GROUPS OF *-AUTOMORPHISMS AND INVARIANT MAPS OF VON NEUMANN ALGEBRAS

Kazuyuki Saitō
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Let $M$ be a von Neumann algebra and let $G$ be a group acting on $M$ by *-automorphisms of $M$. $M$ is $G$-finite if for every nonnegative element $a$ in $M$ with $a \neq 0$, there exists a $G$-invariant normal state $\phi$ such that $\phi(a) \neq 0$. The main result in this paper asserts that $M$ is $G$-finite if and only if for every weakly relatively compact subset $K$ of the predual of $M$, the orbit of $K$ under $G$ is also weakly relatively compact.

Given a noncommutative dynamical system, that is, pairs $(M, G)$ where $M$ is a von Neumann algebra and $G$ is a group of *-automorphisms of $M$, one can ask whether or not there are sufficiently many $G$-invariant normal states (we call such a case that $(M, G)$ is $G$-finite [9])?

First result along these lines is due to I. Kovacs and J. Szücs [9] who obtained that $(M, G)$ is $G$-finite if and only if there is a $G$-invariant faithful normal projection of norm one from $M$ onto the fixed subalgebra $M^G$ under $G$ (see also [11, 14]).

Recently, using results of Akemann [1] and Takesaki [15] concerning the predual of a von Neumann algebra, together with the Ryll-Nardzewski fixed point theorem ([5, 10]), F. J. Yeadon gave an elegant proof of the existence of a trace in a finite von Neumann algebra [16].

In this paper, we will give a Banach space like characterization of the $G$-finiteness of $(M, G)$ using weakly relatively compact subsets of the predual $M_*$ of $M$ which is a noncommutative extension of a theorem of Hajian and Kakutani ([7, 8]) and in case where $G$ is the inner automorphisms of $M$, includes the result of F. J. Yeadon (see also [16]). The result in this paper can be easily extended to groups of identity preserving isometries of $M$.

2. Notations and a statement of a theorem. Let $(M, G)$ be a noncommutative dynamical system and $M_*$ be the predual of $M$, that is, the Banach space of all bounded normal (or $\sigma$-weakly continuous) linear functionals on $M([3, 12])$. Let $(T_g \varphi)(a) = \varphi(a^g)$, $a \in M$, $g \in G$ and $\varphi \in M_*$, then $T_g$ is a linear isometry of $M_*$ onto $M_*$. We say that $(M, G)$ is $G$-finite if $M$ has sufficiently many normal states in the sense that for every nonnegative element $a$ in $M$ with $a \neq 0$, there exists a $G$-invariant normal state $\phi$ (that is, $T_g \phi = \phi$, $g \in G$) such that $\phi(a) \neq 0$. 
Now we state our main theorem.

**THEOREM.** Let \((M, G)\) be a noncommutative dynamical system, then \((M, G)\) is \(G\)-finite if and only if for every weakly relatively compact (w.r.c.) subset \(K\) of \(M^*_\ast\), the orbit of \(K\) under \(G\), that is, the set \(\{T_g\phi; g \in G, \phi \in K\}\) is also w.r.c.

3. Proof of Theorem. "If" part of Theorem is valid under a weaker assumption, more precisely to say that if for every \(\phi\) in \(M^*_\ast\) with \(\phi \not\equiv 0\), \(\{T_g\phi; g \in G\}\) is w.r.c., then \((M, G)\) is \(G\)-finite. However, this is an easy consequence of lemma in [14] (see also [11]). To prove the converse, we need the following lemma which concerns with the continuity of the map \((\Phi, \omega) \mapsto \omega \cdot \Phi\) from \(L_\ast(M) \times M_\ast \to M_\ast\) where \(L_\ast(M)\) is the \(\sigma\)-weakly continuous bounded linear maps of \(M\) into \(M\) equipped with the weak operator topology and \(M_\ast\) has the \(W^\ast\)-topology. For the later discussions, we state it in the following form.

**LEMMA 1.** Let \(N\) be a von Neumann algebra with a set \(H\) of normal \(*\)-homomorphisms of \(N\) into \(N\). Suppose that for every \(\phi\) \(N_\ast\) (the predual of \(N\)) with \(\phi \not\equiv 0\), and every sequence \(\{b_n\}\) in the nonnegative part of the unit sphere \(S\) of \(N\) such that \(b_n \to 0\) \((\sigma\)-weakly\), \(\phi(\Phi(b_n)) \to 0 (n \to \infty)\) uniformly for \(\Phi \in H\). Let \(\{\phi_n\}\) be a sequence in \(N_\ast\) which converges weakly to some \(\phi_0\) in \(N_\ast\) and \(\{a_n\}\) be a sequence of self-adjoint element in \(S\) which converges strongly to 0, then \(\phi_j(\Phi(a_n)) \to 0 (n \to \infty)\) uniformly not only for \(\Phi \in H\) but also for \(j\).

**Proof.** Observe first that the \(\sigma\)-weak topology restricted on \(S\) is a compact Hausdorff topology with the neighborhood basis which consists of all possible sets \(\{a; a \in S, |\psi_i(a) - \psi_i(a_0)| < \varepsilon, i = 1, 2, \ldots, n\}\) with \(a_0 \in S, \varepsilon > 0\) (real number) and \(\psi_i \in N_\ast(\psi_i \geq 0)\). Let \(H_i = \{a \in S; |(\phi_j - \phi_0)(a)| \leq \varepsilon\ \text{for all} \ j \geq i\}, \text{then} \ H_i \text{is \(\sigma\)-weakly closed subset of} \ S \text{for each} \ i \text{and} \ S = \bigcup_{i=1}^\infty H_i\). Now Baire's category theorem says that there are a natural numbers \(i(0), m, \) an element \(a_0\) in \(S\) and \(\psi_i(i = 1, 2, \ldots, m)\) in \(N_\ast\) with \(\psi_i \geq 0\) for all \(i\) such that

\[
\bigcap_{i=1}^m \{a; a \in S, |\psi_i(a) - \psi_i(a_0)| < 1\} \subset H_{i(0)}.
\]

Since \(a_n \to 0(n \to \infty)\) strongly, by the spectral theorem, for any given positive number \(\varepsilon\), there is a sequence \(\{e_n\}\) of projections in \(M\) such that \(e_n \to 1\) (strongly) and \(\|a_n e_n\| \leq \varepsilon/6\) for each \(n\). By the uniform boundedness theorem, we may assume that \(\sup_{j} (||\phi_j||, ||\phi_0||) = 1\) without loss of generality. For each \(\Phi \in H\), we have \(\|\Phi(e_n a_n e_n)\| \leq \|a_n e_n\| \leq \varepsilon/6\), \(\|\Phi((1-e_n) a_n e_n)\| \leq \|a_n e_n\| \leq \varepsilon/6\) and \(\|\Phi((1-e_n) a_n e_n)\| \leq \varepsilon/6\) for each \(n\).
\|a_n e_n\| \leq \varepsilon/6 for each n. Thus we have
\[
| (\phi_j - \phi_0)(\phi(a_n)) | \leq \varepsilon
+ | (\phi_j - \phi_0)(\phi(e_n a_n(1 - e_n))) |
+ | (\phi_j - \phi_0)(\phi((1 - e_n)a_n(1 - e_n))) |
\leq \varepsilon + | (\phi_j - \phi_0)(\phi((1 - e_n)a_n(1 - e_n))) |.
\]

Put \(b_n(\phi) = \phi((1 - e_n)a_n(1 - e_n)) + \phi(e_n)a_n\phi(e_n)\), then, since \(b_n(\phi) - a_0 = (1 - \phi(e_n))\phi(a_n)(1 - \phi(e_n)) - (1 - \phi(e_n))\phi(a_n) - \phi(a_n(1 - \phi(e_n)))\), we have, by Schwarz' inequality,
\[
|\psi_i(b_n(\phi) - a_0)| \leq \psi_i(\phi(1 - e_n)) + 3 \psi_i(\phi(1 - e_n))^{1/2}.
\]

Similarly, we have
\[
|\psi_j(\phi(e_n)a_n \phi(e_n) - a_0)| \leq \psi_i(\phi(1 - e_n)) + 2 \psi_i(\phi(1 - e_n))^{1/2}.
\]

Since, by the assumption, \(\psi_j(\phi(1 - e_n)) \to 0(n \to \infty)\) uniformly for \(\phi \in H\) and \(i = 1, 2, \ldots, m\), we can choose a natural number \(n(\varepsilon)\) (depends only on \(\varepsilon\)) such that \(b_n(\phi), \phi(e_n)a_n\phi(e_n) \in H_{(0)}\) for all \(n \geq n(\varepsilon)\). Thus, we have
\[
| (\phi_j - \phi_0)(\phi((1 - e_n)a_n(1 - e_n))) | < 2\varepsilon
\]
for all \(j \geq i(0), \phi \in H\) and all \(n \geq n(\varepsilon)\). Since, for each \(j(= 1, 2, \ldots, i(0) - 1)\)
\[
| (\phi_j - \phi_0)(\phi(a_n)) | = | (\phi_j - \phi_0)(\phi(a_n) v_j) |
\leq \sqrt{| (\phi_j - \phi_0)(\phi(a_n)) |^2} \sqrt{| (\phi_j - \phi_0) |^2}
\leq 2^{1/2} | (\phi_j - \phi_0) | (\phi(a_n))^{1/2}
\]
and
\[
| \phi_0(\phi(a_n)) | = | \phi_0(\phi(a_n) v) | \leq | \phi_0(\phi(a_n)) |^{1/2}
\]
where \(\phi_j - \phi_0 = R v_j, \phi_j - \phi_0 \) (resp. \(\phi_0 = R v_0 \)) is the polar decomposition of \(\phi_j - \phi_0\) (resp. \(\phi_0\)) ([12]), \(a_n^* \to 0\) weakly implies, by the assumption, that there is a positive integer \(n(\varepsilon)'\) (depending only on \(\varepsilon\)) such that
\[
| (\phi_j - \phi_0)(\phi(a_n)) | < \varepsilon \text{ and } | \phi_0(\phi(a_n)) | < \varepsilon \text{ for all } \phi \in H, j = 1, 2, \ldots, i(0) - 1 \text{ and all } n \geq n(\varepsilon)'.
\]
Combining the above estimations, we have
\[
| \phi_j(\phi(a_n)) | < 4\varepsilon \text{ for all } n \geq \max(n(\varepsilon), n(\varepsilon)') , \text{ all } j
\]
and all \(\phi \in H\). This completes the proof of Lemma 1.

Before going into the proof of theorem, we prepare the following:
LEMMA 2. Keep the notations in theorem. If \((M, G)\) is \(G\)-finite, then, for every sequence \(\{a_n\}\) of nonnegative elements in the unit sphere \(S\) of \(M\) which converges weakly to 0, and every \(\phi\) in \(M^*\), \((T_\phi)(a_n) \to 0\) uniformly for \(g \in G\).

**Proof.** If not, there exists a positive number \(\varepsilon_0\) such that for each positive integer \(n\), we can choose a positive integer \(k(k(n) \uparrow \infty)\) and \(g(n) \in G\) such that

\[
|T_{g(n)} \phi(a_{k(n)})| \geq \varepsilon_0.
\]

Put \(a_{k(n)} = b(n)\) then since \(\{(b(n))^{\varphi(n)}\}\) is a \(\sigma\)-weakly relatively compact subset of \(S \cap M^+\) (where \(M^+\) is the positive portion of \(M\)), there is a \(\sigma\)-weakly cluster point \(a(a \geq 0)\) of \(\{(b(n))^{\varphi(n)}\}\). Thus for every positive number \(\delta\), every \(G\)-invariant normal state \(\rho\) and every positive integer \(n\), there is a natural number \(i(n)(i(n) > n)\) such that

\[
|\rho(a) - \rho(b(n))^{\varphi(i(n))}| < \delta \quad n = 1, 2, \ldots.
\]

Since \(\rho\) is \(G\)-invariant, \(\rho((b(i(n)))^{\varphi(i(n))}) = \rho(b(i(n))) \to 0(i(n) \to \infty)\). Thus \(\rho(a) \leq \delta\) for every \(\delta\) and the \(G\)-finiteness of \((M, G)\) implies \(a = 0\). Hence this contradicts with the inequality (\(*\)). Thus \((T_\phi)(a_n) \to 0(n \to \infty)\) uniformly for \(g \in G\) and the proof is completed.

**Proof of Theorem.** Suppose \((M, G)\) is \(G\)-finite. We will prove that for every w.r.c. subset \(K\) of \(M^*\), \(\{T_\phi; \phi \in Kg \in G\}\) is also w.r.c. To prove this, we have only to show that for every orthogonal sequence \(\{e(n)\}\) of projections, \(\lim_{n \to \infty} T_{\phi}(e(n)) = 0\) uniformly for \(g \in G\) and \(\phi \in K\). If not, there is a positive number \(\varepsilon\) such that for each positive integer \(k\), there are a natural number \(n(k)(n(k) \uparrow \infty)\), \(g(k) \in G\) and \(\phi_k \in K\) such that

\[
|T_{g(k)} \phi(k)(e(n(k)))| \geq \varepsilon.
\]

By Eberlein-Šmulian’s theorem ([4]), there is a subsequence \(\{\phi_{p(n)}\}\) of \(\{\phi_k\}(k(\uparrow \infty)\) such that \(\phi_{p(n)} \to \phi_0\) weakly \((p \to \infty)\) for some \(\phi_0\) in \(M^*\). Now \(e(n(k(p))) \to 0(p \to \infty)\) strongly, which implies by Lemma 2 and Lemma 1, that \(|T_{g(k(p))} \phi_{k(p)}(e(n(k(p))))| \to 0(p \to \infty)\) and this contradicts with the inequality (\(\ast\)). This completes the proof of theorem.

4. Remarks. Theorem is a generalization of [11]. We should remark that the result of theorem can be easily extended to groups of Jordan Automorphisms of \(M\). [13] When \(G\) is a semi-group of normal Jordan homomorphisms ([13]) of \(M\) into \(M\), by an easy modification of Lemma 1 and Lemma 2, "only if" part of theorem is valid,
however, as the following example shows, the converse assertion does not hold in general, even if $G$ is a semi group of $*$-isomorphisms of $M$ into $M$.

Let $M = L^\infty([0, 1))$ be the abelian von Neumann algebra of essentially bounded complex-valued functions on $[0, 1)$ with respect to Lebesgue measure $\mu$. Let us consider two measurable transformations $g_1$ and $g_2$ defined as follows ([2, 8]): $g_1(\omega) = 3\omega(\text{mod } 1)$, $\omega \in [0, 1)$, $g_2(\omega) = 2\omega + 1/3(\text{resp. } = (\omega - 1/3)/2)$, $\omega \in [0, 1/3)$ (resp. $\omega \in [1/3, 1)$). For each $f \in M$, let $(\Phi_1 f)(\omega) = f(g_1(\omega))$, $\omega \in [0, 1)$ and $(\Phi_2 f)(\omega) = f(g_2(\omega))$, $\omega \in [0, 1)$. Let $H$ be the semi-group of normal $*$-homomorphisms of $M$ into $M$ generated by $\Phi_1$ and $\Phi_2$. Then by [2] and [8], we can easily check that for each $\phi \in M_*$ = $L^\infty([0, 1))$, $\{\phi \circ \Phi, \Phi \in H\}$ is w. r. c. Thus by [6] and Lemma 1, for every w. r. c. subset $K$ of $M_*$, $\{\phi \circ \Phi, \Phi \in H, \phi \in K\}$ is also w. r. c. However, since $g_1$ is ergodic with respect to $\mu$ and $\mu$ is not invariant under $g_2$, $(M, H)$ has no $H$-invariant functional in $M_*$. The above example implies that the Ryll-Nardzewski fixed point theorem is not valid in general without the assumption of distal action of $H$.

**References**


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