

# Pacific Journal of Mathematics

**GROUPS OF \*-AUTOMORPHISMS AND INVARIANT MAPS OF  
VON NEUMANN ALGEBRAS**

KAZUYUKI SAITÔ

## GROUPS OF \*-AUTOMORPHISMS AND INVARIANT MAPS OF VON NEUMANN ALGEBRAS

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**Let  $M$  be a von Neumann algebra and let  $G$  be a group acting on  $M$  by \*-automorphisms of  $M$ .  $M$  is  $G$ -finite if for every nonnegative element  $a$  in  $M$  with  $a \neq 0$ , there exists a  $G$ -invariant normal state  $\phi$  such that  $\phi(a) \neq 0$ . The main result in this paper asserts that  $M$  is  $G$ -finite if and only if for every weakly relatively compact subset  $K$  of the predual of  $M$ , the orbit of  $K$  under  $G$  is also weakly relatively compact.**

Given a noncommutative dynamical system, that is, pairs  $(M, G)$  where  $M$  is a von Neumann algebra and  $G$  is a group of \*-automorphisms of  $M$ , one can ask whether or not there are sufficiently many  $G$ -invariant normal states (we call such a case that  $(M, G)$  is  $G$ -finite [9])?

First result along these lines is due to I. Kovacs and J. Szücs [9] who obtained that  $(M, G)$  is  $G$ -finite if and only if there is a  $G$ -invariant faithful normal projection of norm one from  $M$  onto the fixed subalgebra  $M^G$  under  $G$  (see also [11, 14]).

Recently, using results of Akemann [1] and Takesaki [15] concerning the predual of a von Neumann algebra, together with the Ryll-Nardzewski fixed point theorem ([5, 10]), F. J. Yeadon gave an elegant proof of the existence of a trace in a finite von Neumann algebra [16].

In this paper, we will give a Banach space like characterization of the  $G$ -finiteness of  $(M, G)$  using weakly relatively compact subsets of the predual  $M_*$  of  $M$  which is a noncommutative extension of a theorem of Hajian and Kakutani ([7, 8]) and in case where  $G$  is the inner automorphisms of  $M$ , includes the result of F. J. Yeadon (see also [16]). The result in this paper can be easily extended to groups of identity preserving isometries of  $M$ .

2. Notations and a statement of a theorem. Let  $(M, G)$  be a noncommutative dynamical system and  $M_*$  be the predual of  $M$ , that is, the Banach space of all bounded normal (or  $\sigma$ -weakly continuous) linear functionals on  $M$  ([3, 12]). Let  $(T_g\varphi)(a) = \varphi(a^g)$ ,  $a \in M$ ,  $g \in G$  and  $\varphi \in M_*$ , then  $T_g$  is a linear isometry of  $M_*$  onto  $M_*$ . We say that  $(M, G)$  is  $G$ -finite if  $M$  has sufficiently many normal states in the sense that for every nonnegative element  $a$  in  $M$  with  $a \neq 0$ , there exists a  $G$ -invariant normal state  $\phi$  (that is,  $T_g\phi = \phi$ ,  $g \in G$ ) such that  $\phi(a) \neq 0$ .

Now we state our main theorem.

**THEOREM.** *Let  $(M, G)$  be a noncommutative dynamical system, then  $(M, G)$  is  $G$ -finite if and only if for every weakly relatively compact (w.r.c.) subset  $K$  of  $M_*$ , the orbit of  $K$  under  $G$ , that is, the set  $\{T_g\phi; g \in G, \phi \in K\}$  is also w.r.c.*

**3. Proof of Theorem.** "If" part of Theorem is valid under a weaker assumption, more precisely to say that if for every  $\phi$  in  $M_*$  with  $\phi \geq 0$ ,  $\{T_g\phi; g \in G\}$  is w. r. c., then  $(M, G)$  is  $G$ -finite. However, this is an easy consequence of lemma in [14] (see also [11]). To prove the converse, we need the following lemma which concerns with the continuity of the map  $(\Phi, \omega) \rightarrow \omega \circ \Phi$  from  $L_*(M) \times M_* \rightarrow M_*$  where  $L_*(M)$  is the  $\sigma$ -weakly continuous bounded linear maps of  $M$  into  $M$  equipped with the weak operator topology and  $M_*$  has the  $W^*$ -topology. For the later discussions, we state it in the following form.

**LEMMA 1.** *Let  $N$  be a von Neumann algebra with a set  $H$  of normal  $*$ -homomorphisms of  $N$  into  $N$ . Suppose that for every  $\phi \in N_*$  (the predual of  $N$ ) with  $\phi \geq 0$ , and every sequence  $\{b_n\}$  in the nonnegative part of the unit sphere  $S$  of  $N$  such that  $b_n \rightarrow 0$  ( $\sigma$ -weakly),  $\phi(\Phi(b_n)) \rightarrow 0$  ( $n \rightarrow \infty$ ) uniformly for  $\Phi \in H$ . Let  $\{\phi_n\}$  be a sequence in  $N_*$  which converges weakly to some  $\phi_0$  in  $N_*$  and  $\{a_n\}$  be a sequence of self-adjoint element in  $S$  which converges strongly to 0, then  $\phi_j(\Phi(a_n)) \rightarrow 0$  ( $n \rightarrow \infty$ ) uniformly not only for  $\Phi \in H$  but also for  $j$ .*

*Proof.* Observe first that the  $\sigma$ -weak topology restricted on  $S$  is a compact Hausdorff topology with the neighborhood basis which consists of all possible sets  $\{a; a \in S, |\psi_i(a) - \psi_i(a_0)| < \varepsilon, i = 1, 2, \dots, n\}$  with  $a_0 \in S, \varepsilon > 0$  (real number) and  $\psi_i \in N_*(\psi_i \geq 0)$ . Let  $H_i = \{a \in S; |(\phi_j - \phi_0)(a)| \leq \varepsilon \text{ for all } j \geq i\}$ , then  $H_i$  is  $\sigma$ -weakly closed subset of  $S$  for each  $i$  and  $S = \bigcup_{i=1}^{\infty} H_i$ . Now Baire's category theorem says that there are a natural numbers  $i(0), m$ , an element  $a_0$  in  $S$  and  $\psi_i (i = 1, 2, \dots, m)$  in  $N_*$  with  $\psi_i \geq 0$  for all  $i$  such that

$$\bigcap_{i=1}^m \{a; a \in S; |\psi_i(a) - \psi_i(a_0)| < 1\} \subset H_{i(0)}.$$

Since  $a_n \rightarrow 0$  ( $n \rightarrow \infty$ ) strongly, by the spectral theorem, for any given positive number  $\varepsilon$ , there is a sequence  $\{e_n\}$  of projections in  $M$  such that  $e_n \rightarrow 1$  (strongly) and  $\|a_n e_n\| \leq \varepsilon/6$  for each  $n$ . By the uniform boundedness theorem, we may assume that  $\text{Sup}_j \{\|\phi_j\|, \|\phi_0\|\} = 1$  without loss of generality. For each  $\Phi \in H$ , we have  $\|\Phi(e_n a_n e_n)\| \leq \|a_n e_n\| \leq \varepsilon/6$ ,  $\|\Phi(e_n a_n (1 - e_n))\| \leq \|a_n e_n\| \leq \varepsilon/6$  and  $\|\Phi((1 - e_n) a_n e_n)\| \leq$

$\| \alpha_n e_n \| \leq \varepsilon/6$  for each  $n$ . Thus we have

$$\begin{aligned} |(\phi_j - \phi_0)(\Phi(\alpha_n))| &\leq |(\phi_j - \phi_0)(\Phi(e_n \alpha_n e_n))| \\ &\quad + |(\phi_j - \phi_0)(\Phi(e_n \alpha_n (1 - e_n)))| \\ &\quad + |(\phi_j - \phi_0)(\Phi((1 - e_n) \alpha_n e_n))| \\ &\quad + |(\phi_j - \phi_0)(\Phi((1 - e_n) \alpha_n (1 - e_n)))| \\ &\leq \varepsilon + |(\phi_j - \phi_0)(\Phi((1 - e_n) \alpha_n (1 - e_n)))|. \end{aligned}$$

Put  $b_n(\Phi) = \Phi((1 - e_n) \alpha_n (1 - e_n)) + \Phi(e_n) \alpha_0 \Phi(e_n)$ , then, since  $b_n(\Phi) - \alpha_0 = (1 - \Phi(e_n))\Phi(\alpha_n)(1 - \Phi(e_n)) - (1 - \Phi(e_n))\alpha_0\Phi(e_n) - \Phi(e_n)\alpha_0(1 - \Phi(e_n)) - (1 - \Phi(e_n))\alpha_0(1 - \Phi(e_n))$ , we have, by Schwarz' inequality,

$$|\psi_j(b_n(\Phi) - \alpha_0)| \leq \psi_i(\Phi(1 - e_n)) + 3 \|\psi_i\| \psi_i(\Phi(1 - e_n))^{1/2}.$$

Similarly, we have

$$|\psi_j(\Phi(e_n) \alpha_0 \Phi(e_n) - \alpha_0)| \leq \psi_i(\Phi(1 - e_n)) + 2 \|\psi_i\| \psi_i(\Phi(1 - e_n))^{1/2}.$$

Since, by the assumption,  $\psi_i(\Phi(1 - e_n)) \rightarrow 0 (n \rightarrow \infty)$  uniformly for  $\Phi \in H$  and  $i = 1, 2, \dots, m$ , we can choose a natural number  $n(\varepsilon)$  (depends only on  $\varepsilon$ ) such that  $b_n(\Phi), \Phi(e_n) \alpha_0 \Phi(e_n) \in H_{i(0)}$  for all  $n \geq n(\varepsilon)$ . Thus, we have

$$|(\phi_j - \phi_0)(\Phi((1 - e_n) \alpha_n (1 - e_n)))| < 2\varepsilon$$

for all  $j \geq i(0)$ , all  $\Phi \in H$  and all  $n \geq n(\varepsilon)$ . Since, for each  $j(j = 1, 2, \dots, i(0) - 1)$

$$\begin{aligned} |(\phi_j - \phi_0)(\Phi(\alpha_n))| &= |(\phi_j - \phi_0)(\Phi(\alpha_n) v_j)| \\ &\leq \{|\phi_j - \phi_0| (\{\Phi(\alpha_n)\}^2)\}^{1/2} \|\phi_j - \phi_0\|^{1/2} \\ &\leq 2^{1/2} \{|\phi_j - \phi_0| (\Phi(\alpha_n^2))\}^{1/2} \end{aligned}$$

and

$$|\phi_0(\Phi(\alpha_n))| = |\phi_0(\Phi(\alpha_n) v)| \leq \{|\phi_0| (\Phi(\alpha_n^2))\}^{1/2}$$

where  $\phi_j - \phi_0 = R_{v_j} |\phi_j - \phi_0|$  (resp.  $\phi_0 = R_v |\phi_0|$ ) is the polar decomposition of  $\phi_j - \phi_0$  (resp.  $\phi_0$ ) ([12]),  $\alpha_n^2 \rightarrow 0$  weakly implies, by the assumption, that there is a positive integer  $n(\varepsilon)'$  (depending only on  $\varepsilon$ ) such that  $|(\phi_j - \phi_0)(\Phi(\alpha_n))| < \varepsilon$  and  $|\phi_0(\Phi(\alpha_n))| < \varepsilon$  for all  $\Phi \in H, j = 1, 2, \dots, i(0) - 1$  and all  $n \geq n(\varepsilon)'$ .

Combining the above estimations, we have

$$|\phi_j(\Phi(\alpha_n))| < 4\varepsilon \text{ for all } n \geq \max(n(\varepsilon), n(\varepsilon)'), \text{ all } j$$

and all  $\Phi \in H$ . This completes the proof of Lemma 1.

Before going into the proof of theorem, we prepare the following:

LEMMA 2. *Keep the notations in theorem. If  $(M, G)$  is  $G$ -finite, then, for every sequence  $\{a_n\}$  of nonnegative elements in the unit sphere  $S$  of  $M$  which converges weakly to 0, and every  $\phi$  in  $M_*$ ,  $(T_g\phi)(a_n) \rightarrow 0$  uniformly for  $g \in G$ .*

*Proof.* If not, there exists a positive number  $\varepsilon_0$  such that for each positive integer  $n$ , we can choose a positive integer  $k(k(n) \uparrow \infty)$  and  $g(n) \in G$  such that

$$(*) \quad |T_{g(n)}\phi(a_{k(n)})| \geq \varepsilon_0.$$

Put  $a_{k(n)} = b(n)$  then since  $\{(b(n))^{g(n)}\}$  is a  $\sigma$ -weakly relatively compact subset of  $S \cap M^+$  (where  $M^+$  is the positive portion of  $M$ ), there is a  $\sigma$ -weakly cluster point  $a(a \geq 0)$  of  $\{(b(n))^{g(n)}\}$ . Thus for every positive number  $\delta$ , every  $G$ -invariant normal state  $\rho$  and every positive integer  $n$ , there is a natural number  $i(n)(i(n) > n$  and  $i(n) \uparrow \infty)$  such that

$$|\rho(a) - \rho(b(n))^{g(i(n))}| < \delta \quad n = 1, 2, \dots$$

Since  $\rho$  is  $G$ -invariant,  $\rho((b(i(n)))^{g(i(n))}) = \rho(b(i(n))) \rightarrow 0(i(n) \rightarrow \infty)$ . Thus  $|\rho(a)| \leq \delta$  for every  $\delta$  and the  $G$ -finiteness of  $(M, G)$  implies  $a = 0$ . Hence this contradicts with the inequality (\*). Thus  $(T_g\phi)(a_n) \rightarrow 0(n \rightarrow \infty)$  uniformly for  $g \in G$  and the proof is completed.

*Proof of Theorem.* Suppose  $(M, G)$  is  $G$ -finite. We will prove that for every w.r.c. subset  $K$  of  $M_*$ ,  $\{T_g\phi; \phi \in Kg \in G\}$  is also w.r.c. To prove this, we have only to show that for every orthogonal sequence  $\{e(n)\}$  of projections,  $\lim_{n \rightarrow \infty} T_g\phi(e(n)) = 0$  uniformly for  $g \in G$  and  $\phi \in K$ . If not, there is a positive number  $\varepsilon$  such that for each positive integer  $k$ , there are a natural number  $n(k)(n(k) \uparrow \infty)$ ,  $g(k) \in G$  and  $\phi_k \in K$  such that

$$(**) \quad |T_{g(k)}\phi_k(e(n(k)))| \geq \varepsilon.$$

By Eberlein-Šmulian's theorem ([4]), there is a subsequence  $\{\phi_{k(p)}\}$  of  $\{\phi_k\}(k(p) \uparrow \infty)$  such that  $\phi_{k(p)} \rightarrow \phi_0$  weakly ( $p \rightarrow \infty$ ) for some  $\phi_0$  in  $M_*$ . Now  $e(n(k(p))) \rightarrow 0(p \rightarrow \infty)$  strongly, which implies by Lemma 2 and Lemma 1, that  $|T_{g(k(p))}\phi_{k(p)}(e(n(k(p))))| \rightarrow 0(p \rightarrow \infty)$  and this contradicts with the inequality (\*\*). This completes the proof of theorem.

4. Remarks. Theorem is a generalization of [11]. We should remark that the result of theorem can be easily extended to groups of Jordan Automorphisms of  $M$ . [13] When  $G$  is a semi-group of normal Jordan homomorphisms ([13]) of  $M$  into  $M$ , by an easy modification of Lemma 1 and Lemma 2, "only if" part of theorem is valid,

however, as the following example shows, the converse assertion does not hold in general, even if  $G$  is a semi group of \*-isomorphisms of  $M$  into  $M$ .

Let  $M = L^\infty([0, 1])$  be the abelian von Neumann algebra of essentially bounded complex-valued functions on  $[0, 1]$  with respect to Lebesgue measure  $\mu$ . Let us consider two measurable transformations  $g_1$  and  $g_2$  defined as follows ([2, 8]):  $g_1(\omega) = 3\omega \pmod{1}$ ,  $\omega \in [0, 1]$ ,  $g_2(\omega) = 2\omega + 1/3$  (resp.  $= (\omega - 1/3)/2$ ,  $\omega \in [0, 1/3]$  (resp.  $\omega \in [1/3, 1]$ ). For each  $f \in M$ , let  $(\Phi_1 f)(\omega) = f(g_1\omega)$ ,  $\omega \in [0, 1]$  and  $(\Phi_2 f)(\omega) = f(g_2\omega)$ ,  $\omega \in [0, 1]$ . Let  $H$  be the semi-group of normal \*-homomorphisms of  $M$  into  $M$  generated by  $\Phi_1$  and  $\Phi_2$ . Then by [2] and [8], we can easily check that for each  $\phi \in M_*(= L^1([0, 1]))$ ,  $\{\phi \circ \Phi, \Phi \in H\}$  is w. r. c.. Thus by [6] and Lemma 1, for every w. r. c. subset  $K$  of  $M_*$ ,  $\{\phi \circ \Phi, \Phi \in H, \phi \in K\}$  is also w. r. c. However, since  $g_1$  is ergodic with respect to  $\mu$  and  $\mu$  is not invariant under  $g_2$ ,  $(M, H)$  has no  $H$ -invariant functionals in  $M_*$ .

The above example implies that the Ryll-Nardzewski fixed point theorem is not valid in general without the assumption of distal action of  $H$ .

#### REFERENCES

1. C. A. Akemann, *The dual space of an operator algebra*, Trans. Amer. Math. Soc., **126** (1967), 286-302.
2. J. R. Blum and N. Friedman, *On invariant measures for classes of transformations*, Z. Wahrschein. verw. Geb., **8** (1967), 301-305.
3. J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien*, Gauthier-Villars, Paris, 1969.
4. N. Dunford and J. T. Schwartz, *Linear Operators 1*, Interscience, New York, 1963.
5. F. P. Greenleaf, *Invariant Means on Topological Groups*, Van Nostrand, New York, 1969.
6. A. Grothendieck, *Sur les applications lineaires faiblement compacts d'espaces du type  $C(K)$* , Canad. J. Math., **5** (1953), 129-173.
7. A. B. Hajian and S. Kakutani, *Weakly wandering sets and invariant measures*, Trans. Amer. Math. Soc., **110** (1964), 136-151.
8. A. B. Hajian and Y. Itô, *Weakly wandering sets and invariant measures for a group of transformations*, J. Math. and Mech., **18** (1969), 1203-1216.
9. I. Kovacs and J. Szücs, *Ergodic type theorems in von Neumann algebras*, Acta Sci. Math. (Szeged), **27** (1966), 233-246.
10. I. Namioka and E. Asplund, *A geometric proof of Ryll-Nardzewski's fixed point theorem*, Bull. Amer. Math. Soc., **73** (1967), 443-445.
11. K. Saitô, *Automorphism groups of von Neumann algebras and ergodic type theorems*, Acta Sci. Math. (Szeged), **36** (1974), 119-130.
12. S. Sakai, *C\*-algebras and W\*-algebras*, Springer, Berlin, Heidelberg, New York, 1971.
13. E. Størmer, *On the Jordan structures of C\*-algebras*, Trans. Amer. Math. Soc., **120** (1965), 438-447.
14. ———, *Invariant states of von Neumann algebras*, Math. Scand., **30** (1972), 253-256.

15. M. Takesaki, *On the conjugate space of an operator algebra*, Tôhoku Math. J., **10** (1958), 194-203.
16. F. J. Yeadon, *A new proof of the existence of a trace in a finite von Neumann algebra*, Bull. Amer. Math. Soc., **77** (1971), 257-260.

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