = \prod_{n=1}^{\infty} * Z(r_n)$  where  $r_n \in \mathbb{N}$ ,  $r_n \geq 2$  for all  $n$  and  $\lim_{n \rightarrow \infty} r_n = \infty$ .

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