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# MATRIX RINGS OVER POLYNOMIAL IDENTITY RINGS II

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**If $A$ is a ring satisfying a polynomial identity, what identity is satisfied by the matrix ring $A_n$? Theorem: If $A$ satisfies the standard identity of degree $k$, then $A_n$ satisfies the standard identity of degree $2kn^2 - n^2 + 1$.**

Definition: Suppose that $\{r_1, \cdots, r_q\}$ is a sequence of elements of a ring. *To parenthesize the sequence into j clumps* is to insert $j$ pairs of adjacent, nonoverlapping parentheses. The subsequence within one pair of parentheses constitutes a *clump.* It is odd or even, depending on the number of entries. The *value* of the clump is the product of the entries. If the value is zero, the clump *vanishes.*

In the following let $Z$ represent the integers.

LEMMA 1. *Let $k, m$, and $n$ be positive integers. Let $\{u_1, \cdots, u_m\}$ be a nonvanishing sequence of matrix units $e_{ij}$ in $Z_n$.*

(i) *If $m = kn$, there exists $i$ such that the sequnce can be parenthesized into $k$ clumps, each of value $e_{ii}$.*

(ii) *If $m = (kn - n + 1)n$, there exist $i$ and $j$ such that the sequence can be parenthesized into $k$ clumps, each of value $e_{ii}$, and each beginning with $e_{ij}$.*

*Proof of (i). Case* 1. Suppose there exists $i$ such that at least $k + 1$ of the entries in the sequence have $i$ as initial subscript. Call the first $k + 1$ such entries $y_1, y_2, \cdots, y_{k+1}$. Then parenthesize the sequence as follows: start with $y_1$. Enclose it in parentheses, together with all entries to the right, if any, up to $y_2$. Next parenthesize $y_2$ with all entries up to $y_3$, etc. We form $k$ clumps, each beginning with a $y$. Since each clump has to the right an entry with $i$ as initial subscript, and the sequence is nonvanishing, each clump has value $e_{ii}$.

*Case* 2. Suppose that for all $i$, at most $k$ of the entries have $i$ as initial subscript. Since the sequence has $kn$ entries, every $i$ from $l$ through $n$ occurs exactly $k$ times as an initial subscript.

*Case* 2a. The last entry is an idempotent $e_{ii}$. There are previous entries $y_1, \cdots, y_{k-1}$, each with $i$ as initial subscript. Start with $y_1$ and enclose it in parentheses with all entries to the right, up to $y_2$. Continue, forming $k-1$ clumps, each of value $e_{ii}$. Then form a final clump consisting of the single $e_{ii}$ at the end.

*Case* 2b. The last entry is $e_{ij}$, with $i \neq j$. Then there are $k$ previous entries $y_1, \cdots, y_k$ with $j$ as initial subscript. Parenthesize, forming $k-1$ clumps, beginning with $y_1, y_2, \cdots, y_{k-1}$, respectively. Then form a final clump, beginning with $y_k$ and ending with the last $e_{ij}$. The result is $k$ clumps, wach of value $e_{ij}$.

*Proof of* (ii). Let $m = (kn - n + 1)n$. Let $\{u_1, \cdots, u_m\}$ be a nonvanishing sequence of matrix units. Let $t = kn - n + 1$. By (i) there exists $i$ such that the sequence can be parenthesized into $t$ clumps, each of value $e_{ii}$. Let $y_1, \cdots, y_t$ be the first entries in these clumps. Each $y$ has $i$ as initial subscript. The second subscript can be any integer from 1 through $n$. Now

$$t = kn - n + 1 = (k-1)n + 1.$$

Thus for some $j$, at least $k$ of the $y$'s have $j$ as second subscript. Suppose that $y_{f(1)}, \cdots, y_{f(k)}$ are all $e_{ij}$. Make new clumps as follows: start with $y_{f(1)}$ and enclose it in parentheses together with all entries to the right, up to $y_{f(2)}$. Continue, forming $k-1$ clumps. In the old parenthesizing $y_{f(k)}$ was the initial entry in a clump of value $e_{ii}$. Let this old clump be the $k$th clump in the new parenthesizing. The result is $k$ clumps, each of value $e_{ii}$, and each beginning with $e_{ij}$.

Theorem 3.2 of **[2]** established that if $A$ is an algebra satisfying a standard identity, so is $A_n$. The following theorem improves this result in three ways: (1) the degree of the identity satisfied by $A_n$ is much lower. (2) The theorem holds for rings, not just algebras over fields. (3) The proof is simpler.

THEOREM 1. *If $A$ is a ring satisfying the standard identity of degree $k$, then $A_n$ satisfies thestandard identity of degree $2kn^2 - n^2 + 1$.*

*Proof.* Let

$$t = 2kn^2 - n^2 + 1 = (2k-1)n^2 + 1.$$

Choose $t$ simple tensors in $A \otimes Z_n$ of form $a \otimes e_{ij}$, where $a \in A$, and $e_{ij}$ is a matrix unit. Evaluate on these simple tensors the standard polynomial of degree $t$. Consider only nonvanishing terms.

*Case* 1. Suppose that for some $i$, at least $k$ simple tensors have form

$$a_1 \otimes e_{ii}, \cdots, a_k \otimes e_{ii}.$$

Let $y = e_{ii}$. Call the remaining elements

$$b_1 \otimes z_1, b_2 \otimes z_2, \cdots.$$

Insert parentheses on the right side of each term: start with the first $y$ and enclose it with all $z$'s to the right, if any. Similarly parenthesize the next $y$ with its $z$'s, etc. The last $y$ forms a singleton clump. Thus $k$ clumps are created, each beginning with $e_{ii}$, and each of value $e_{ii}$. If there are any $z$'s in the clump, call them the *z sub-clump*. It also has value $e_{ii}$.

Let $V$ be the number of even clumps, and let $D$ be the number of odd clumps. Then $V + D = k$. Each even clump yields two new odd clumps: the initial $y$ and the $z$ sub-clump. The result is $2V + D$ adjacent odd clumps, each of value $e_{ii}$. Note that $2V + D \geqq V + D = k$.

In each term find the first set of $k$ adjacent odd clumps of value $e_{ii}$. Create a corresponding set of clumps on the left side. Call two terms equivalent if the following conditions hold on their left sides:

1. The elements to the left of the clumps are the same elements in the same order.
2. The $k$ clumps are the same, but in any order.
3. The elements to the right of the clumps are the same elements in the same order.

Consider a fixed equivalence class. The sum of the terms in the class is a simple tensor whose right side has the common value for the class. The left side is the product of the following:

1. The product of all elements left of the clumps.
2. The standard polynomial of degree $k$, evaluated on the values of the $k$ clumps, in some order.
3. The product of all the elements right of the clumps. (Because all these clumps are odd, Corollary to Lemma 4 of [4] ensures correctness of signs of terms.) Since the second factor vanishes, the conclusion follows.

*Case* 2. Suppose that Case 1 does not hold. Since there are $(2k-1)n^2 + 1$ simple tensors, by Lemma 1 (ii) there exist $i$ and $j$ such that at least $2k$ simple tensors have form $a \otimes e_{ij}$. Evidently, $i \neq j$. Let $w_{ii} = e_{ii} + e_{ij}$. Then $w_{ii}$ is idempotent, and

$$e_{ij} = e_{ii} + e_{ij} - e_{ii} = w_{ii} - e_{ii}.$$

In each term replace $e_{ij}$ by $w_{ii} - e_{ii}$. Let $N$ be the original number of $e_{ij}$'s. Each old term, upon expansion, yields $2^N$ new terms. Every new term has on the right a monomial in $w$'s and $e$'s. If there are at least $k$ of the $e_{ii}$'s in the term, it is suitable for Case 1. Otherwise there are at least $k$ of the $w_{ii}$'s. In this case, define new elements as follows:

$$w_{jj} = -e_{ij} + e_{jj}$$

$$w_{ji} = -e_{ii} - e_{ij} + e_{ji} + e_{jj}.$$

If $i \neq p \neq j$, let

$$w_{pi} = e_{pi} + e_{pj}$$

$$w_{jp} = -e_{ip} + e_{jp}.$$

For the remaining integers from 1 through $n$, let $w_{pq} = e_{pq}$.

The $w$'s constitute another set of matrix units in $Z_n$. Each old matrix unit $e_{pq}$ is a linear combination of the $w$'s with integral coefficients. Replace all the $e$'s by $w$'s. The conclusion follows by the linearity of the standard polynomial and by Case 1.

DEFINITION. The *unitary identity of degree k is*

$$\sum_{\pi} x_{\pi(1)} \cdots x_{\pi(k)} = 0,$$

where the sum is over all permutations $\pi$ of the integers 1 through $k$.

THEOREM 2. *If $A$ is a ring satisfying the unitary identity of degree $k$, then $A_n$ satisfies the unitary identity of degree $kn$.*

*Proof.* The proof uses Lemma 1 (i) and is similar to Theorem 1 of **[4]**.

THEOREM 3. *If $A$ is an algebra over a field with at least $k$ elements, and $A$ satisfies $x^k = 0$, then $A_n$ satisfies $x^{kn} = 0$.*

*Proof.* The proof uses Lemma 1(i) and is similar to Theorem 1.2 of **[3]**. Note: That paper uses without definition the term "homogeneous component" of a polynomial. If $f(x_1, \cdots, x_j)$ is a polynomial, the homogeneous component of degree $n_1$ in $x_1$, degree $n_2$ in $x_2$, etc., is the sum of all terms with degree $n_1$ *in* $x_1$, etc.

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## CONTENTS

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