MATRIX RINGS OVER POLYNOMIAL IDENTITY RINGS. II

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If $A$ is a ring satisfying a polynomial identity, what identity
is satisfied by the matrix ring $A_n$? Theorem: If $A$ satisfies the
standard identity of degree $k$, then $A_n$ satisfies the standard
identity of degree $2kn^2 - n^2 + 1$.

Definition: Suppose that $\{r_1, \cdots, r_q\}$ is a sequence of elements of a
ring. To parenthesize the sequence into $j$ clumps is to insert $j$ pairs of
adjacent, nonoverlapping parentheses. The subsequence within one
pair of parentheses constitutes a clump. It is odd or even, depending
on the number of entries. The value of the clump is the product of the
entries. If the value is zero, the clump vanishes.

In the following let $\mathbb{Z}$ represent the integers.

**Lemma 1.** Let $k, m,$ and $n$ be positive integers. Let $\{u_1, \cdots, u_m\}$
be a nonvanishing sequence of matrix units $e_{ij}$ in $\mathbb{Z}_n$.

(i) If $m = kn$, there exists $i$ such that the sequence can be parenthe-
sized into $k$ clumps, each of value $e_{ii}$.

(ii) If $m = (kn - n + 1)n$, there exist $i$ and $j$ such that the sequence
can be parenthesized into $k$ clumps, each of value $e_{ij}$, and each beginning
with $e_{ij}$.

**Proof of (i).** Case 1. Suppose there exists $i$ such that at least
$k + 1$ of the entries in the sequence have $i$ as initial subscript. Call the
first $k + 1$ such entries $y_1, y_2, \cdots, y_{k+1}$. Then parenthesize the sequence
as follows: start with $y_1$. Enclose it in parentheses, together with all
entries to the right, if any, up to $y_2$. Next parenthesize $y_2$ with all
entries up to $y_3$, etc. We form $k$ clumps, each beginning with a
$y$. Since each clump has to the right an entry with $i$ as initial subscript,
and the sequence is nonvanishing, each clump has value $e_{ii}$.

Case 2. Suppose that for all $i$, at most $k$ of the entries have $i$ as
initial subscript. Since the sequence has $kn$ entries, every $i$ from $l$
through $n$ occurs exactly $k$ times as an initial subscript.

Case 2a. The last entry is an idempotent $e_{ii}$. There are previous
entries $y_1, \cdots, y_{k-1}$, each with $i$ as initial subscript. Start with $y_1$ and
enclose it in parentheses with all entries to the right, up to $y_2$. Continue, forming $k - 1$ clumps, each of value $e_{ii}$. Then form a final clump consisting of the single $e_{ii}$ at the end.

Case 2b. The last entry is $e_{ij}$, with $i \neq j$. Then there are $k$ previous entries $y_1, \ldots, y_k$ with $j$ as initial subscript. Parenthesize, forming $k - 1$ clumps, beginning with $y_1, y_2, \ldots, y_{k-1}$, respectively. Then form a final clump, beginning with $y_k$ and ending with the last $e_{ij}$. The result is $k$ clumps, each of value $e_{ij}$.

Proof of (ii). Let $m = (kn - n + 1)n$. Let $\{u_1, \ldots, u_m\}$ be a non-vanishing sequence of matrix units. Let $t = kn - n + 1$. By (i) there exists $i$ such that the sequence can be parenthesized into $t$ clumps, each of value $e_{ii}$. Let $y_1, \ldots, y_t$ be the first entries in these clumps. Each $y$ has $i$ as initial subscript. The second subscript can be any integer from 1 through $n$. Now

$$t = kn - n + 1 = (k - 1)n + 1.$$ 

Thus for some $j$, at least $k$ of the $y$'s have $j$ as second subscript. Suppose that $y_{f(1)}, \ldots, y_{f(k)}$ are all $e_{ii}$. Make new clumps as follows: start with $y_{f(1)}$ and enclose it in parentheses together with all entries to the right, up to $y_{f(2)}$. Continue, forming $k - 1$ clumps. In the old parenthesizing $y_{f(k)}$ was the initial entry in a clump of value $e_{ii}$. Let this old clump be the $k$th clump in the new parenthesizing. The result is $k$ clumps, each of value $e_{ii}$, and each beginning with $e_{ii}$.

Theorem 3.2 of [2] established that if $A$ is an algebra satisfying a standard identity, so is $A_n$. The following theorem improves this result in three ways: (1) the degree of the identity satisfied by $A_n$ is much lower. (2) The theorem holds for rings, not just algebras over fields. (3) The proof is simpler.

**Theorem 1.** If $A$ is a ring satisfying the standard identity of degree $k$, then $A_n$ satisfies the standard identity of degree $2kn^2 - n^2 + 1$.

**Proof.** Let

$$t = 2kn^2 - n^2 + 1 = (2k - 1)n^2 + 1.$$ 

Choose $t$ simple tensors in $A \otimes Z_n$ of form $a \otimes e_{ij}$, where $a \in A$, and $e_{ij}$ is a matrix unit. Evaluate on these simple tensors the standard polynomial of degree $t$. Consider only nonvanishing terms.
Case 1. Suppose that for some \( i \), at least \( k \) simple tensors have form

\[
a_i \otimes e_{ii}, \ldots, a_k \otimes e_{ii}.
\]

Let \( y = e_{ii} \). Call the remaining elements

\[
b_1 \otimes z_1, b_2 \otimes z_2, \ldots.
\]

Insert parentheses on the right side of each term: start with the first \( y \) and enclose it with all \( z \)'s to the right, if any. Similarly parenthesize the next \( y \) with its \( z \)'s, etc. The last \( y \) forms a singleton clump. Thus \( k \) clumps are created, each beginning with \( e_{ii} \), and each of value \( e_{ii} \). If there are any \( z \)'s in the clump, call them the \( z \) sub-clump. It also has value \( e_{ii} \).

Let \( V \) be the number of even clumps, and let \( D \) be the number of odd clumps. Then \( V + D = k \). Each even clump yields two new odd clumps: the initial \( y \) and the \( z \) sub-clump. The result is \( 2V + D \) adjacent odd clumps, each of value \( e_{ii} \). Note that \( 2V + D \geq V + D = k \).

In each term find the first set of \( k \) adjacent odd clumps of value \( e_{ii} \). Create a corresponding set of clumps on the left side. Call two terms equivalent if the following conditions hold on their left sides:

1. The elements to the left of the clumps are the same elements in the same order.
2. The \( k \) clumps are the same, but in any order.
3. The elements to the right of the clumps are the same elements in the same order.

Consider a fixed equivalence class. The sum of the terms in the class is a simple tensor whose right side has the common value for the class. The left side is the product of the following:

1. The product of all elements left of the clumps.
2. The standard polynomial of degree \( k \), evaluated on the values of the \( k \) clumps, in some order.
3. The product of all the elements right of the clumps. (Because all these clumps are odd, Corollary to Lemma 4 of [4] ensures correctness of signs of terms.) Since the second factor vanishes, the conclusion follows.

Case 2. Suppose that Case 1 does not hold. Since there are \((2k - 1)n^2 + 1\) simple tensors, by Lemma 1 (ii) there exist \( i \) and \( j \) such that at least \( 2k \) simple tensors have form \( a \otimes e_{ij} \). Evidently, \( i \neq j \). Let \( w_{ii} = e_{ii} + e_{ij} \). Then \( w_{ii} \) is idempotent, and

\[
e_{ij} = e_{ii} + e_{ij} - e_{ii} = w_{ii} - e_{ii}.
\]
In each term replace $e_i$ by $w_i - e_i$. Let $N$ be the original number of $e_i$’s. Each old term, upon expansion, yields $2^N$ new terms. Every new term has on the right a monomial in $w$’s and $e$’s. If there are at least $k$ of the $e_i$’s in the term, it is suitable for Case 1. Otherwise there are at least $k$ of the $w_i$’s. In this case, define new elements as follows:

$$w_{ij} = -e_i + e_{ij}$$

$$w_{ji} = -e_i - e_j + e_{ji} + e_{ij}.$$  

If $i \neq p \neq j$, let

$$w_{pi} = e_{pi} + e_{pj}$$

$$w_{jp} = -e_{ip} + e_{ip}.$$ 

For the remaining integers from 1 through $n$, let $w_{pq} = e_{pq}$.

The $w$’s constitute another set of matrix units in $\mathbb{Z}_n$. Each old matrix unit $e_{pq}$ is a linear combination of the $w$’s with integral coefficients. Replace all the $e$’s by $w$’s. The conclusion follows by the linearity of the standard polynomial and by Case 1.

**Definition.** The **unitary identity of degree $k$** is

$$\sum_{\pi} x_{\pi(1)} \cdots x_{\pi(k)} = 0,$$

where the sum is over all permutations $\pi$ of the integers 1 through $k$.

**Theorem 2.** If $A$ is a ring satisfying the unitary identity of degree $k$, then $A_n$ satisfies the unitary identity of degree $kn$.

**Proof.** The proof uses Lemma 1 (i) and is similar to Theorem 1 of [4].

**Theorem 3.** If $A$ is an algebra over a field with at least $k$ elements, and $A$ satisfies $x^k = 0$, then $A_n$ satisfies $x^{kn} = 0$.

**Proof.** The proof uses Lemma 1(i) and is similar to Theorem 1.2 of [3]. Note: That paper uses without definition the term “homogeneous component” of a polynomial. If $f(x_1, \cdots, x_j)$ is a polynomial, the homogeneous component of degree $n_1$ in $x_1$, degree $n_2$ in $x_2$, etc., is the sum of all terms with degree $n_1$ in $x_1$, etc.
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