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**HYPONORMAL CONTRACTIONS AND STRONG POWER  
CONVERGENCE**

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## HYPONORMAL CONTRACTIONS AND STRONG POWER CONVERGENCE

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Let  $T^*$  be a hyponormal contraction on a Hilbert space, so that  $TT^* - T^*T = D \geq 0$  and  $\|T\| \leq 1$ . It is shown that if, in addition,  $T^*$  is completely hyponormal, then the sequence  $\{T^n\}_{n=1,2,\dots}$  converges strongly to 0 as  $n \rightarrow \infty$ . The result is obtained as a consequence of properties of the solution  $w(z)$  of  $(T - zI)w(z) = x$ , where  $x$  is a certain vector in the range of  $D$ .

1. Let  $T$  be a bounded operator on a Hilbert space  $\mathfrak{H}$  with spectrum  $\sigma(T)$  and point spectrum  $\Pi_0(T)$ . The range and null space of  $T$  will be denoted by  $R(T)$  and  $N(T)$  respectively. If  $A$  is any linear manifold in  $\mathfrak{H}$ , its closure will be denoted by  $[A]$ . Also, we shall consider the set of numbers  $z$  for which  $\bar{z} \in \Pi_0(T^*)$  and which will be denoted by  $(\Pi_0(T^*))^*$ .

Let  $T_z = T - zI$  for any complex number  $z$  and let  $D$  be a nonnegative self-adjoint operator satisfying

$$(1.1) \quad T_z T_z^* \geq D \geq 0 \quad \text{for all } z \text{ in } C.$$

It was shown in Putnam [8] that if  $D$  has the spectral resolution

$$(1.2) \quad D = \int_0^\infty u dF_u$$

and if  $x$  is any vector satisfying

$$(1.3) \quad x = F((s, \infty))x, \quad s > 0,$$

then  $T_z^{-1}x$  is bounded and weakly continuous on  $C - P$ , where  $P = \{z: z \in \Pi_0(T) \text{ or } \bar{z} \in \Pi_0(T^*)\}$ . (Actually, the set  $P$  occurring in [8] was defined differently but should have been defined as above.) This result will be strengthened below to the following

**THEOREM 1.** *Suppose (1.1), (1.2) and that  $x \in \mathfrak{H}$  satisfies*

$$(1.4) \quad k_x \equiv \int_{+0}^\infty u^{-1} d \|F_u x\|^2 < \infty.$$

*Then there exists a vector-valued function  $w(z)$  on  $C$  satisfying*

$$(1.5) \quad T_z w(z) = x \text{ and } \|w(z)\| \leq k_x^{1/2}, \quad z \in C,$$

and having the following properties. At every point  $z_0 \notin \Pi_0(T)$ ,  $w(z)$  is weakly continuous, that is, for every  $f$  in  $\mathfrak{S}$ ,  $(w(z), f)$  is continuous at  $z_0$ . Further, if  $\mathfrak{S}$  is separable then, for every  $f$  in  $\mathfrak{S}$ , the function  $(w(z), f)$  is Lebesgue planar measurable on the set  $C - (\Pi_0(T^*))^*$ . In addition, if  $\alpha$  is any rectifiable curve in  $C$  with arc length measure  $m_\alpha$  and if  $m_\alpha(\alpha \cap (\Pi_0(T^*))^*) = 0$  then  $(w(z), f)$  is  $m_\alpha$ -measurable as well as  $dz (= dx + idy)$ -measurable on  $\alpha$ .

REMARKS. Note that if  $z \in \Pi_0(T)$  then necessarily  $w(z) = T_z^{-1}x$ , and that, for any  $f$  in  $\mathfrak{S}$ ,  $(w(z), f)$  is analytic in  $C - \sigma(T)$ . Further, it is clear that all vectors  $x$  of (1.3) satisfy (1.4) and hence that the set of vectors  $x$  satisfying (1.4) is dense in  $R(D)$ .

That the set  $\Pi_0(T^*)$  occurring in the statement of Theorem 1 and, more generally, the point spectrum of any bounded operator on a separable Hilbert space, is Lebesgue planar measurable follows from a result of Dixmier and Foaïş [3] as Nikolskaya [7]. We are indebted to K. F. Clancey for informing us of these facts.

Recall that a bounded operator  $S$  is said to be hyponormal if  $S^*S - SS^* \geq 0$  and completely hyponormal if, in addition, there does not exist any non-trivial reducing subspace of  $S$  on which its restriction is normal. If  $S_z = S - zI$ , then  $S_z^*S_z - S_zS_z^* = S^*S - SS^*$ . Clearly, if  $S$  is hyponormal then  $\Pi_0(S) \subset (\Pi_0(S^*))^*$  and any eigenvector of  $S$  belonging to  $z$  is also an eigenvector of  $S^*$  belonging to  $\bar{z}$ . In particular,  $\Pi_0(S)$  must be empty whenever  $S$  is completely hyponormal. Further, it is easy to see that if  $T^*$  is hyponormal then (1.1) holds with  $D = TT^* - T^*T$ . Consequently, in view of Theorem 1, we have the following

**THEOREM 2.** *Let  $T^*$  be completely hyponormal on  $\mathfrak{S}$  and let  $D = TT^* - T^*T (\geq 0)$  have the spectral resolution (1.2). If  $x \in \mathfrak{S}$  satisfies (1.4) then there exists a vector-valued function  $w(z)$  on  $C$  satisfying the conditions of Theorem 1. Thus, relation (1.5) holds and  $w(z)$  is weakly continuous at all points  $z_0 \notin \Pi_0(T)$ . If  $\mathfrak{S}$  is separable, then, since  $\Pi_0(T^*)$  is now empty,  $(w(z), f)$  is Lebesgue planar measurable in  $C$  and is measurable with respect to arc length measure and to the  $dz = dx + idy$  measure on all rectifiable curves in  $C$ .*

As a consequence of Theorem 2 there will be proved

**THEOREM 3.** *Let  $T^*$  be completely hyponormal on  $\mathfrak{S}$  and suppose that  $T$  is a contraction, that is,  $\|T\| \leq 1$ . Then  $\{T^n\}_{n=1,2,\dots}$  converges*

strongly to 0 as  $n \rightarrow \infty$ , that is,  $\|T^n f\| \rightarrow 0$  as  $n \rightarrow \infty$  for every  $f$  in  $\mathfrak{S}$ .

REMARKS. It follows from Theorem 3 that if  $T^*$  is any hyponormal contraction then  $T$  can be written as the direct sum  $T = T_1 \oplus N$ , where  $T_1^*$  is completely hyponormal,  $T_1^n \rightarrow 0$  strongly as  $n \rightarrow \infty$ , and  $N$  is normal. Clearly,  $N = \int z dK_z$  can be further decomposed as  $N = \int_{|z| < 1} z dK_z + \int_{|z|=1} z dK_z = N_1 \oplus N_2$ , where  $N_1^n \rightarrow 0$  strongly as  $n \rightarrow \infty$  and  $N_2$  is unitary. Hence, one has the following

COROLLARY 1 OF THEOREM 3. *Let  $T^*$  be any hyponormal contraction on a Hilbert space. Then  $T = T_2 \oplus U$  where  $T_2^n \rightarrow 0$  strongly as  $n \rightarrow \infty$  and  $U$  is unitary, where it is understood that either component of the direct sum may be missing.*

Thus, if  $T^*$  is any completely nonunitary (cf. Sz.-Nagy and Foiaş [11], p. 72) hyponormal contraction, then  $T^n \rightarrow 0$  strongly as  $n \rightarrow \infty$ , so that  $T$  is of class  $C_0$ . (cf. [11], p. 72). It was shown in [8], p. 167, that if  $T^*$  is a hyponormal contraction for which  $T^n \not\rightarrow 0$  then  $T$  has a nontrivial invariant subspace. The above Corollary yields the stronger result that  $T^*$  (hence  $T$ ) even has a unitary part. Also, it follows from the Corollary that if  $T^*$  is a hyponormal contraction for which  $T^n f \not\rightarrow 0$  as  $n \rightarrow \infty$  whenever  $f \neq 0$ , then  $T$  must be unitary. In case  $T^*$  is also subnormal, this last result was obtained by Stampfli [10].

COROLLARY 2 OF THEOREM 3. *Let  $T$  be a completely hyponormal contraction on a Hilbert space. Then  $T^*$  is (unitarily equivalent to) the restriction of the adjoint of a unilateral shift to an invariant subspace.*

*Proof.* Actually, every contraction  $S$  satisfying  $S^n \rightarrow 0$  strongly as  $n \rightarrow \infty$  is unitarily equivalent to the restriction of the adjoint of a unilateral shift to an invariant subspace (Foiaş [4], de Branges and Rovnyak [1, 2]. See also Halmos [5], problem 121, and Sz.-Nagy and Foiaş [11] p. 95. Note that the unilateral shift in question is, in general, not the simple unilateral shift.

2. *Proof of Theorem 1.* The proof will be an extension and refinement of the argument given in [8]. Let  $z$  be fixed and let  $T_z T_z^*$  have the spectral resolution

$$(2.1) \quad T_z T_z^* = \int_0^\infty u dE_u^{(z)} .$$

Then, by an argument like that on pp. 165-166 of [8],

$$\int_0^\infty \lim_{t \rightarrow 0^+} (u+t)^{-1} d \| E_u^{(z)} x \|^2 \leq k_x ,$$

where  $k_x$  is defined by (1.4). It follows that

$$(2.2) \quad E^{(z)}(\{0\})x = 0 \quad \text{and} \quad \int_{+0}^\infty u^{-1} d \| E_u^{(z)} x \|^2 \leq k_x ,$$

and hence, for any  $z$  in  $C$ ,

$$(2.3) \quad y(z) = \int_{+0}^\infty u^{-1/2} d E_u^{(z)} x \text{ is defined and } \| y(z) \|^2 \leq k_x .$$

Next, let  $T_z = T - zI$  have the polar factorization (see Kato [6], pp. 334-335)

$$(2.4) \quad T_z = U(z)G(z) ,$$

where  $G(z) = (T_z^* T_z)^{1/2}$  and  $U(z)$  is partially isometric with initial set  $[R(G(z))]$  and final set  $[R(T_z)]$ . Then  $T_z U^*(z)y(z) = (U(z)G(z)U^*(z))y(z) = (T_z T_z^*)^{1/2} y(z) = x$ . On putting

$$(2.5) \quad w(z) = U^*(z)y(z) ,$$

one sees that (1.5) follows from (2.3).

Next, it will be shown that the above defined bounded vector-valued function  $w(z)$  on  $C$  is weakly continuous at every point  $z_0$  not in  $\Pi_0(T)$ . It must be shown that  $w(z)$  converges weakly to  $w(z_0)$ , that is, for any  $f$  in  $\mathfrak{H}$ ,  $(w(z), f) \rightarrow (w(z_0), f)$  as  $z \rightarrow z_0$ . If this limit relation did not hold however, then, since  $w(z)$  is bounded, there would exist a  $z_0$  and a sequence  $\{z_n\}$  such that  $w(z_n) \rightarrow p$  (weakly) as  $z_n \rightarrow z_0$  with  $p \neq w(z_0)$ . It follows from the relation  $T_z w(z) = x$ , on letting  $z = z_n$  and noting that  $\| T - T_n \| \rightarrow 0$ , that  $T_{z_0} p = x$  and, since  $T_{z_0} w(z_0) = x$ , that

$$(2.6) \quad T_{z_0}(p - w(z_0)) = 0 .$$

Since  $z_0 \notin \Pi_0(T)$ , then  $p = w(z_0)$ , a contradiction.

There remains then to establish the measurability of  $w(z)$  in the sense described in Theorem 1, at least if  $\mathfrak{H}$  is separable. To this end, we first shall show that, whether or not  $\mathfrak{H}$  is separable, if  $T$  is any operator with the polar factorization of (2.4), then

$$(2.7) \quad U(z) \rightarrow U(z_0) \text{ strongly as } z \rightarrow z_0 \text{ whenever } z_0 \notin \Pi_0(T) .$$

Assume then that  $z_0 \notin \Pi_0(T)$ . Note (cf. [6], pp. 334-335) that

$U(z)$  is defined for vectors in  $R(G(z))$  by  $U(z): G(z)u \rightarrow T_z u$  and that  $U(z)$  is then extended by continuity to be isometric on  $[R(G(z))]$ . For  $y$  in  $N(G(z)) (= N(T_z))$ ,  $U(z)y = 0$ . Since  $z_0 \notin \Pi_0(T)$ , then  $N(G(z_0)) = 0$  and so  $U(z_0)$  is isometric.

Since  $R(G(z_0))$  is dense in  $\mathfrak{H}$ , relation (2.7) will follow if it is shown that

$$(2.8) \quad U(z)v \longrightarrow U(z_0)v \text{ (strongly) as } z \longrightarrow z_0, \text{ whenever } v \in R(G(z_0)).$$

Suppose then that  $v \in R(G(z_0))$ , so that  $v = G(z_0)u$  for some vector  $u$ . In view of  $U(z)G(z)u = T_z u$  and  $U(z_0)G(z_0)u = T_{z_0} u$ , we have  $U(z)v - U(z_0)v = (T_z - T_{z_0})u - U(z)(G(z) - G(z_0))u$ . Since  $\|T_z - T_{z_0}\| \rightarrow 0$ , hence also  $\|G(z) - G(z_0)\| \rightarrow 0$ , as  $z \rightarrow z_0$ , relation (2.8), hence also (2.7), follows. By symmetry, we have also

$$(2.9) \quad U^*(z) \longrightarrow U^*(z_0) \text{ strongly as } z \longrightarrow z_0 \text{ whenever } \bar{z}_0 \notin \Pi_0(T^*).$$

Henceforth, it will be supposed that  $T$  is the operator occurring in the statement of Theorem 1. By (2.2) and (2.3),  $y(z) \in [R(T_z T_z^*)^{1/2}] = [R(T_z)]$ , and this set is the initial set of  $U^*(z)$ . Since  $w(z) = U^*(z)y(z)$ , it follows that  $U(z)w(z) = U(z)U^*(z)y(z) = y(z)$ . We shall show that

$$(2.10) \quad y(z) \longrightarrow y(z_0) \text{ weakly as } z \longrightarrow z_0, \bar{z}_0 \notin \Pi_0(T^*).$$

If (2.10) did not hold then, since  $y(z)$  is (uniformly) bounded in  $C$ , there would exist a sequence  $\{z_n\}$  for which  $z_n \rightarrow z_0$  and  $y(z_n) \rightarrow q$  (weakly) as  $n \rightarrow \infty$  with  $q \neq y(z_0)$ . Since  $w(z)$  is also bounded, we may choose a subsequence of  $\{z_n\}$ , which will also be denoted by  $\{z_n\}$ , such that  $w(z_n) \rightarrow p$  (weakly).

Let  $f$  be arbitrary in  $\mathfrak{H}$ . Then  $(y(z_n)f) \rightarrow (q, f)$  and also  $(y(z_n), f) = (U(z_n)w(z_n), f) = (w(z_n), U^*(z_n)f)$ . In view of (2.9), we have  $(w(z_n), U^*(z_n)f) \rightarrow (p, U^*(z_0)f) = (U(z_0)p, f)$ , and hence  $q = U(z_0)p$ . Since  $y(z_0) = U(z_0)w(z_0)$ , we see that  $q - y(z_0) = U(z_0)(p - w(z_0))$ . But, as noted earlier,  $T_{z_0}(p - w(z_0)) = 0$  (cf. (2.6)), so that  $p - w(z_0) \in N(G(z_0))$  and hence  $U(z_0)(p - w(z_0)) = 0$ . Thus  $q = y(z_0)$ , a contradiction, and so (2.10) is proved.

In summary, we see that the vector-valued function  $w(z)$  on  $C$  is weakly continuous at  $z_0 \notin \Pi_0(T)$ . The vector-valued function  $y(z)$  is weakly continuous at  $z_0$  if  $\bar{z}_0 \notin \Pi_0(T^*)$ . Also, the operator-valued function  $U^*(z)$  on  $C$  is strongly continuous and, hence,  $U(z)$  is weakly continuous at  $z_0$  if  $\bar{z}_0 \notin \Pi_0(T^*)$ .

Suppose now that  $\mathfrak{H}$  is separable. Then, as noted earlier,  $\Pi_0(T^*)$

(hence also  $(\Pi_0(T^*))^*$ ) is Lebesgue planar measurable. It will be shown that for any  $f$  in  $\mathfrak{H}$ , the function  $(w(z), f)$  is Lebesgue planar measurable on  $C - (\Pi_0(T^*))^*$ . For, let  $\{\phi_n\}(n = 1, 2, \dots)$  be any complete orthonormal system for  $\mathfrak{H}$ . Then  $(w(z), f) = (y(z), U(z)f) = \sum_{n=1}^{\infty} (y(z), \phi_n)(\phi_n, U(z)f)$ . But each term of the summation is a function continuous at all points  $z$  for which  $\bar{z}$  is not in  $\Pi_0(T^*)$ . In particular, each such term, and hence the sum, is (planar) measurable on  $C - \Pi_0(T^*)^*$ . (The argument is similar to that used in [9], p. 384, in connection with the proof of Stone's theorem on unitary groups.)

Finally, a similar argument establishes the assertion of the last part of Theorem 1 and the proof is now complete.

3. *Proof of Theorem 3.* Without loss of generality it may be supposed that  $\mathfrak{H}$  is separable. It follows from Theorem 2 that if  $w(z)$  is defined by (2.5) and if  $f$  is arbitrary in  $\mathfrak{H}$ , then  $(w(z), f)$  is (bounded and) measurable with respect to arc length and to the measure  $dz = dx + idy$  on every circle  $C_r = \{z: |z| = r\}$ ,  $0 < r < \infty$ . Let

$$(3.1) \quad y(r) = -(2\pi i)^{-1} \int_{C_r} w(z) dz \left( = -r(2\pi i)^{-1} \int_C w(rt) dt \right),$$

where  $C = C_1$  and all circles are oriented positively. It is understood, of course, that  $y(r)$  is defined in terms of the relation  $(y(r), f) = -(2\pi i)^{-1} \int_{C_r} (w(z), f) dz$  for any  $f$  in  $\mathfrak{H}$  and that the latter integral is a Lebesgue integral. A similar remark applies to the other integrals of this section.

The set  $\Pi_0(T) \cap \{z: |z| = 1\}$  is empty; otherwise,  $T$  would have a normal part. (In fact, if  $T$  is any contraction and if  $z$  is an eigenvalue of  $T$  satisfying  $|z| = 1$  with eigenvector  $f$  then  $\bar{z}$  is an eigenvalue of  $T^*$  with the same eigenvector  $f$ ; cf. [11], p. 8.) If  $z$  is fixed and  $|z| = 1$ , then, by Theorem 2,  $w(rz) \rightarrow w(z)$  (weakly) as  $r \rightarrow 1 - 0$ . For any fixed  $f$  in  $\mathfrak{H}$ , it follows from (3.1) and the uniform boundedness convergence theorem that

$$\begin{aligned} (y(r), f) &= -(2\pi i)^{-1} \int_{C_r} (w(z), f) dz \\ &\longrightarrow -(2\pi i)^{-1} \int_C (w(z), f) dz \text{ as } r \longrightarrow 1 - 0, \end{aligned}$$

Thus,  $y(r) \rightarrow y(1)$  (weakly) as  $r \rightarrow 1 - 0$ . Similarly,  $y(r) \rightarrow y(1)$  (weakly) as  $r \rightarrow 1 + 0$ . But, if  $r > 1$ ,  $-(2\pi i)^{-1} \int_{C_r} T_z^{-1} v dz = v$ , for arbitrary  $v$ , so that, if  $x$  is any vector satisfying (1.4),  $y(r) = x$  for  $r > 1$  and hence  $y(1) = x$ . Hence, we have for such vectors  $x$ ,

$$(3.2) \quad y(r) \longrightarrow x \text{ (weakly) as } r \longrightarrow 1 - 0 .$$

In view of (1.5),

$$\begin{aligned} Ty(r) &= -(2\pi i)^{-1} \int_{c_r} Tw(z)dz \\ &= -(2\pi i)^{-1} \int_{c_r} (T - z + z)w(z)dz , \\ &= -(2\pi i)^{-1} \int_{c_r} zw(z)dz . \end{aligned}$$

Similarly, one sees that  $T^n y(r) = -(2\pi i)^{-1} \int_{c_r} z^n w(z)dz$  for  $n = 1, 2, \dots$ , and hence

$$(3.3) \quad T^n y(r) \longrightarrow 0 \text{ (strongly) as } n \longrightarrow \infty, \text{ for } r < 1 .$$

Next, let  $\mathfrak{M} = \{v: T^n v \rightarrow 0 \text{ (strongly) as } n \rightarrow \infty\}$ . Since  $T$  is a contraction,  $\mathfrak{M}$  is a subspace invariant under  $T$ . Also, by (3.3), each  $y(r)$ ,  $r < 1$ , belongs to  $\mathfrak{M}$ . Hence, by (3.2), if  $u$  is any vector in  $\mathfrak{M}^\perp$ ,  $0 = (y(r), u) \rightarrow (x, u)$  as  $r \rightarrow 1 - 0$ , and so  $x \in \mathfrak{M}$ , where  $x$  is any vector satisfying (1.4). Since such vectors are dense in  $R(D)$ ,  $R(D) \subset \mathfrak{M}$ .

Let now  $\mathfrak{S}$  denote the least subspace containing  $R(D)$  and reducing  $T$ . It will be shown that

$$(3.4) \quad \mathfrak{S} \subset \mathfrak{M} .$$

To see this, note that if  $u \in \mathfrak{M}$  then  $Tu \in \mathfrak{M}$ . Also,  $TT^* - TT^* = D$  and hence  $T^n T^* u = T^{n-1} T^* Tu + T^{n-1} Du$ . Since  $Du \in \mathfrak{M}$ , then  $T^{n-1} Du \rightarrow 0$  as  $n \rightarrow \infty$  and hence  $\limsup_{n \rightarrow \infty} \|T^n T^* u\| \leq \|Tu\|$ . Repetition of this argument shows that  $\limsup_{n \rightarrow \infty} \|T^n T^* u\| \leq \|T^k u\|$  for  $k = 1, 2, \dots$ , and hence  $T^n Tu \rightarrow 0$  as  $n \rightarrow \infty$ , so that  $T^* u \in \mathfrak{M}$ . Thus, whenever  $u$  is in  $\mathfrak{M}$  so also are  $Tu$  and  $T^* u$ . Since  $R(D) \subset \mathfrak{M}$ , the desired relation (3.4) follows.

It is clear that  $\mathfrak{S}^\perp$  also reduces  $T$  and that  $T|_{\mathfrak{S}^\perp}$  is normal. Since  $T^*$  is completely hyponormal then  $\mathfrak{S}^\perp = 0$ , and so by (3.4),  $\mathfrak{M} = \mathfrak{S}$ . This completes the proof of Theorem 3.

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