ON MULTIPLY PERFECT NUMBERS WITH A SPECIAL PROPERTY

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If \( m \) is a multiply perfect number and \( m = p^a n \) where \( p \) is prime and \( n \mid \sigma(p^a) \), then \( m = 120, 672, 523776 \), or \( m \) is an even perfect number.

1. Introduction. Suppose \( p \) is a prime \( a \), \( n \) are natural numbers, and

\[
\begin{align*}
    p^a & \mid \sigma(n), \quad n \mid \sigma(p^a)
\end{align*}
\]

where \( \sigma \) is the sum of the divisors function. Then \( 1 = (p^a, \sigma(p^a)) = (p^a, n) \), so that \( p^a n \mid \sigma(p^a) \sigma(n) = \sigma(p^a n) \); that is \( p^a n \) is a multiply perfect number. In this paper we identify all multiply perfect numbers which arise in this fashion.

Let \( M \) be the set of Mersenne exponents, that is, \( M = \{ k : 2^k - 1 \text{ is prime} \} \). We shall prove

**Theorem 1.1.** If \( p, a, n \) is a solution of (1.1) where \( p \) is prime, then either

\[
\begin{align*}
    (1.2) & \quad p^a = 2^k - 1, \quad n = 2^{k-1} \quad \text{for some } k \in M \\
    (1.3) & \quad p^a = 2^{k-1}, \quad n = 2^k - 1 \quad \text{for some } k \in M \\
    (1.4) & \quad p^a = 2^3, \quad n = 15 \\
    (1.5) & \quad p^a = 2^5, \quad n = 21 \\
    (1.6) & \quad p^a = 2^9, \quad n = 1023.
\end{align*}
\]

**Corollary 1.1.** If \( m \) is a multiply perfect number and \( m = p^a n \) where \( p \) is prime and \( n \mid \sigma(p^a) \), then \( m = 120, 672, 523776 \), or \( m \) is an even perfect number.

Note that in [2] all solutions of (1.1) with \( p^a = \sigma(n) \) are enumerated: they are (1.2) and (1.5). Hence in the proof of Theorem 1.1, we may assume \( p^a < \sigma(n) \).

We recall that a natural number \( n \) is said to be super perfect if \( \sigma(\sigma(n)) = 2n \). In [2] and Suryanarayana [8] it is shown that if \( n \) is super perfect and if either \( n \) or \( \sigma(n) \) is a prime power, then \( n = 2^{k-1} \) for \( k \in M \). Here we will say \( n \) is super multiply perfect if \( \sigma(\sigma(n))/n \) is an integer.
Corollary 1.2. If \( n \) is super multiply perfect, and if \( n \) or \( \sigma(n) \) is a prime power, then \( n = 8, 21, 512 \), or \( n = 2^k \) for some \( k \in M \).

If \( p \) is a prime, denote by \( \sigma_p(n) \) the sum of all those divisors of \( n \) which are powers of \( p \). Then \( \sigma_p(n) | \sigma(n) \).

Corollary 1.3. If \( n > 1 \) and \( n | \sigma_p(\sigma(n)) \) for some prime \( p \), then \( p = 2 \) and \( n = 15, 21, \) or \( 1023 \) or \( p = 2^k - 1 \) for some \( k \in M \) and \( n = 2^{k-1} \).

We remark that in general the super multiply perfect numbers appear to be quite intractable. Partly complicating matters is that for every \( K \), \( \sigma(\sigma(n))/n \geq K \) on a set of density 1. Professor David E. Penney of the University of Georgia, in a computer search, found that there are exactly 37 super multiply perfect numbers \( \leq 150000 \). Of these, the only odd ones are 1, 15, 21, 1023, and 29127.

Recently, Guy and Selfridge [4], p. 104, published a proof of a stronger version of Theorem 1.1 for the special case \( p = 2 \).

2. Preliminaries. If \( n \) is a natural number, we let \( \omega(n) \) be the number of distinct prime factors of \( n \), and we let \( \tau(n) \) be the number of natural divisors of \( n \). If \( a, b \) are natural numbers with \( (a, b) = 1 \), we let \( \text{ord}_a(b) \) be the least positive integer \( k \) for which \( a | b^k - 1 \). If \( p \) is a prime and \( x \) is a natural number, then \( \sigma(p^x) = (p^{x+1} - 1)/(p - 1) < (p/(p - 1))p^x \).

Theorem 2.1 (Bang [1]). If \( p \) is a prime, \( a \) is a natural number, and \( 1 < d | a + 1 \), then there is a prime \( q | \sigma(p^a) \) with \( \text{ord}_q(p) = d \), unless

(i) \( p = 2 \) and \( d = 6 \), or

(ii) \( p \) is a Mersenne prime and \( d = 2 \).

Corollary 2.1.

\[
\omega(\sigma(p^a)) \geq \begin{cases} 
\tau(a + 1) - 2 , & \text{if } p = 2 \text{ and } 6 | a + 1 \\
\tau(a + 1) , & \text{if } p > 2 \text{ is not Mersenne and } 2 | a + 1 \\
\tau(a + 1) - 1 , & \text{otherwise}.
\end{cases}
\]

The following is a weaker form of a lemma from [2].

Lemma 2.1. Suppose \( p, q \) are primes with \( q > 2 \) and \( x, y, b, c \) are natural numbers with \( \sigma(q^x) = p^y \) and \( q^y | \sigma(p^b) \). Then \( q^{y+1} | c + 1 \).
3. The start of the proof. Suppose \( p, a, n \) is a solution of (1.1) where \( p \) is prime. Then there are integers \( s, t \) with
\[
\sigma(n) = sp^a, \quad \sigma(p^a) = tn.
\]
As we remarked, we have already studied these equations in the case \( s = 1 \) (in [2]), so here we assume \( s > 1 \). We have
\[
(3.1) \quad st = \frac{\sigma(p^a)}{p^a} \cdot \frac{\sigma(n)}{n},
\]
Considering the unique prime factorization of \( n \), we write \( n_i \) for the product of those prime powers \( q^k \) for which \( \sigma(q^k) \) is divisible by a prime \( \neq p \), and we write \( n_2 \) for the product of those prime powers \( q^k \) for which \( \sigma(q^k) \) is a power of \( p \). Then \( (n_1, n_2) = 1 \), \( n_1n_2 = n \), and \( \sigma(n_i) \) is a power of \( p \). Let \( \omega_i \) be the number of distinct odd prime factors of \( n_i \) for \( i = 1, 2 \). Let \( \omega_3 \) be the number of distinct prime factors of \( t \) which do not divide \( n \). Hence
\[
(3.2) \quad \omega(\sigma(p^a)) = \omega(tn) = \begin{cases} \omega_1 + \omega_2 + \omega_3, & \text{if } n \text{ is odd} \\ 1 + \omega_1 + \omega_2 + \omega_3, & \text{if } n \text{ is even} \end{cases}.
\]
We write
\[
n_1 = 2^{b_1} \prod_{i=1}^{\omega_1} p_i^{g_i}, \quad n_2 = 2^{b_2} \prod_{i=1}^{\omega_2} q_i^{h_i},
\]
where \( k_1, k_2 = 0 \) and the \( p_i \) and \( q_i \) are distinct odd primes.

4. The case \( p > 2 \). Since each \( \sigma(q_i^{b_i}) \) is a power of \( p \), and since \( p \) is odd, we have each \( b_i \) even. Since also each \( q_i^{b_i} | \sigma(p^a) \), Lemma 2.1 implies
\[
\prod_{i=1}^{\omega_2} q_i | a + 1.
\]
Suppose \( n \) is even. Then also \( 2 | a + 1 \), so that \( \tau(a + 1) \geq 2^{\omega_2+1} \). It follows from (3.2) and Corollary 2.1 that
\[
(4.1) \quad \omega_1 + \omega_2 \geq 2^{\omega_2+1} - \omega_2 - 2.
\]
Suppose \( k_i > 0 \). Then \( (\sigma(2^{k_i}), s) \geq 3 \) and for
\[
1 \leq i \leq \omega_1, \quad (\sigma(p_i^{g_i}), s) \geq 2.
\]
Then \( s \geq 3 \cdot 2^{\omega_1} \). Also every prime counted by \( \omega_3 \) is odd, so \( t \geq 3^{\omega_2} \). Hence from (3.1) we have
\[
3 \cdot \left(\frac{5}{4}\right)^{3w_1 + 4w_2} < 3 \cdot 2^{w_1} \cdot 3^{w_2}
\]
\[
\leq st = \frac{\sigma(p^s)}{p^s} \cdot \frac{\sigma(n)}{n} < \frac{p}{p-1} \cdot 2 \cdot \prod_{i=1}^{w_1} \frac{p_i}{p_i - 1} \cdot \prod_{i=1}^{w_2} \frac{q_i}{q_i - 1}
\]
\[
\leq 3 \cdot \left(\frac{5}{4}\right)^{w_1 + w_2}
\]
so that
\[
\omega_2 > 2\omega_1 + 4\omega_3 \geq 2(\omega_1 + \omega_3).
\]
Hence (4.1) implies that
\[
\omega_2 > 2^{w_2+2} - 2\omega_2 - 4
\]
which fails for all \( \omega_2 \geq 0 \). This contradiction shows \( k_2 = 0 \).

Suppose \( k_2 > 0 \). Then \( \sigma(2^{k_2}) \) is a power of \( p \), so that \( \sigma(2^{k_2}) = p \) (Gerono [3]). Now \( 2 \mid a + 1 \), so \( 2^{k_2+1} = \sigma(p) \mid \sigma(p^s) \). Hence \( 2 \mid t \), so that \( t \geq 2 \cdot 3^{w_3} \). Also \( (\sigma(p^s), s) \geq 2 \), so \( s \geq 2^{w_3} \). Hence
\[
\left(\frac{5}{4}\right)^{3w_1 + 4w_2} < \frac{1}{2} \cdot \frac{p}{p-1} \cdot \prod_{i=1}^{w_1} \frac{p_i}{p_i - 1} \cdot \prod_{i=1}^{w_2} \frac{q_i}{q_i - 1} \leq \frac{3}{2} \left(\frac{5}{4}\right)^{w_1 + w_2}
\]
so that
\[
(4.2) \quad \omega_2 > 2\omega_1 + 4\omega_3 - 2 \geq 2(\omega_1 + \omega_3) - 2.
\]

It follows from (4.1) that
\[
\omega_2 > 2^{w_2+2} - 2\omega_2 - 6,
\]
which implies \( \omega_2 \leq 1 \). Then (4.2) implies \( \omega_1 \leq 1 \). Since \( s > 1 \) and \( 2 \mid t \), we have
\[
4 \leq st < \frac{\sigma(2^{k_2})}{2^{k_2}} \cdot \frac{p}{p-1} \cdot \frac{p_1}{p_1 - 1} \cdot \frac{q_1}{q_1 - 1} < \frac{2ppq_1}{(p-1)(p_1-1)(q_1-1)}
\]
so that \( \max\{p, p_1, q_1\} = 13 \). But \( \sigma(2^{k_2}) = p \), so \( k_2 \leq 2 \). Then
\[
4 < \frac{\sigma(2^{k_2})}{2^{k_2}} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} < 4,
\]
so \( k_2 = 0 \).

Thus we have \( n \) odd, so \( p^n n \) is an odd multiply perfect number.
It follows from Hagis [5] and McDaniel [6] that
\[
(4.3) \quad 1 + \omega_1 + \omega_2 = 1 + \omega(n) = \omega(p^n n) \geq 8.
\]
From (3.2) and Corollary 2.1 we have
\[(4.4) \quad \omega_1 + \omega_2 \geq 2^w_2 - \omega_2 - 1.\]

Now \(s \geq 2^n_1, t \geq 2^n_3\) so that
\[
\left(\frac{5}{4}\right)^{3w_1 + 3w_3} < st < \frac{p}{p-1} \cdot \prod \frac{p_i}{p_i - 1} \cdot \prod \frac{q_i}{q_i - 1}
\]
\[
\leq \frac{3}{2} \left(\frac{5}{4}\right)^{w_1 + w_2} \leq \left(\frac{5}{4}\right)^{w_1 + w_2 + 2}.
\]

Hence
\[(4.5) \quad \omega_2 > 2\omega_1 + 3\omega_3 - 2 \geq 2(\omega_1 + \omega_3) - 2,\]
so that (4.4) implies
\[
\omega_2 > 2^{\omega_1 + 1} - 2\omega_3 - 4,
\]
which implies \(\omega_2 \leq 2\). Then (4.5) implies \(\omega_1 \leq 1\), contradicting (4.3).

5. The case \(p = 2\). Since \(\sigma(n_2)\) is a power of 2, it follows that \(n_2\) is a product of distinct Mersenne primes (Sierpiński [7]), say
\[n_2 = \prod_{i=1}^{w_2} (2^{c_i} - 1)\]
where each \(c_i\) and \(q_i = 2^{c_i} - 1\) is prime, and \(c_1 < c_2 < \cdots < c_{w_2}\).

Suppose \(n_1 = 1\). Then \(s\) is a power of 2, say \(s = 2^r\). Then
\[2^{c+s} = \sigma(n) = \sigma(n_2) = 2^{\sum c_i}\]
so that \(c + a = \sum c_i\). But \(2^{c_i} - 1 \mid \sigma(2^c)\), so \(c_i \mid a + 1\). Since \(c_i, c_2, \ldots, c_{w_2}\) are distinct primes, \(\prod c_i \mid a + 1\). Since \(1 < s = 2^c\), we have \(c \geq 1\). Hence \(\prod c_i \leq a + 1 \leq \sum c_i\), so
\[\prod_{i=1}^{w_2} c_i - \sum_{i=1}^{w_2} c_i \leq 0.\]

This only for \(\omega_2 = 1\), which gives solution (1.3).

We now assume \(n_1 > 1\). Then \(s\) is divisible by an odd prime; in fact, \(s \geq 3^{\omega_1} \geq 3\). Also \(t\) is odd, so \(t \geq 3^{\omega_3}\). As above, \(\prod c_i \mid a + 1\), so \(\tau(a + 1) \geq 2^{w_3}\). Hence from (3.2) and Corollary 2.1 we have
\[(5.1) \quad \omega_1 + \omega_3 \geq 2^{w_3} - \omega_3 - 2.\]

Also from (3.1) we have
so that
\[(5.2) \quad \omega_2 > 3 \omega_1 + 4 \omega_3 - 3 \geq 3(\omega_1 + \omega_3) - 3.
\]
Then (5.1) implies
\[
\omega_2 > 3 \cdot 2^\omega_2 - 3 \omega_2 - 9
\]
so that \(\omega_2 \leq 2\). Then from (5.2) and the fact that \(\omega_i \geq 1\), we have \(\omega_1 = 1\), \(\omega_2 = 0\), and \(\omega_3 > 0\). Hence we have two choices for \(\omega_1, \omega_2, \omega_3\): \(1, 1, 0\) and \(1, 2, 0\). Also since
\[
5 > \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} > \frac{\sigma(2^\omega)}{2^\omega} \cdot \frac{\sigma(n)}{n} = st
\]
and since \(s \geq 3\), \(s \neq 4\), we have \(s = 3\), \(t = 1\).

Suppose \(\omega_2 = 1\). Then \(\sigma(2^\omega) = p_1^\omega(2^\omega - 1)\). Then \(c_1\) is a proper divisor of \(a + 1\). But \(\omega(\sigma(2^\omega)) = 2\), so Corollary 2.1 implies \(a + 1 = 6\) or \(a + 1 = c_1^t\). The first choice gives \(n = 63\), but \(\sigma(63) \neq 3 \cdot 2^s\). Hence \(a + 1 = c_1^t\). Then Theorem 2.1 implies \(\text{ord}_{p_1}(2) = c_1^t\), so that \(p_1 \equiv 1 \pmod{c_1^t}\). If \(c_1 \geq 3\), then \(p_1 \geq 19\), \(q_1 = 2^s - 1 \geq 7\), so that
\[
3 = st < \frac{2}{1} \cdot \frac{7}{6} \cdot \frac{19}{18} < 3,
\]
a contradiction. Hence \(c_1 = 2\), \(a + 1 = 4\), \(n = 15\), and we have solution (1.4).

Our last case is \(\omega_2 = 2\). Then \(\sigma(2^\omega) = p_1^\omega(2^s - 1)(2^t - 1)\), so that \(c_1c_2 | a + 1\). Now \(\omega(\sigma(2^\omega)) = 3\), so that Corollary 2.1 implies \(c_1c_2 = a + 1\), where \(c_1c_2 \neq 6\). We also have \(\sigma(p_1^\omega(2^s - 1)(2^t - 1)) = 3 \cdot 2^s\). Then \(\sigma(p_1^\omega)\) is 3 times a power of 2. Now \(\sigma(p_1^\omega) \neq 3\), so \(\sigma(p_1^\omega)\) is even. Hence \(2 | a_1 + 1\). Now Theorem 2.1 implies \(\text{ord}_{p_1}(2) = c_1c_2\), a composite number. Hence \(p_1\) is not Mersenne. Also, \(p_1 \equiv 1 \pmod{c_1c_2}\). From Corollary 2.1 and the fact that \(\omega(\sigma(p_1^\omega)) = 2\) we have \(a_1 = 1\). Hence for some \(d\) we have \(p_1 = 3 \cdot 2^d - 1\). If \(c_1 > 2\), then \(q_1 = 2^s - 1 \geq 7\), \(q_2 = 2^t - 1 \geq 31\), \(p_1 \geq 2c_1c_2 + 1 \geq 31\). Then
\[
3 = st < \frac{2}{1} \cdot \frac{31}{30} \cdot \frac{31}{30} \cdot \frac{7}{6} < 3,
\]
so that we must have \(c_1 = 2\). Then
\[
2^{c_1} - 1 = (3 \cdot 2^d - 1)(2^s - 1)(2^t - 1)
\]
where \( c_2 \geq 3 \). Looking at this equation mod 8, we obtain 
\[ 3 \cdot 2^d - 1 \equiv 2^i - 1 \text{ (mod 8)} \]
Hence \( d = 2, p_1 = 11 \). Then 
\[ a + 1 = 2c_2 = \text{ord}_{p_1}(2) = 10 \]
This gives solution (1.6).

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