POLYNOMIAL CONSTRAINTS FOR FINITENESS OF SEMISIMPLE RINGS

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Suppose $R$ is an associative ring with Jacobson radical $J$. Suppose that for each sequence $x_1, \ldots, x_n$ in $R$ there exists a polynomial $p$ homogeneous (of bounded degree) in each $x_i$ and a monomial $w$ in the $x$'s, in which some $x_i$ is missing, such that $p = w$. Then $R/J$ is finite. It is also shown that if the above polynomial $p$ is a monomial, then $R/J$ is finite and $J$ is nil of bounded index.

In a recent paper, the authors proved the following theorem:

Suppose $R$ is an associative ring with Jacobson radical $J$. Suppose further, that, for all $x_1, \ldots, x_n$ in $R$, there exists a word $w(x_1, \ldots, x_n)$, depending on $x_1, \ldots, x_n$, in which at least one $x_i$ ($i$ varies) is missing, and such that

\begin{equation}
    x_1 \cdots x_n = w(x_1, \ldots, x_n).
\end{equation}

Then $J$ is a nil ring of bounded index and $R/J$ is finite. In the present paper, we consider the structure of an associative ring $R$ which satisfies, instead of the identity (1) above, an identity of the form

\begin{equation}
    p(x_1, \ldots, x_n) = w(x_1, \ldots, x_n).
\end{equation}

In particular, we take a closer look at the structure of $R$ in those cases where (i) $p(x_1, \ldots, x_n)$ is any fixed word involving each of the variables $x_1, \ldots, x_n$ at least once, or (ii) $p(x_1, \ldots, x_n)$ is a variable polynomial in $x_1, \ldots, x_n$ with integer coefficients such that each $x_i$ is of the same degree in each term of $p(x_1, \ldots, x_n)$, and where these degrees are bounded. We show, for example, that if $p(x_1, \ldots, x_n)$ is as in (i) above, then the Jacobson radical $J$ of $R$ is nil of bounded index and $R/J$ is finite. Moreover, we show that, if $p(x_1, \ldots, x_n)$ is as in (ii) above, then $R/J$ is still finite. We conclude by giving some examples of the rings under consideration.

In establishing the results of this paper, we use the structure theory of rings, starting with the division ring case, then the primitive ring case, followed by the semisimple ring case.

1. Main results. Throughout $R$ will denote an associative ring, $Z$ will denote the ring of integers, and $n$ will denote a fixed positive integer $>1$. We now introduce the following.

**Definition 1.** Let $Z[x_1, \ldots, x_n]$ be the ring of polynomials in $n$
noncommuting indeterminates $x_i, \ldots, x_n$ over $\mathbb{Z}$. Let $\mathcal{V}_n$ be the subset of $\mathbb{Z}[x_1, \ldots, x_n]$ consisting of polynomials $p(x_1, \ldots, x_n)$ such that each $x_i$ appears in every term in $p(x_1, \ldots, x_n)$. By a word $w(x_1, \ldots, x_n)$ we mean a product in which each factor is $x_i$, for some $i$. Let $X \subseteq \mathcal{V}_n$. An associative ring $R$ is called an $X$-ring if, for all $a_1, \ldots, a_n$ in $R$, there exists a polynomial $p(x_1, \ldots, x_n)$ in $X$ and a word $w(x_1, \ldots, x_n)$ with some $x_j (j$ varies) missing from $w(x_1, \ldots, x_n)$, such that

$$p(a_1, \ldots, a_n) = w(a_1, \ldots, a_n), \text{ some } a_j \text{ missing from } w(a_1, \ldots, a_n).$$

A division ring (respectively, primitive ring, semisimple ring) which is also an $X$-ring is called an $X$-division ring (respectively, $X$-primitive ring, $X$-semisimple ring).

The following lemma is immediate from the definition of an $X$-ring.

**Lemma 1.** (a) If $X_1 \subseteq X_2$, then any $X_1$-ring is also an $X_2$-ring. 
(b) Any subring and any homomorphic image of an $X$-ring is also an $X$-ring.

In preparation for the proofs of the main results, we first establish the following lemmas.

**Lemma 2.** Suppose that $X \subseteq \mathcal{V}_n$ and $D$ is an $X$-division ring. Then $D$ is of prime characteristic.

**Proof.** Suppose that the characteristic of the $X$-division ring $D$ is zero. Then $D$ contains the rationals. Suppose that $q_1, \ldots, q_n$ are the first $n$ primes. Then, by hypothesis,

$$p(q_1, \ldots, q_n) = w(q_1, \ldots, q_n),$$

where each term in the polynomial $p(x_1, \ldots, x_n)$ involves every $x_i$, while some $x_j$ is missing from the word $w(x_1, \ldots, x_n)$. Thus, $q_j$ divides the left side of (4) but $q_j$ does not divide the right side of (4), a contradiction. This contradiction proves the lemma.

**Lemma 3.** Let $R$ be a $\mathcal{V}_n$-primitive ring. Then $R$ is a complete matrix ring $D_q$ over a division ring $D$ of prime characteristic.

**Proof.** Let $R$ be a $\mathcal{V}_n$-primitive ring. Then, by Jacobson's Density Theorem [3; p. 33], either (i) $R \cong$ a complete matrix ring $D_q$ over a division ring $D$, or (ii) for every positive integer $l$, there
exists a subring $S$ of $R$ such that $S$ maps homomorphically onto $D_i$ for some division ring $D$. We now show that (ii) never holds. For, suppose (ii) is true. Then, by taking $l = n$ in (ii) and by using Lemma 1(b), we readily obtain

$$(5) \quad D_n \text{ is a } \mathcal{V}_n\text{-ring }.$$ 

Now, let

$$(6) \quad a_i = \begin{bmatrix} 1 & 1 & \cdots & 0 & \circ & \circ & \cdots & \circ \\ \circ & 0 & 1 & 1 & \cdots & \circ & \cdots & \circ \\ \circ & \circ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \end{bmatrix}, \ (i = 1, \ldots, n),$$

where $a_i$ is a diagonal $n \times n$ matrix over $D$ with an entry of 0 in the $(i, i)$ position and with entries of 1 elsewhere on the main diagonal. By hypothesis, there exists a polynomial $p(x_1, \ldots, x_n)$ and a word $w(x_1, \ldots, x_n)$ such that

$$(7) \quad \begin{cases} p(a_1, \ldots, a_n) = w(a_1, \ldots, a_n); \text{ each } a_i \text{ appears in every} \\ \text{term in } p(a_1, \ldots, a_n); \text{ some } a_j \text{ is missing from } w(a_1, \\ \ldots, a_n). \end{cases}$$

Keeping in mind (6), and comparing the entries in the $(j, j)$ positions of the two matrices in the equation in (7), we obtain $0 = 1$, a contradiction. This contradiction shows that (ii) above never holds, and hence (i) above holds always. We have thus shown that

$$(8) \quad \text{The ground } \mathcal{V}_n\text{-primitive ring } R \text{ is a complete matrix}$$

ring $D_n$ over a division ring $D$.

Moreover, $D$, as a subring of $R(=D_n)$, is a $\mathcal{V}_n$-division ring (see Lemma 1(b)), and hence, by Lemma 2, $D$ is of prime characteristic. This proves the lemma.

**Lemma 4.** The direct sum of $n$ rings $R_i$ each of which has a unit-element is not a $\mathcal{V}_n$-ring.

**Proof.** Suppose that

$$(9) \quad R \cong R_1 + \cdots + R_n; \text{ each } R_i \text{ has a unit-element }.$$

Let

$$a_i = (1, 1, \ldots, 1, 0, 1, 1, \ldots, 1); \text{ 0 is in the } i\text{th position }.$$ 

An argument similar to the one given in the proof of Lemma 3
shows that we obtain a contradiction if we assume that \( R \) is a \( \mathcal{Y}_n \)-ring. This proves the lemma.

We are now in a position to prove the following

**Theorem 1.** Let \( X \subseteq \mathcal{Y}_n \). Then every \( X \)-semisimple ring is finite if and only if every \( X \)-division ring of prime characteristic is finite.

**Proof.** The "only if" part of the theorem being obvious, we now proceed to prove the "if" part. Thus, suppose that

(10) Every \( X \)-division ring of prime characteristic is finite.

Suppose, further, that \( R \) is an \( X \)-semisimple ring which is not finite. We shall show that this leads to a contradiction. Since \( R \) is semisimple, there exist ideals \( I_\alpha, \alpha \in \Omega \) of \( R \) such that [3; p. 14]

\[ \bigcap_{\alpha \in \Omega} I_\alpha = (0); \text{ each } R/I_\alpha \text{ is primitive.} \]

Now, by Lemma 1 and Lemma 3, it readily follows that \( R/I_\alpha \) is a complete matrix ring \( D_q \) over a division ring \( D \) of prime characteristic. Since \( D \) is a subring of \( R/I_\alpha (= D_\alpha) \), it follows, by Lemma 1(b), that \( D \) is an \( X \)-division ring of prime characteristic, and hence \( D \) is finite, by (10). Therefore,

(11) \( R/I_\alpha (= D_\alpha) \) is finite.

Now, choose \( \alpha_i \in R \), and having chosen \( \alpha_i, \ldots, \alpha_k \) so that

(12) \[ \sum_{i=1}^{k} R/I_{\alpha_i} \cong R/\bigcap_{i=1}^{k} I_{\alpha_i}, \]

choose \( \alpha_{k+1} \in \Omega \) such that \( \bigcap_{i=1}^{k} I_{\alpha_i} \supseteq I_{\alpha_{k+1}} \). That such \( \alpha_{k+1} \) can always be so chosen is proved as follows: suppose no such \( \alpha_{k+1} \) exists. Then

(0) \[ \bigcap_{\alpha \in \Omega} I_{\alpha} = \bigcap_{i=1}^{k} I_{\alpha_i}, \text{ and hence (see (12))} \]

\[ R \cong R/\bigcap_{i=1}^{k} I_{\alpha_i} \cong \sum_{i=1}^{k} R/I_{\alpha_i}. \]

Thus, using (11), we see that \( R \) is finite, a contradiction. This contradiction shows that there exists \( \alpha_{k+1} \in \Omega \) such that \( \bigcap_{i=1}^{k} I_{\alpha_i} \supseteq I_{\alpha_{k+1}} \). Now, as we have seen in (11), \( R/I_{a_{k+1}} \) is simple. Since, moreover,

\[ \bigcap_{i=1}^{k} I_{\alpha_i} \supseteq I_{\alpha_{k+1}}, \text{ we have } \bigcap_{i=1}^{k} I_{\alpha_i} + I_{\alpha_{k+1}} = R. \]

Hence, by applying the second isomorphism theorem, we readily verify that

\[ R/\bigcap_{i=1}^{k+1} I_{\alpha_i} \cong R/\bigcap_{i=1}^{k} I_{\alpha_i} + R/I_{\alpha_{k+1}} \cong \sum_{i=1}^{k+1} R/I_{\alpha_i}, \]
by (12). In particular, we have

$$\sum_{i=1}^{n} R/I_{\alpha_i} \cong R/\bigcap_{i=1}^{n} R/I_{\alpha_i}.$$  

Hence, using Lemma 1(b), $\sum_{i=1}^{n} R/I_{\alpha_i}$ is an $X$-ring (and thus a $\%$-ring) also. This, however, contradicts Lemma 4 (see (11)). This contradiction shows that $R$ is finite, and the theorem is proved.

We call a field $F$ periodic if for every $x$ in $F$, we have $x^m = x^n$ for some positive integers $m, n, m \neq n$. A periodic field which is also an $X$-ring is called an $X$-periodic field. We now prove the following

**Theorem 2.** Let $X \subseteq \%$. Suppose that there exists a fixed integer $N$ such that, for all polynomials $p(x, \ldots, x_n)$ in $X$, the degree in $x_i, \ldots, x_n$ of every term in $p(x, \ldots, x_n)$ is less than $N$. Then, every $X$-semisimple ring is finite if and only if every $X$-periodic field is finite.

**Proof.** The "only if" part of the theorem being obvious, we now proceed to prove the "if" part. Thus, suppose that $R$ is an $X$-semisimple ring. Now, in view of Theorem 1, it suffices to show that

(13) Every $X$-division ring $D$ of prime characteristic is a periodic field.

Thus, suppose that $D$ is an $X$-division ring of prime characteristic $p$, and suppose $a \in D$. We first show that

(14) $a$ is algebraic over $GF(p)$.

Clearly, we may assume that $a \neq 0$. Now, suppose that

(15) $q_1, \ldots, q_n$ are fixed distinct primes; each $q_i > N$, and

(16) $h_i = (q_1 \cdots q_n)/q_i; (i = 1, \ldots, n)$.

Then

(17) $q_j$ divides $h_i$ if and only if $i \neq j$.

Let $b_i = a^{h_i}$. Then since $D$ is an $X$-ring there exists a polynomial $p(x, \ldots, x_n) \in X$, and a word $w(x, \ldots, x_n)$ with some $x_k$ missing such that

(18) $p(b_1, \ldots, b_n) = w(b_1, \ldots, b_n)$.

Since $x_k$ is missing from $w(x, \ldots, x_n)$ we have by (17),
\begin{equation}
\sum r m_r a^{r} \text{ where } c_r = \sum d_{i} h_{i} \text{ and } m_r \text{ are some integers.}
\end{equation}

By hypothesis each \( d_{i} < N \) and thus, by (15), \( q_k > d_{i} \). We therefore have, by (17), that \( q_k \not| c_r \) for every \( r \). In particular, by (19), \( c_r \neq t \) for any \( r \). We now have, by a combination of (18), (19) and (20),
\begin{equation}
\sum r m_r a^{r} = a^t, \quad c_r \neq t \text{ for all } r.
\end{equation}

Hence \( a \) is algebraic over \( GF(p) \) and (14) is proved.

Now, consider the field \((GF(p))(a)\). Since, by (14), \( a \) is algebraic over \( GF(p) \), it is easily seen that \((GF(p))(a)\) is a finite field, and hence
\[
a^m = a^n; \quad m, n \text{ positive integers; } m \neq n; (a \in D).\]

Thus, by Jacobson's Theorem [3], \( D \) is a periodic field. The theorem now follows from Theorem 1.

In preparation for the proof of the next theorem, we now introduce the following notations and lemmas.

Suppose \( a \) and \( b \) are positive integers, \( a > b \), which are relatively prime, and suppose
\begin{equation}
V_n = a^n - b^n, \quad (a, b) = 1.
\end{equation}

Let \( n_1, n_2, \ldots, n_k \) be all the distinct positive divisors of \( n \) which are less than \( n \). Then \( V_n \) is divisible by \( V_{n_1}, V_{n_2}, \ldots, V_{n_k} \). A divisor of \( V_n \) which is relatively prime to all of the \( V_{n_i} (i = 1, \ldots, k) \) is called a primitive divisor of \( V_n \). For example, 5 is a primitive divisor of \( 2^4 - 1^4 \).

The following lemma was proved by Birkhoff and Vandiver [1]:

**Lemma 5.** Let \( n \) be a positive integer, \( n \neq 2 \), and let \( V_n \) be as in (22). Then \( V_n \) has at least one primitive divisor other than unity, with the single exception \( V_2 = 2^1 - 1^1 \).

Next we introduce the following

**Notation.** \( Z^+ \) will denote the set of all positive integers. Let \( s \in Z^+ \). Then,
\[
D(s) = \{m | m \in Z^+, \text{ } m \text{ divides } s\};
\]
\[
P(s) = \{m | m \in D(s), \text{ } m \text{ is prime}\}.
\]
If $S$ is any nonempty subset of $\mathbb{Z}^+$, then
$$D(S) = \bigcup_{s \in S} D(s) \quad \text{and} \quad P(S) = \bigcup_{s \in S} P(s).$$

The following lemma is an immediate consequence of Lemma 5.

**Lemma 6.** Let $p \in \mathbb{Z}^+$, $p > 1$, and let $\{k_i | i \in \mathbb{Z}^+\}$ be a strictly increasing sequence of positive integers such that $k_i$ divides $k_{i+1}$ for each $i$. Let
$$S = \{p^{k_i} - 1 | i \in \mathbb{Z}^+\}.$$Then $P(S)$ is infinite.

We are now in a position to prove the following

**Theorem 3 (Principal Theorem).** Let $X \subseteq \mathbb{F}_n$. Suppose that there exists a fixed integer $N$ such that, for all polynomials $p(x_1, \ldots, x_n)$ in $X$, the degree in $x_\ell$, $x_n$ of every term in $p(x_1, \ldots, x_n)$ is less than $N$. Suppose, further, that for all polynomials $p(x_1, \ldots, x_n)$ in $X$, each $x_\ell$ is of the same degree in each term in $p(x_1, \ldots, x_n)$. Then every $X$-semisimple ring is finite.

**Proof.** In view of Theorem 2, it suffices to show that every $X$-periodic field $F$ is finite. Suppose not; that is, suppose that $F$ is an infinite $X$-periodic field. Then $F$ is of prime characteristic $p$, since $F$ is periodic. Moreover, the subfield $\langle x \rangle$ generated by a single element $x$ in $F$ is finite, and hence
$$(23) \quad x^{p^k} = x \text{ for some positive integer } k = k(x).$$Now, for each $j \in \mathbb{Z}^+$, define
$$(24) \quad F_j = \{x | x \in F, x^{p^j} = x\}.$$Then, in view of (23) and (24), we have (since if $x \in F$ satisfies (23), then $x \in F_i$)
$$(25) \quad F_i = F_j \subseteq F_3 \subseteq \cdots; \text{ each } F_i \text{ is a finite subfield of } F; \bigcup_{i \in \mathbb{Z}^+} F_i = F.$$Now, since $F$ is infinite, we can find a subsequence of (25) such that
$$(26) \quad F_{i_1} \subseteq F_{i_2} \subseteq F_{i_3} \subseteq \cdots, \text{ and again } \bigcup_{k \in \mathbb{Z}^+} F_{i_k} = F.$$Moreover, the order of $F_{i_\sigma} = p^{k_\sigma}(\sigma \in \mathbb{Z}^+)$. Next, let
$$S = \{p^{k_\sigma} - 1 | \sigma \in \mathbb{Z}^+\}.$$
Then, as is well known, $k_{\sigma} < k_{\sigma+1}$ for each $\sigma \in \mathbb{Z}^+$, and $k_{\sigma} < k_{\sigma+1}$, by (26). Hence by Lemma 6, $P(S)$ is infinite, and there, therefore, exist $n$ distinct primes $q_1, \ldots, q_n$ in $P(S)$, such that

\[(27) \quad q_i > N + p, \quad (i = 1, \ldots, n).\]

Thus, there exists $m_j \in \mathbb{Z}^+$ such that

\[(28) \quad q_j \in P(p^{k_{m_j}} - 1), \quad (j = 1, \ldots, n).\]

Now, since the nonzero elements of the field $F_{m_j}$ form a multiplicative group of order $p^{k_{m_j}} - 1$, and since the prime $q_j | p^{k_{m_j}} - 1$, it follows, by Cauchy’s theorem, that there exists $a_j \in F_{m_j} (\cong F)$ such that

\[(29) \quad \text{order of } a_j = q_j; \quad a_j \neq 0 \quad (j = 1, \ldots, n).\]

Now, since $F$ is an $X$-ring, there exists a polynomial $p(x_1, \ldots, x_n)$ in $X$ and a word $w(x_1, \ldots, x_n)$ such that

\[(30) \quad \begin{cases} p(a_1, \ldots, a_n) = w(a_1, \ldots, a_n); \text{each } x_i \text{ appears in every} \cr \text{term in } p(x_1, \ldots, x_n); \text{some } x_j \text{ is missing from } w(x_1, \ldots, x_n); \cr \text{all coefficients in } p(x_1, \ldots, x_n) \text{ are integers}. \end{cases}\]

Moreover, recalling that $F$ is commutative, and using the hypothesis regarding the degrees of the $x_i$’s in the various terms of $p(x_1, \ldots, x_n)$, we see that

\[(31) \quad m^p = 1.\]

Now, let $c_j$ be the degree of $x_j$ in the word $w_i(x_1, \ldots, x_n)$. Then, by hypothesis,

\[(32) \quad c_j < N; \quad (j = 1, \ldots, n).\]

Let

\[(33) \quad M = \frac{q_1 \cdots q_n}{q_j} (p - 1).\]

Then, by (29), (30),

\[(34) \quad (m w_i(a_1, \ldots, a_n))^\mu = (w(a_1, \ldots, a_n))^\mu.\]

Hence by (31), (28), (33), and the fact that $a_j$ is missing from the
word \( w(a_1, \ldots, a_n) \), the above equality reduces to
\[
(a_j^e)^w = 1.
\]
Therefore (see (28)), \( q_j \) divides \( c_j M \). This is absurd, however, since \( q_j \) does not divide \( c_j \) (recall that \( q_j > N > c_j \); see (27), (32)), and \( q_j \) does not divide \( M \) (recall that \( q_j > p \), by (27); also see (33)). This contradiction proves the theorem.

Next, we prove the following

**Theorem 4.** Let \( X \) be a collection of words in \( \mathcal{V}_n \) each of which is of degree \( < N \) in \( x_1, \ldots, x_n \). If \( R \) is an \( X \)-ring with Jacobson radical \( J \), then \( J \) is a nil ring of bounded index, and \( R/J \) is finite.

**Proof.** Clearly, \( X \) satisfies the hypotheses of Theorem 3, and hence the \( X \)-semisimple ring \( R/J \) is finite. Now, to prove that \( J \) is nil, let \( a \in J \), and suppose that

\[
q_1, \ldots, q_n \text{ are distinct primes, each } q_i > N.
\]

Let

\[
h_i = \frac{q_1 \cdots q_n}{q_i}; \quad (i = 1, \ldots, n),
\]

and let

\[
b_i = a^{h_i}; \quad (i = 1, \ldots, n).
\]

Then, since \( R \) is an \( X \)-ring, there exist words \( w_i(x_1, \ldots, x_n) \) and \( w(x_1, \ldots, x_n) \) such that

\[
\begin{cases}
w_i(b_1, \ldots, b_n) = w(b_1, \ldots, b_n); w_i(x_1, \ldots, x_n) \in X; \\
some \ x_j \text{ is missing from } w(x_1, \ldots, x_n).
\end{cases}
\]

Now, let

\[
degree \ of \ x_i \ in \ w_i(x_1, \ldots, x_n) = c_i; \quad (i = 1, \ldots, n).
\]

Then, by (38), (37), (39), we obtain

\[
a^{e_1 h_1 + \cdots + e_n h_n} = a^t.
\]

Moreover, since \( x_j \) is missing from the word \( w(x_1, \ldots, x_n) \), it is easily seen (see (36)) that

\[
q_j \text{ divides } t.
\]

On the other hand, since \( q_j > N \), by (35), and \( c_j < N \) (since \( w_i(x_1, \ldots, x_n) \in X \)), \( q_j \) does not divide \( c_j \). Also, by (36), \( q_j \) does not divide
and hence the prime $q_j$ does not divide $c_j h_j$. However, by (36), $q_j$ divides $h_i$ for each $i \neq j$. Therefore,

$$q_j \text{ does not divide } c_i h_i + \cdots + c_n h_n. \tag{42}$$

Comparing (41), (42), we see that

$$c_i h_i + \cdots + c_n h_n \neq t. \tag{43}$$

Now, let $c_i h_i + \cdots + c_n h_n = l$, and let

$$M = Nh_i + \cdots + Nh_n; \quad (M > l, \text{ since each } c_i < N). \tag{44}$$

Then, by (40), (43), (44), it is easily seen that

$$a^s = a^s, \text{ for some positive integer } s; s \neq M. \tag{45}$$

Now, if in (45), $s < M$, by iterating in (45), we can eventually make $s > M$. We have thus shown that

$$a^s = a^s; s > M > 0; M \text{ fixed}. \tag{46}$$

Equation (46) readily implies that a suitable power of $a$ is an idempotent element in $J$ (recall that $a$ is in $J$), and hence by (46), $a^s = 0$, $(M \text{ fixed})$. Thus $J$ is nil of bounded index, and the theorem is proved.

In view of Theorem 4, it follows that $J$ is locally nilpotent [2; p. 28].

The following corollary is an immediate consequence of Theorem 4 as well as Theorem 3.

**Corollary 1.** Let $X$ consist of a single fixed word involving each of the variables $x_1, \ldots, x_n$. Then every $X$-semisimple ring is finite.

If, further, we let $X$ consist of the single fixed word $x_1 \cdots x_n$, we obtain, as a further corollary of Theorem 4, the following result which has already been proved by the authors [5]:

**Corollary 2.** Let $R$ be an associative ring with Jacobson radical $J$ and with the property that, for all $x_1, \ldots, x_n$ in $R$, there exists a word $w(x_1, \ldots, x_n)$ depending on $x_1, \ldots, x_n$, in which at least one $x_i$ (i varies) is missing, and such that $x_1 \cdots x_n = w(x_1, \ldots, x_n)$. Then $J$ is a nil ring of bounded index and $R/J$ is finite.

2. Examples and remarks. In the following examples, we show that the class of $X$-rings subsumes all finite rings and all nilpotent rings. We also give an example of an $X$-ring which is neither finite
nor nilpotent.

**EXAMPLE 1.** Let \( R \) be any finite ring with exactly \( m \) elements. Let \( x_1, \ldots, x_{m+1} \) be any elements of \( R \). Then \( x_i = x_j \) for some \( i > j \), and hence

\[
x_1 \cdots x_{m+1} = x_1 \cdots x_j \cdots x_{i-1} x_ix_{i+1} \cdots x_{m+1} = u(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+1}).
\]

Thus \( R \) is an \( X \)-ring, where \( X = \{x_1 \cdots x_{m+1}\} \).

**EXAMPLE 2.** Let \( R \) be any nilpotent ring, say \( R^m = (0) \). Then, for all elements \( x_1, \ldots, x_{m+1} \) of \( R \), we have

\[
x_1 \cdots x_{m+1} = 0 = x_1 \cdots x_m.
\]

Thus \( R \) is an \( X \)-ring, where \( X = \{x_1 \cdots x_{m+1}\} \).

**EXAMPLE 3.** Let \( R_0 \) be an infinite field of characteristic 2, and let

\[
R = \left\{ \begin{pmatrix} a & u \\ 0 & 0 \end{pmatrix} \mid a \in GF(2), \ u \in R_0 \right\}.
\]

Let \( x_1, x_2, x_3 \) be any elements of \( R \). Then, as is readily verified,

\[
x_1x_2x_3 = x_2x_3 \quad \text{if} \quad x_1 = \begin{pmatrix} 1 & u \\ 0 & 0 \end{pmatrix};
\]

\[
x_1x_2x_3 = x_1x_2 \quad \text{if} \quad x_1 = \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}.
\]

Hence \( R \) is an \( X \)-ring, where \( X = \{x_1x_2x_3\} \). Observe that \( R \) is neither finite nor nilpotent. In fact, \( R \) is not isomorphic to any finite direct sum of finite or nilpotent rings.

Returning to our Principal Theorem (Theorem 3), we have the following

**REMARK.** In the proof of Theorem 3, we showed that every \( X \)-periodic field is finite. We claim that the group-theoretic analogue of this result is false. To see this, consider the group \( Z(p^n) \), which consists of the set of all \( p^n \)-th roots of unity, where \( p \) is a fixed prime and \( n = 0, 1, 2, \ldots \) \([4, \text{p. 4}]\). Suppose that \( x_1, x_2 \in Z(p^n) \). Then, for some integer \( n, x_1, x_2 \in Z(p^n) \), where \( Z(p^n) \) is the group of all \( p^n \)-th roots of unity. Let \( \sigma \) be a generator of \( Z(p^n) \). Then

\[
x_1 = \sigma^r, \ x_2 = \sigma^s; \ 1 \leq r \leq p^n, \ 1 \leq s \leq p^n.
\]
Now let
\[ r = r_0 p^i, \quad s = s_0 p^j; \quad (r_0, p) = 1, \quad (s_0, p) = 1, \]
and suppose, without loss of generality, that \( i \leq j \). Since \((r/p^i, p) = 1\), there exists a solution \( x \) to
\[ (r/p^i)x \equiv s/p^i \mod p^n, \]
and hence \( rxp^{j-i} \equiv s \mod p^n \). Thus, \( r + s \equiv r(1 + xp^{j-i}) \mod p^n \), and hence
\[ \sigma^{r+s} = (\sigma^r)^{1+xp^{j-i}}, \]

since \( \sigma^{p^n} = 1 \). Therefore \( x_1x_2 = (x_1)^{1+xp^{j-i}} \). Note that \( \mathbb{Z}(p^n) \) is an infinite group.

We leave as an open question whether or not Corollary 1 is true when \( X \) consists of a single fixed polynomial in which each term involves every variable \( x_1, \ldots, x_n \). In view of Theorem 2, this question reduces to whether or not an \( X \)-periodic field is finite in this case.

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