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## **AN ASYMPTOTIC ANALYSIS OF AN ODD ORDER LINEAR DIFFERENTIAL EQUATION**

DAVID LOWELL LOVELADY

## AN ASYMPTOTIC ANALYSIS OF AN ODD ORDER LINEAR DIFFERENTIAL EQUATION

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Let  $q$  be a continuous function from  $[0, \infty)$  to  $(0, \infty)$ , and let  $n$  be a positive integer. With respect to the equation  $u^{(2n+1)} + qu = 0$ , we study the relationship between the existence of oscillatory solutions and the asymptotic behavior of nonoscillatory solutions.

There is no additional hypothesis on  $q$  which will ensure that every solution of

$$(1) \quad u^{(2n+1)} + qu = 0$$

is oscillatory. In particular, it follows from a result of P. Hartman and A. Wintner [5] that there is a solution  $u$  of (1) such that

$$(2) \quad (-1)^k u^{(k)}(t) > 0$$

whenever  $t \geq 0$  and  $0 \leq k \leq 2n$ . We shall call a solution of  $u$  of (1) *strongly decreasing* if and only if there is  $c \geq 0$  such that (2) is true whenever  $t \geq c$  and  $0 \leq k \leq 2n$ . Since we know that (1) has a strongly decreasing solution, the best result one can hope for in an oscillation theorem is that every eventually positive solution of (1) is strongly decreasing. G. V. Anan'eva and V. I. Balaganskii [1] (see also C. A. Swanson [7, p. 175]) have shown that if

$$(3) \quad \int_0^\infty t^{2n-1} q(t) dt = \infty,$$

then every eventually positive solution of (1) is strongly decreasing. Our first result extends this.

**THEOREM 1.** *If (3) fails and the second order equation*

$$(4) \quad w''(t) + \frac{1}{(2n-2)!} \left( \int_t^\infty (s-t)^{2n-2} q(s) ds \right) w(t) = 0$$

*is oscillatory, then every eventually positive solution of (1) is strongly decreasing.*

Although the conclusion of Theorem 1 limits the asymptotic behavior of nonoscillatory solutions of (1) (if  $u$  is nonoscillatory then either  $u$  or  $-u$  is eventually positive), it does not in fact ensure the existence of oscillatory solutions.

**THEOREM 2.** *Suppose that every eventually positive solution of (1) is strongly decreasing. Then if  $u$  is a solution of (1), and if any of  $u, u', u'', \dots, u^{(2n)}$  has a zero in  $[0, \infty]$ ,  $u$  is oscillatory.*

**COROLLARY 1.** *With the hypotheses of Theorem 2, the solution space  $Q$  of (1) has a basis each member of which is oscillatory, and  $Q$  has a  $2n$ -dimensional subspace each member of which is oscillatory.*

**COROLLARY 2.** *With the hypotheses of Theorem 1, the conclusions of Corollary 1 hold.*

Finally, we offer a comparison theorem.

**THEOREM 3.** *Suppose that  $p$  is a continuous function from  $[0, \infty)$  to  $(0, \infty)$  with  $p(t) \geq q(t)$  whenever  $t \geq 0$ . Suppose also that every eventually positive solution of (1) is strongly decreasing. Then every eventually positive solution of*

$$(5) \quad u^{(2n+1)} + pu = 0$$

*is strongly decreasing.*

A. C. Lazer has shown [6, Theorem 1.2] that in the third order case the existence of a nontrivial oscillatory solution of (1) implies that every eventually positive solution of (1) is strongly decreasing. The following example shows that this is not true in general.

**EXAMPLE.** Suppose  $1 < r < 2$ . Now

$$r(r-1)(r-2)(r-3)(r-4) > (r+2)(r+1)r(r-1)(r-2),$$

so  $\alpha > \gamma$ , where

$$\alpha = \min \{r(r-1)(r-2)(r-3)(r-4): 1 \leq r \leq 2\}$$

and

$$\begin{aligned} \gamma &= \min \{(r+2)(r+1)r(r-1)(r-2): 1 \leq r \leq 2\} \\ &= \min \{r(r-1)(r-2)(r-3)(r-4): 3 \leq r \leq 4\}. \end{aligned}$$

Let  $n = 2$ , and, noting that  $\alpha$  and  $\gamma$  are negative, let  $\beta$  be a positive number such that  $\alpha > -\beta < \gamma$ . Let  $q$  be given by  $q(t) = \beta(t+1)^{-5}$ . The polynomial equation

$$(6) \quad r(r-1)(r-2)(r-3)(r-4) + \beta = 0$$

has two complex roots, so (1) has nontrivial oscillatory solutions. On the other hand, (6) has a solution  $r$  in the interval  $(3, 4)$ , and  $u$ ,

given by  $u(t) = (t + 1)^r$ , satisfies (1) and is not strongly decreasing. The example is complete.

**LEMMA.** *Suppose  $u$  is a solution of (1),  $c \geq 0$ , and  $(-1)^k u^{(k)}(c) > 0$  for  $k = 0, \dots, 2n$ . Then (2) is true for  $0 \leq t \leq c$  and  $k = 0, \dots, 2n$ .*

*Proof.* Let  $v$  be given on  $[-c, 0]$  by  $v(t) = u(-t)$ . If  $k = 0, \dots, 2n + 1$  then  $v^{(k)}(t) = (-1)^k u^{(k)}(-t)$ , so

$$v^{(2n+1)}(t) - q(-t)v(t) = 0$$

and  $v^{(k)}(-c) > 0$  for  $-c \leq t \leq 0$  and  $k = 0, \dots, 2n$ . Thus

$$(7) \quad v(t) = v(-c) + \sum_{m=1}^{2n} \frac{(t+c)^m}{m!} v^{(m)}(-c) + \int_{-c}^t \frac{(t-s)^{2n}}{(2n)!} q(-s)v(s)ds$$

if  $-c \leq t \leq 0$ . But clearly the solution of (7) is positive, so  $v(t) > 0$  if  $-c \leq t \leq 0$ . Now, if  $k = 0, \dots, 2n$  and  $-c \leq t \leq 0$ ,

$$v^{(k)}(t) = v^{(k)}(-c) + \sum_{m=k+1}^{2n} \frac{(t+c)^{m-k}}{(m-k)!} v^{(m)}(-c) + \int_{-c}^t \frac{(t-s)^{2n-k}}{(2n-k)!} q(-s)v(s)ds,$$

so  $v^{(k)}(t) > 0$ , i.e.,  $(-1)^k u^{(k)}(-t) > 0$ . The proof is complete.

*Proof of Theorem 1.* Assume that (3) fails. We shall show that if there is an eventually positive solution of (1) which is not strongly decreasing then (4) is nonoscillatory. Let  $u$  be an eventually positive solution of (1) which is not strongly decreasing. Find  $a \geq 0$  such that  $u(t) > 0$  if  $t \geq a$ . Now  $u^{(2n+1)} < 0$  on  $[a, \infty)$ , so  $u^{(2n)}$  is eventually one-signed. Since  $u^{(2n)}$  is eventually one-signed,  $u^{(2n-1)}$  is eventually one-signed. Continuing this, we see that there is  $c \geq a$  such that none of  $u, u', \dots, u^{(2n)}$  has a zero in  $[c, \infty)$ . Let  $j$  be the largest integer such that  $u^{(j)} > 0$  on  $[c, \infty)$  if  $i \leq j$  (we write  $u = u^{(0)}$ ). Note that  $j \neq 2n + 1$ . Now  $u^{(j+1)} < 0$  on  $[c, \infty)$ , so  $u^{(j)}$  is bounded. Thus, if  $j \leq k \leq 2n$ ,  $u^{(k)} u^{(k+1)} < 0$  on  $[c, \infty)$ . But  $u^{(2n+1)} < 0$ , so if  $j \leq k \leq 2n$  then  $u^{(k)} > 0$  on  $[c, \infty)$  if  $k$  is even and  $u^{(k)} < 0$  on  $[c, \infty)$  if  $k$  is odd. Since  $u^{(j+1)} < 0$  (recall how  $j$  was chosen), this says  $j + 1$  is odd and  $j$  is even. By hypothesis,  $j \neq 0$ . Suppose  $j < 2n$ . Now

$$-u^{(j+1)}(t) = \frac{1}{(2n-j-1)!} \int_t^\infty (s-t)^{2n-j-1} q(s)u(s)ds$$

if  $t \geq c$ . Also,  $u^{(j-1)}$  is increasing on  $[c, \infty)$  since  $u^{(j)} > 0$ , so, if  $s \geq t \geq c$ ,

$$\begin{aligned} u(s) &\geq \frac{1}{(j-2)!} \int_c^s (s-\xi)^{j-2} u^{(j-1)}(\xi) d\xi \\ &\geq \frac{1}{(j-2)!} \int_t^s (s-\xi)^{j-2} u^{(j-1)}(\xi) d\xi \\ &\geq \frac{u^{(j-1)}(t)}{(j-2)!} \int_t^s (s-\xi)^{j-2} d\xi = \frac{u^{(j-1)}(t)}{(j-1)!} (s-t)^{j-1}. \end{aligned}$$

Since  $(2n-j-1)!(j-1)! \leq (2n-2)!$ , this says

$$\begin{aligned} (8) \quad -u^{(j+1)}(t) &\geq \frac{u^{(j-1)}(t)}{(2n-2)!} \int_t^\infty (s-t)^{2n-2} q(s) ds, \\ u^{(j+1)}(t)/u^{(j-1)}(t) &\leq -\frac{1}{(2n-2)!} \int_t^\infty (s-t)^{2n-2} q(s) ds \end{aligned}$$

if  $t \geq c$ . Let  $v$  be given on  $[c, \infty)$  by  $v = u^{(j)}/u^{(j-1)}$ , and note that  $v > 0$  on  $[c, \infty)$ . Now if  $t > c$ ,

$$v'(t) = u^{(j+1)}(t)/u^{(j-1)}(t) - v(t)^2,$$

so (8) says

$$(9) \quad v'(t) + v(t)^2 \leq -\frac{1}{(2n-2)!} \int_t^\infty (s-t)^{2n-2} q(s) ds.$$

But a classical result of M. Bôcher [2], [3] (see also C. de la Vallée Poussin [8], A. Wintner [9], C. A. Swanson [7, Theorem 2.15, p. 63], and W. A. Coppel [4, Theorem 4, p. 6]) says that the existence of a positive solution of (9) implies that (4) is nonoscillatory. The proof is complete, if  $j < 2n$ .

Suppose  $j = 2n$ . Now

$$\begin{aligned} u^{(2n)}(t) &\geq \int_t^\infty q(s)u(s)ds \\ &\geq \frac{1}{(2n-2)!} \int_t^\infty q(s) \left( \int_c^s (s-\xi)^{2n-2} u^{(2n-1)}(\xi) d\xi \right) ds \\ &\geq \frac{1}{(2n-2)!} \int_t^\infty q(s) \left( \int_t^s (s-\xi)^{2n-2} u^{(2n-1)}(\xi) d\xi \right) ds \end{aligned}$$

if  $t \geq c$ . But this and standard iteration methods say that there is a continuously differentiable function  $w$  from  $[c, \infty)$  to  $[u^{(2n-1)}(c), \infty)$  such that  $w(c) = u^{(2n-1)}(c)$  and

$$w'(t) = \frac{1}{(2n-2)!} \int_t^\infty q(s) \left( \int_t^s (s-\xi)^{2n-2} w(\xi) d\xi \right) ds$$

if  $t \geq c$ . But  $w$  clearly satisfies (4) on  $[c, \infty)$ , and can be extended to a nonoscillatory solution of (4) on  $[0, \infty)$ , so the proof is complete.

*Proof of Theorem 2.* Let  $u$  be a nonoscillatory solution of (1). If  $u$  is eventually negative, we may replace  $u$  by  $-u$ , so we assume that  $u$  is eventually positive. Now there is  $c \geq 0$  such that (2) holds whenever  $t \geq c$  and  $k = 0, \dots, 2n$ . Now our lemma says that if  $k = 0, \dots, 2n$  then  $u^{(k)}$  has no zeroes in  $[0, c]$  and thus has no zeros at all. The proof is complete.

*Proof of Corollary 1.* If  $k$  is an integer in  $[1, 2n + 1]$ , let  $z_k$  be the solution of (1) such that  $z_k^{(j)}(0) = 0$  if  $j \neq k - 1$  and  $z_k^{(k-1)}(0) = 1$ . Clearly  $\{z_1, \dots, z_{2n+1}\}$  is a basis for  $Q$ , and Theorem 3 says that each  $z_k$  is oscillatory. Also, if  $u$  is in the  $2n$ -dimensional subspace spanned by  $\{z_2, \dots, z_{2n+1}\}$ , then  $u(0) = 0$  so  $u$  is oscillatory. The proof is complete.

Corollary 2 is now immediate from Theorem 2 and Corollary 1.

*Proof of Theorem 3.* We shall assume the existence of an eventually positive solution of (5) which is not strongly decreasing, and show the existence of an eventually positive solution of (1) which is not strongly decreasing. Let  $v$  be an eventually positive solution of (5) which is not strongly decreasing. Let  $c \geq 0$  be such that none of  $v, v', \dots, v^{(2n)}$  has a zero in  $[c, \infty)$ , and let  $j$  be the largest integer such that  $v^{(i)} > 0$  on  $[c, \infty)$  if  $i \leq j$ . By hypothesis,  $j \neq 0$ , and we know that  $j$  is even. Now

$$v^{(j)}(t) \geq \frac{1}{(2n - j)!} \int_t^\infty (s - t)^{2n-j} p(s) v(s) ds$$

if  $t \geq c$ , and

$$v(t) \geq v(c) + \frac{1}{(j - 1)!} \int_c^t (t - s)^{j-1} v^{(j)}(s) ds$$

if  $t \geq c$ , so

$$\begin{aligned} (10) \quad v(t) &\geq v(c) + \frac{1}{(j - 1)!(2n - j)} \int_c^t (t - s)^{j-1} \left( \int_s^\infty (\xi - s)^{2n-j} p(\xi) v(\xi) d\xi \right) ds \\ &\geq v(c) + \frac{1}{(j - 1)!(2n - j)!} \int_c^t (t - s)^{j-1} \left( \int_s^\infty (\xi - s)^{2n-j} q(\xi) v(\xi) d\xi \right) ds \end{aligned}$$

if  $t \geq c$ . Now (10) and standard iteration techniques say that there is a continuous function  $u$  from  $[c, \infty)$  to  $[0, \infty)$  such that  $u(t) \leq v(t)$  whenever  $t \geq c$  and such that

$$\begin{aligned} (11) \quad u(t) &= v(c) \\ &+ \frac{1}{(j - 1)!(2n - j)!} \int_c^t (t - s)^{j-1} \left( \int_s^\infty (\xi - s)^{2n-j} q(\xi) u(\xi) d\xi \right) ds \end{aligned}$$

if  $t \geq c$ . The fact that  $u$  has only nonnegative values, together with (11), says  $u(t) \geq v(c)$  whenever  $t \geq c$ ; in particular,  $u$  has no zeros in  $[c, \infty)$ . Differentiation of (11) yields that  $u$  satisfies (1) on  $[c, \infty)$ , and  $u' > 0$  on  $(c, \infty)$ , i.e., (2) is not true for  $k = 1$  and  $t > c$ . Clearly  $u$  can be extended to a solution of (1) on  $[0, \infty)$ , and this solution is eventually positive but not strongly decreasing. The proof is complete.

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Norman Larrabee Alling, <i>On Cauchy's theorem for real algebraic curves with boundary</i> .....	315
Daniel D. Anderson, <i>A remark on the lattice of ideals of a Prüfer domain</i> .....	323
Dennis Neal Barr and Peter D. Miletta, <i>A necessary and sufficient condition for uniqueness of solutions to two point boundary value problems</i> .....	325
Ladislav Beran, <i>On solvability of generalized orthomodular lattices</i> .....	331
L. Carlitz, <i>A three-term relation for some sums related to Dedekind sums</i> .....	339
Arthur Herbert Copeland, Jr. and Albert Oscar Shar, <i>Images and pre-images of localization maps</i> .....	349
G. G. Dandapat, John L. Hunsucker and Carl Pomerance, <i>Some new results on odd perfect numbers</i> .....	359
M. Edelstein and L. Keener, <i>Characterizations of infinite-dimensional and nonreflexive spaces</i> .....	365
Francis James Flanigan, <i>On Levi factors of derivation algebras and the radical embedding problem</i> .....	371
Harvey Friedman, <i>Provable equality in primitive recursive arithmetic with and without induction</i> .....	379
Joseph Braucher Fugate and Lee K. Mohler, <i>The fixed point property for tree-like continua with finitely many arc components</i> .....	393
John Norman Ginsburg and Victor Harold Saks, <i>Some applications of ultrafilters in topology</i> .....	403
Arjun K. Gupta, <i>Generalisation of a "square" functional equation</i> .....	419
Thomas Lee Hayden and Frank Jones Massey, <i>Nonlinear holomorphic semigroups</i> .....	423
V. Kannan and Thekkedath Thrivikraman, <i>Lattices of Hausdorff compactifications of a locally compact space</i> .....	441
J. E. Kerlin and Wilfred Dennis Pepe, <i>Norm decreasing homomorphisms between group algebras</i> .....	445
Young K. Kwon, <i>Behavior of <math>\Phi</math>-bounded harmonic functions at the Wiener boundary</i> .....	453
Richard Arthur Levaro, <i>Projective quasi-coherent sheaves of modules</i> .....	457
Chung Lin, <i>Rearranging Fourier transforms on groups</i> .....	463
David Lowell Lovelady, <i>An asymptotic analysis of an odd order linear differential equation</i> ...	475
Jerry Malzan, <i>On groups with a single involution</i> .....	481
J. F. McClendon, <i>Metric families</i> .....	491
Carl Pomerance, <i>On multiply perfect numbers with a special property</i> .....	511
Mohan S. Putcha and Adil Mohamed Yaqub, <i>Polynomial constraints for finiteness of semisimple rings</i> .....	519
Calvin R. Putnam, <i>Hyponormal contractions and strong power convergence</i> .....	531
Douglas Conner Ravenel, <i>Multiplicative operations in <math>BP^*BP</math></i> .....	539
Judith Roitman, <i>Attaining the spread at cardinals which are not strong limits</i> .....	545
Kazuyuki Saitô, <i>Groups of *-automorphisms and invariant maps of von Neumann algebras</i> ...	553
Brian Kirkwood Schmidt, <i>Homotopy invariance of contravariant functors acting on smooth manifolds</i> .....	559
Kenneth Barry Stolarsky, <i>The sum of the distances to N points on a sphere</i> .....	563
Mark Lawrence Teply, <i>Semiprime rings with the singular splitting property</i> .....	575
J. Pelham Thomas, <i>Maximal connected Hausdorff spaces</i> .....	581
Charles Thomas Tucker, II, <i>Concerning <math>\sigma</math>-homomorphisms of Riesz spaces</i> .....	585
Rangachari Venkataraman, <i>Compactness in abelian topological groups</i> .....	591
William Charles Waterhouse, <i>Basically bounded functors and flat sheaves</i> .....	597
David Westreich, <i>Bifurcation of operator equations with unbounded linearized part</i> .....	611
William Robin Zame, <i>Extendibility, boundedness and sequential convergence in spaces of holomorphic functions</i> .....	619