

Pacific Journal of Mathematics

**GROUPS OF *-AUTOMORPHISMS AND INVARIANT MAPS OF
VON NEUMANN ALGEBRAS**

KAZUYUKI SAITÔ

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Let M be a von Neumann algebra and let G be a group acting on M by *-automorphisms of M . M is G -finite if for every nonnegative element a in M with $a \neq 0$, there exists a G -invariant normal state ϕ such that $\phi(a) \neq 0$. The main result in this paper asserts that M is G -finite if and only if for every weakly relatively compact subset K of the predual of M , the orbit of K under G is also weakly relatively compact.

Given a noncommutative dynamical system, that is, pairs (M, G) where M is a von Neumann algebra and G is a group of *-automorphisms of M , one can ask whether or not there are sufficiently many G -invariant normal states (we call such a case that (M, G) is G -finite [9])?

First result along these lines is due to I. Kovacs and J. Szücs [9] who obtained that (M, G) is G -finite if and only if there is a G -invariant faithful normal projection of norm one from M onto the fixed subalgebra M^G under G (see also [11, 14]).

Recently, using results of Akemann [1] and Takesaki [15] concerning the predual of a von Neumann algebra, together with the Ryll-Nardzewski fixed point theorem ([5, 10]), F. J. Yeadon gave an elegant proof of the existence of a trace in a finite von Neumann algebra [16].

In this paper, we will give a Banach space like characterization of the G -finiteness of (M, G) using weakly relatively compact subsets of the predual M_* of M which is a noncommutative extension of a theorem of Hajian and Kakutani ([7, 8]) and in case where G is the inner automorphisms of M , includes the result of F. J. Yeadon (see also [16]). The result in this paper can be easily extended to groups of identity preserving isometries of M .

2. Notations and a statement of a theorem. Let (M, G) be a noncommutative dynamical system and M_* be the predual of M , that is, the Banach space of all bounded normal (or σ -weakly continuous) linear functionals on M ([3, 12]). Let $(T_g\varphi)(a) = \varphi(a^g)$, $a \in M$, $g \in G$ and $\varphi \in M_*$, then T_g is a linear isometry of M_* onto M_* . We say that (M, G) is G -finite if M has sufficiently many normal states in the sense that for every nonnegative element a in M with $a \neq 0$, there exists a G -invariant normal state ϕ (that is, $T_g\phi = \phi$, $g \in G$) such that $\phi(a) \neq 0$.

Now we state our main theorem.

THEOREM. *Let (M, G) be a noncommutative dynamical system, then (M, G) is G -finite if and only if for every weakly relatively compact (w.r.c.) subset K of M_* , the orbit of K under G , that is, the set $\{T_g\phi; g \in G, \phi \in K\}$ is also w.r.c.*

3. Proof of Theorem. "If" part of Theorem is valid under a weaker assumption, more precisely to say that if for every ϕ in M_* with $\phi \geq 0$, $\{T_g\phi; g \in G\}$ is w. r. c., then (M, G) is G -finite. However, this is an easy consequence of lemma in [14] (see also [11]). To prove the converse, we need the following lemma which concerns with the continuity of the map $(\Phi, \omega) \rightarrow \omega \circ \Phi$ from $L_*(M) \times M_* \rightarrow M_*$ where $L_*(M)$ is the σ -weakly continuous bounded linear maps of M into M equipped with the weak operator topology and M_* has the W^* -topology. For the later discussions, we state it in the following form.

LEMMA 1. *Let N be a von Neumann algebra with a set H of normal $*$ -homomorphisms of N into N . Suppose that for every ϕ in N_* (the predual of N) with $\phi \geq 0$, and every sequence $\{b_n\}$ in the nonnegative part of the unit sphere S of N such that $b_n \rightarrow 0$ (σ -weakly), $\phi(\Phi(b_n)) \rightarrow 0$ ($n \rightarrow \infty$) uniformly for $\Phi \in H$. Let $\{\phi_n\}$ be a sequence in N_* which converges weakly to some ϕ_0 in N_* and $\{a_n\}$ be a sequence of self-adjoint element in S which converges strongly to 0, then $\phi_j(\Phi(a_n)) \rightarrow 0$ ($n \rightarrow \infty$) uniformly not only for $\Phi \in H$ but also for j .*

Proof. Observe first that the σ -weak topology restricted on S is a compact Hausdorff topology with the neighborhood basis which consists of all possible sets $\{a; a \in S, |\psi_i(a) - \psi_i(a_0)| < \varepsilon, i = 1, 2, \dots, n\}$ with $a_0 \in S, \varepsilon > 0$ (real number) and $\psi_i \in N_*(\psi_i \geq 0)$. Let $H_i = \{a \in S; |(\phi_j - \phi_0)(a)| \leq \varepsilon \text{ for all } j \geq i\}$, then H_i is σ -weakly closed subset of S for each i and $S = \bigcup_{i=1}^{\infty} H_i$. Now Baire's category theorem says that there are a natural numbers $i(0), m$, an element a_0 in S and ψ_i ($i = 1, 2, \dots, m$) in N_* with $\psi_i \geq 0$ for all i such that

$$\bigcap_{i=1}^m \{a; a \in S; |\psi_i(a) - \psi_i(a_0)| < 1\} \subset H_{i(0)}.$$

Since $a_n \rightarrow 0$ ($n \rightarrow \infty$) strongly, by the spectral theorem, for any given positive number ε , there is a sequence $\{e_n\}$ of projections in M such that $e_n \rightarrow 1$ (strongly) and $\|a_n e_n\| \leq \varepsilon/6$ for each n . By the uniform boundedness theorem, we may assume that $\text{Sup}_j \{\|\phi_j\|, \|\phi_0\|\} = 1$ without loss of generality. For each $\Phi \in H$, we have $\|\Phi(e_n a_n e_n)\| \leq \|a_n e_n\| \leq \varepsilon/6$, $\|\Phi(e_n a_n (1 - e_n))\| \leq \|a_n e_n\| \leq \varepsilon/6$ and $\|\Phi((1 - e_n) a_n e_n)\| \leq$

$\|a_n e_n\| \leq \varepsilon/6$ for each n . Thus we have

$$\begin{aligned} |(\phi_j - \phi_0)(\Phi(a_n))| &\leq |(\phi_j - \phi_0)(\Phi(e_n a_n e_n))| \\ &\quad + |(\phi_j - \phi_0)(\Phi(e_n a_n (1 - e_n)))| \\ &\quad + |(\phi_j - \phi_0)(\Phi((1 - e_n) a_n e_n))| \\ &\quad + |(\phi_j - \phi_0)(\Phi((1 - e_n) a_n (1 - e_n)))| \\ &\leq \varepsilon + |(\phi_j - \phi_0)(\Phi((1 - e_n) a_n (1 - e_n)))|. \end{aligned}$$

Put $b_n(\Phi) = \Phi((1 - e_n) a_n (1 - e_n)) + \Phi(e_n) a_0 \Phi(e_n)$, then, since $b_n(\Phi) - a_0 = (1 - \Phi(e_n)) \Phi(a_n) (1 - \Phi(e_n)) - (1 - \Phi(e_n)) a_0 \Phi(e_n) - \Phi(e_n) a_0 (1 - \Phi(e_n)) - (1 - \Phi(e_n)) a_0 (1 - \Phi(e_n))$, we have, by Schwarz' inequality,

$$|\psi_j(b_n(\Phi) - a_0)| \leq \psi_i(\Phi(1 - e_n)) + 3 \|\psi_i\| \psi_i(\Phi(1 - e_n))^{1/2}.$$

Similarly, we have

$$|\psi_j(\Phi(e_n) a_0 \Phi(e_n) - a_0)| \leq \psi_i(\Phi(1 - e_n)) + 2 \|\psi_i\| \psi_i(\Phi(1 - e_n))^{1/2}.$$

Since, by the assumption, $\psi_i(\Phi(1 - e_n)) \rightarrow 0 (n \rightarrow \infty)$ uniformly for $\Phi \in H$ and $i = 1, 2, \dots, m$, we can choose a natural number $n(\varepsilon)$ (depends only on ε) such that $b_n(\Phi), \Phi(e_n) a_0 \Phi(e_n) \in H_{i(0)}$ for all $n \geq n(\varepsilon)$. Thus, we have

$$|(\phi_j - \phi_0)(\Phi((1 - e_n) a_n (1 - e_n)))| < 2\varepsilon$$

for all $j \geq i(0)$, all $\Phi \in H$ and all $n \geq n(\varepsilon)$. Since, for each $j (j = 1, 2, \dots, i(0) - 1)$

$$\begin{aligned} |(\phi_j - \phi_0)(\Phi(a_n))| &= |(\phi_j - \phi_0)(\Phi(a_n) v_j)| \\ &\leq \{|\phi_j - \phi_0| (\{\Phi(a_n)\}^2)\}^{1/2} \|\phi_j - \phi_0\|^{1/2} \\ &\leq 2^{1/2} \{|\phi_j - \phi_0| (\Phi(a_n^2))\}^{1/2} \end{aligned}$$

and

$$|\phi_0(\Phi(a_n))| = |(\phi_0)(\Phi(a_n) v)| \leq \{|\phi_0| (\Phi(a_n^2))\}^{1/2}$$

where $\phi_j - \phi_0 = R_{v_j} |\phi_j - \phi_0|$ (resp. $\phi_0 = R_v |\phi_0|$) is the polar decomposition of $\phi_j - \phi_0$ (resp. ϕ_0) ([12]), $a_n^2 \rightarrow 0$ weakly implies, by the assumption, that there is a positive integer $n(\varepsilon)'$ (depending only on ε) such that $|(\phi_j - \phi_0)(\Phi(a_n))| < \varepsilon$ and $|\phi_0(\Phi(a_n))| < \varepsilon$ for all $\Phi \in H, j = 1, 2, \dots, i(0) - 1$ and all $n \geq n(\varepsilon)'$.

Combining the above estimations, we have

$$|\phi_j(\Phi(a_n))| < 4\varepsilon \text{ for all } n \geq \max(n(\varepsilon), n(\varepsilon)'), \text{ all } j$$

and all $\Phi \in H$. This completes the proof of Lemma 1.

Before going into the proof of theorem, we prepare the following:

LEMMA 2. *Keep the notations in theorem. If (M, G) is G -finite, then, for every sequence $\{a_n\}$ of nonnegative elements in the unit sphere S of M which converges weakly to 0, and every ϕ in M_* , $(T_g\phi)(a_n) \rightarrow 0$ uniformly for $g \in G$.*

Proof. If not, there exists a positive number ε_0 such that for each positive integer n , we can choose a positive integer $k(k(n) \uparrow \infty)$ and $g(n) \in G$ such that

$$(*) \quad |T_{g(n)}\phi(a_{k(n)})| \geq \varepsilon_0.$$

Put $a_{k(n)} = b(n)$ then since $\{(b(n)^{g(n)})\}$ is a σ -weakly relatively compact subset of $S \cap M^+$ (where M^+ is the positive portion of M), there is a σ -weakly cluster point $a(a \geq 0)$ of $\{(b(n)^{g(n)})\}$. Thus for every positive number δ , every G -invariant normal state ρ and every positive integer n , there is a natural number $i(n)(i(n) > n$ and $i(n) \uparrow \infty)$ such that

$$|\rho(a) - \rho(b(n)^{g(i(n))})| < \delta \quad n = 1, 2, \dots.$$

Since ρ is G -invariant, $\rho((b(i(n))^{g(i(n))}) = \rho(b(i(n))) \rightarrow 0(i(n) \rightarrow \infty)$. Thus $|\rho(a)| \leq \delta$ for every δ and the G -finiteness of (M, G) implies $a = 0$. Hence this contradicts with the inequality (*). Thus $(T_g\phi)(a_n) \rightarrow 0(n \rightarrow \infty)$ uniformly for $g \in G$ and the proof is completed.

Proof of Theorem. Suppose (M, G) is G -finite. We will prove that for every w.r.c. subset K of M_* , $\{T_g\phi; \phi \in K, g \in G\}$ is also w.r.c. To prove this, we have only to show that for every orthogonal sequence $\{e(n)\}$ of projections, $\lim_{n \rightarrow \infty} T_g\phi(e(n)) = 0$ uniformly for $g \in G$ and $\phi \in K$. If not, there is a positive number ε such that for each positive integer k , there are a natural number $n(k)(n(k) \uparrow \infty)$, $g(k) \in G$ and $\phi_k \in K$ such that

$$(**) \quad |T_{g(k)}\phi_k(e(n(k)))| \geq \varepsilon.$$

By Eberlein-Šmulian's theorem ([4]), there is a subsequence $\{\phi_{k(p)}\}$ of $\{\phi_k\}(k(p) \uparrow \infty)$ such that $\phi_{k(p)} \rightarrow \phi_0$ weakly ($p \rightarrow \infty$) for some ϕ_0 in M_* . Now $e(n(k(p))) \rightarrow 0(p \rightarrow \infty)$ strongly, which implies by Lemma 2 and Lemma 1, that $|T_{g(k(p))}\phi_{k(p)}(e(n(k(p))))| \rightarrow 0(p \rightarrow \infty)$ and this contradicts with the inequality (**). This completes the proof of theorem.

4. Remarks. Theorem is a generalization of [11]. We should remark that the result of theorem can be easily extended to groups of Jordan Automorphisms of M . [13] When G is a semi-group of normal Jordan homomorphisms ([13]) of M into M , by an easy modification of Lemma 1 and Lemma 2, "only if" part of theorem is valid,

however, as the following example shows, the converse assertion does not hold in general, even if G is a semi group of *-isomorphisms of M into M .

Let $M = L^\infty([0, 1])$ be the abelian von Neumann algebra of essentially bounded complex-valued functions on $[0, 1]$ with respect to Lebesgue measure μ . Let us consider two measurable transformations g_1 and g_2 defined as follows ([2, 8]): $g_1(\omega) = 3\omega \pmod{1}$, $\omega \in [0, 1)$, $g_2(\omega) = 2\omega + 1/3$ (resp. $= (\omega - 1/3)/2$, $\omega \in [0, 1/3)$ (resp. $\omega \in [1/3, 1)$). For each $f \in M$, let $(\Phi_1 f)(\omega) = f(g_1 \omega)$, $\omega \in [0, 1)$ and $(\Phi_2 f)(\omega) = f(g_2 \omega)$, $\omega \in [0, 1)$. Let H be the semi-group of normal *-homomorphisms of M into M generated by Φ_1 and Φ_2 . Then by [2] and [8], we can easily check that for each $\phi \in M_*(= L^1([0, 1]))$, $\{\phi \circ \Phi, \Phi \in H\}$ is w. r. c.. Thus by [6] and Lemma 1, for every w. r. c. subset K of M_* , $\{\phi \circ \Phi, \Phi \in H, \phi \in K\}$ is also w. r. c. However, since g_1 is ergodic with respect to μ and μ is not invariant under g_2 , (M, H) has no H -invariant functionals in M_* .

The above example implies that the Ryll-Nardzewski fixed point theorem is not valid in general without the assumption of distal action of H .

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Norman Larrabee Alling, <i>On Cauchy's theorem for real algebraic curves with boundary</i>	315
Daniel D. Anderson, <i>A remark on the lattice of ideals of a Prüfer domain</i>	323
Dennis Neal Barr and Peter D. Miletta, <i>A necessary and sufficient condition for uniqueness of solutions to two point boundary value problems</i>	325
Ladislav Beran, <i>On solvability of generalized orthomodular lattices</i>	331
L. Carlitz, <i>A three-term relation for some sums related to Dedekind sums</i>	339
Arthur Herbert Copeland, Jr. and Albert Oscar Shar, <i>Images and pre-images of localization maps</i>	349
G. G. Dandapat, John L. Hunsucker and Carl Pomerance, <i>Some new results on odd perfect numbers</i>	359
M. Edelstein and L. Keener, <i>Characterizations of infinite-dimensional and nonreflexive spaces</i>	365
Francis James Flanigan, <i>On Levi factors of derivation algebras and the radical embedding problem</i>	371
Harvey Friedman, <i>Provable equality in primitive recursive arithmetic with and without induction</i>	379
Joseph Braucher Fugate and Lee K. Mohler, <i>The fixed point property for tree-like continua with finitely many arc components</i>	393
John Norman Ginsburg and Victor Harold Saks, <i>Some applications of ultrafilters in topology</i>	403
Arjun K. Gupta, <i>Generalisation of a "square" functional equation</i>	419
Thomas Lee Hayden and Frank Jones Massey, <i>Nonlinear holomorphic semigroups</i>	423
V. Kannan and Thekkedath Thrivikraman, <i>Lattices of Hausdorff compactifications of a locally compact space</i>	441
J. E. Kerlin and Wilfred Dennis Pepe, <i>Norm decreasing homomorphisms between group algebras</i>	445
Young K. Kwon, <i>Behavior of Φ-bounded harmonic functions at the Wiener boundary</i>	453
Richard Arthur Levaro, <i>Projective quasi-coherent sheaves of modules</i>	457
Chung Lin, <i>Rearranging Fourier transforms on groups</i>	463
David Lowell Lovelady, <i>An asymptotic analysis of an odd order linear differential equation</i>	475
Jerry Malzan, <i>On groups with a single involution</i>	481
J. F. McClendon, <i>Metric families</i>	491
Carl Pomerance, <i>On multiply perfect numbers with a special property</i>	511
Mohan S. Putcha and Adil Mohamed Yaqub, <i>Polynomial constraints for finiteness of semisimple rings</i>	519
Calvin R. Putnam, <i>Hyponormal contractions and strong power convergence</i>	531
Douglas Conner Ravenel, <i>Multiplicative operations in BP^*BP</i>	539
Judith Roitman, <i>Attaining the spread at cardinals which are not strong limits</i>	545
Kazuyuki Saitô, <i>Groups of *-automorphisms and invariant maps of von Neumann algebras</i>	553
Brian Kirkwood Schmidt, <i>Homotopy invariance of contravariant functors acting on smooth manifolds</i>	559
Kenneth Barry Stolarsky, <i>The sum of the distances to N points on a sphere</i>	563
Mark Lawrence Teply, <i>Semiprime rings with the singular splitting property</i>	575
J. Pelham Thomas, <i>Maximal connected Hausdorff spaces</i>	581
Charles Thomas Tucker, II, <i>Concerning σ-homomorphisms of Riesz spaces</i>	585
Rangachari Venkataraman, <i>Compactness in abelian topological groups</i>	591
William Charles Waterhouse, <i>Basically bounded functors and flat sheaves</i>	597
David Westreich, <i>Bifurcation of operator equations with unbounded linearized part</i>	611
William Robin Zame, <i>Extendibility, boundedness and sequential convergence in spaces of holomorphic functions</i>	619