

# Pacific Journal of Mathematics

**MAXIMAL CONNECTED HAUSDORFF SPACES**

J. PELHAM THOMAS

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A nowhere neighborhood nested space is one in which no point has a local base which is linearly ordered by set inclusion. An *MI* space is one in which every dense subset is open. In this paper we show that every Hausdorff topology without isolated points has a nowhere neighborhood nested refinement. We show that every maximal connected Hausdorff topology is *MI* and nowhere neighborhood nested, and that every connected, but not maximal connected, Hausdorff topology has a connected, but not maximal connected, nowhere neighborhood nested refinement. Every connected Hausdorff topology has a connected, *MI*, nowhere neighborhood nested refinement.

In [4] the author raised the question of the existence of non-trivial maximal connected Hausdorff spaces. The question remains open.

A topology  $\mathcal{T}'$  on a set  $X$  is said to be finer than, or to be a refinement of, a topology  $\mathcal{T}$  on  $X$  if  $\mathcal{T} \subset \mathcal{T}'$ . It is said to be strictly finer than  $\mathcal{T}$ , if, in addition, we have  $\mathcal{T} \neq \mathcal{T}'$ . We say that  $(X, \mathcal{T})$  (and by abuse of language,  $\mathcal{T}$ ) is maximal connected, if  $(X, \mathcal{T})$  is connected and whenever  $\mathcal{T}'$  is strictly finer than  $\mathcal{T}$ ,  $(X, \mathcal{T}')$  is not connected. An *MI* space (see [2]) is one in which every dense subset is open.

The following result is in the authors thesis [5].

**THEOREM 1.** *Every maximal connected space is an MI space.*

An irresolvable space is one which does not have a dense subset whose complement is also dense. Anderson [1] has shown that every connected Hausdorff space has a connected irresolvable refinement. If in his proof of his Theorem 1, in the fourth paragraph, we simply choose  $D$  to be an  $R^*$ -dense set which is not  $R^*$  open, we will have proved

**THEOREM 2.** *Let  $\tau$  be an infinite cardinal number. Let  $R$  be a connected topology for  $X$  with  $\Delta(R) \geq \tau$ , where  $\Delta(R)$  denotes the dispersion character, or minimum cardinality of an open set of  $R$ . Then there exists a connected *MI* refinement  $R^*$  of  $R$  with  $\Delta(R^*) \geq \tau$ .*

**DEFINITION 1.** Let  $(X, \mathcal{T})$  be a topological space,  $x \in X$ . If there is a "local" base at  $x$  which is linearly ordered under set

inclusion ( $V \leq W$  if  $W \subset V$ ), we say  $\mathcal{T}$  is *neighborhood nested at  $x$* . If  $\mathcal{T}$  is not neighborhood nested at any point of  $X$ , we say  $\mathcal{T}$  is *nowhere neighborhood nested*, abbreviated n.n.n.

**LEMMA 1.** *Let  $(X, \mathcal{T})$  be a Hausdorff space,  $x \in X$ . If there is a base  $\{V_i\}_{i \in I}$  at  $x$  which is linearly ordered under set inclusion, then there is a base  $\{W_k\}_{k \in K}$  at  $x$  which is well ordered under set inclusion and such that if  $\delta, \sigma \in K$ ,  $\delta < \sigma$ , then  $\text{Int}(W_\delta - W_\sigma) \neq \emptyset$ .*

*Proof of Lemma 1.* First recall that (1) every totally ordered set has a cofinal well ordered subset (2) every well ordered set has a cofinal subset which is order isomorphic with a regular cardinal  $A$ . Next, using  $A$ , one can easily construct sets with the desired properties.

**COROLLARY.** *If there is an ordered local base at  $x$ ,  $x$  is adherent to a set  $S$  of isolated points (isolated in  $S$ ).*

*Proof.* Choose one member of  $\text{Int}(W_\iota - W_{\iota+1})$  for each  $\iota$ .

**THEOREM 3.** *Every Hausdorff topology without isolated points has an n.n.n. refinement. Every connected, Hausdorff topology has a connected n.n.n. refinement. Every connected, Hausdorff, but not n.n.n. topology has a connected n.n.n. refinement which is not maximal connected.*

*Proof of Theorem 3.* For a space  $(X, \mathcal{T})$ , denote by  $\mathcal{T}'$  the topology on  $X$  which has as a base  $\mathcal{T} \cup \{D \cap T \mid T \in \mathcal{T} \text{ and } \text{Int}_{(X, \mathcal{T})} D \text{ is dense in } (X, \mathcal{T})\}$ . Then one can show that  $(X, \mathcal{T})$  and  $(X, \mathcal{T}')$  have the same open-and-closed subsets by appealing to the following fact: if  $D$  is a dense subset of  $(X, \mathcal{T})$  and  $U, V \in \mathcal{T}$  with  $U \cap V \neq \emptyset$ , then  $U \cap V \cap D \neq \emptyset$ . Using Lemma 1,  $(X, \mathcal{T}')$ , and, some (or all) nowhere dense subsets of  $(X, \mathcal{T})$ , one obtains the statements of Theorem 3.

The following theorem is an immediate corollary to Theorem 3.

**THEOREM 4.** *Every maximal connected Hausdorff topology is n.n.n.*

**THEOREM 5.** *Every Hausdorff connected topology  $\mathcal{T}_1$  has a Hausdorff, connected, MI, n.n.n. refinement  $\mathcal{T}_3$ .*

*Proof.* By Theorem 2,  $\mathcal{T}_1$  has a connected Hausdorff MI refinement  $\mathcal{T}_2$ . By Theorem 3,  $\mathcal{T}_2$  has a connected, Hausdorff,

n.n.n. refinement  $\mathcal{S}_3$ . It is easy to see that every refinement of an  $MI$  topology is  $MI$ . Thus,  $\mathcal{S}_3$  meets the required conditions.

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WESTERN CAROLINA UNIVERSITY



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