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MATRIX RINGS OVER POLYNOMIAL IDENTITY RINGS. II

ELIZABETH BERMAN

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If A is a ring satisfying a polynomial identity, what identity is satisfied by the matrix ring A_n ? Theorem: If A satisfies the standard identity of degree k , then A_n satisfies the standard identity of degree $2kn^2 - n^2 + 1$.

Definition: Suppose that $\{r_1, \dots, r_q\}$ is a sequence of elements of a ring. *To parenthesize the sequence into j clumps* is to insert j pairs of adjacent, nonoverlapping parentheses. The subsequence within one pair of parentheses constitutes a *clump*. It is odd or even, depending on the number of entries. The *value* of the clump is the product of the entries. If the value is zero, the clump *vanishes*.

In the following let Z represent the integers.

LEMMA 1. *Let k, m , and n be positive integers. Let $\{u_1, \dots, u_m\}$ be a nonvanishing sequence of matrix units e_{ij} in Z_n .*

(i) *If $m = kn$, there exists i such that the sequence can be parenthesized into k clumps, each of value e_{ii} .*

(ii) *If $m = (kn - n + 1)n$, there exist i and j such that the sequence can be parenthesized into k clumps, each of value e_{ii} , and each beginning with e_{ij} .*

Proof of (i). Case 1. Suppose there exists i such that at least $k + 1$ of the entries in the sequence have i as initial subscript. Call the first $k + 1$ such entries y_1, y_2, \dots, y_{k+1} . Then parenthesize the sequence as follows: start with y_1 . Enclose it in parentheses, together with all entries to the right, if any, up to y_2 . Next parenthesize y_2 with all entries up to y_3 , etc. We form k clumps, each beginning with a y . Since each clump has to the right an entry with i as initial subscript, and the sequence is nonvanishing, each clump has value e_{ii} .

Case 2. Suppose that for all i , at most k of the entries have i as initial subscript. Since the sequence has kn entries, every i from 1 through n occurs exactly k times as an initial subscript.

Case 2a. The last entry is an idempotent e_{ii} . There are previous entries y_1, \dots, y_{k-1} , each with i as initial subscript. Start with y_1 and

enclose it in parentheses with all entries to the right, up to y_2 . Continue, forming $k - 1$ clumps, each of value e_{ii} . Then form a final clump consisting of the single e_{ii} at the end.

Case 2b. The last entry is e_{ij} , with $i \neq j$. Then there are k previous entries y_1, \dots, y_k with j as initial subscript. Parenthesize, forming $k - 1$ clumps, beginning with y_1, y_2, \dots, y_{k-1} , respectively. Then form a final clump, beginning with y_k and ending with the last e_{ij} . The result is k clumps, each of value e_{ij} .

Proof of (ii). Let $m = (kn - n + 1)n$. Let $\{u_1, \dots, u_m\}$ be a nonvanishing sequence of matrix units. Let $t = kn - n + 1$. By (i) there exists i such that the sequence can be parenthesized into t clumps, each of value e_{ii} . Let y_1, \dots, y_t be the first entries in these clumps. Each y has i as initial subscript. The second subscript can be any integer from 1 through n . Now

$$t = kn - n + 1 = (k - 1)n + 1.$$

Thus for some j , at least k of the y 's have j as second subscript. Suppose that $y_{f(1)}, \dots, y_{f(k)}$ are all e_{ij} . Make new clumps as follows: start with $y_{f(1)}$ and enclose it in parentheses together with all entries to the right, up to $y_{f(2)}$. Continue, forming $k - 1$ clumps. In the old parenthesizing $y_{f(k)}$ was the initial entry in a clump of value e_{ii} . Let this old clump be the k th clump in the new parenthesizing. The result is k clumps, each of value e_{ij} , and each beginning with e_{ij} .

Theorem 3.2 of [2] established that if A is an algebra satisfying a standard identity, so is A_n . The following theorem improves this result in three ways: (1) the degree of the identity satisfied by A_n is much lower. (2) The theorem holds for rings, not just algebras over fields. (3) The proof is simpler.

THEOREM 1. *If A is a ring satisfying the standard identity of degree k , then A_n satisfies the standard identity of degree $2kn^2 - n^2 + 1$.*

Proof. Let

$$t = 2kn^2 - n^2 + 1 = (2k - 1)n^2 + 1.$$

Choose t simple tensors in $A \otimes Z_n$ of form $a \otimes e_{ij}$, where $a \in A$, and e_{ij} is a matrix unit. Evaluate on these simple tensors the standard polynomial of degree t . Consider only nonvanishing terms.

Case 1. Suppose that for some i , at least k simple tensors have form

$$a_1 \otimes e_{ii}, \dots, a_k \otimes e_{ii}.$$

Let $y = e_{ii}$. Call the remaining elements

$$b_1 \otimes z_1, b_2 \otimes z_2, \dots.$$

Insert parentheses on the right side of each term: start with the first y and enclose it with all z 's to the right, if any. Similarly parenthesize the next y with its z 's, etc. The last y forms a singleton clump. Thus k clumps are created, each beginning with e_{ii} , and each of value e_{ii} . If there are any z 's in the clump, call them the z *sub-clump*. It also has value e_{ii} .

Let V be the number of even clumps, and let D be the number of odd clumps. Then $V + D = k$. Each even clump yields two new odd clumps: the initial y and the z sub-clump. The result is $2V + D$ adjacent odd clumps, each of value e_{ii} . Note that $2V + D \geq V + D = k$.

In each term find the first set of k adjacent odd clumps of value e_{ii} . Create a corresponding set of clumps on the left side. Call two terms equivalent if the following conditions hold on their left sides:

1. The elements to the left of the clumps are the same elements in the same order.
2. The k clumps are the same, but in any order.
3. The elements to the right of the clumps are the same elements in the same order.

Consider a fixed equivalence class. The sum of the terms in the class is a simple tensor whose right side has the common value for the class. The left side is the product of the following:

1. The product of all elements left of the clumps.
2. The standard polynomial of degree k , evaluated on the values of the k clumps, in some order.
3. The product of all the elements right of the clumps. (Because all these clumps are odd, Corollary to Lemma 4 of [4] ensures correctness of signs of terms.) Since the second factor vanishes, the conclusion follows.

Case 2. Suppose that Case 1 does not hold. Since there are $(2k - 1)n^2 + 1$ simple tensors, by Lemma 1 (ii) there exist i and j such that at least $2k$ simple tensors have form $a \otimes e_{ij}$. Evidently, $i \neq j$. Let $w_{ii} = e_{ii} + e_{ij}$. Then w_{ii} is idempotent, and

$$e_{ij} = e_{ii} + e_{ij} - e_{ii} = w_{ii} - e_{ii}.$$

In each term replace e_{ij} by $w_{ii} - e_{ii}$. Let N be the original number of e_{ij} 's. Each old term, upon expansion, yields 2^N new terms. Every new term has on the right a monomial in w 's and e 's. If there are at least k of the e_{ii} 's in the term, it is suitable for Case 1. Otherwise there are at least k of the w_{ii} 's. In this case, define new elements as follows:

$$\begin{aligned}w_{jj} &= -e_{ij} + e_{ji} \\w_{ji} &= -e_{ii} - e_{ij} + e_{ji} + e_{jj}.\end{aligned}$$

If $i \neq p \neq j$, let

$$\begin{aligned}w_{pi} &= e_{pi} + e_{pj} \\w_{jp} &= -e_{ip} + e_{jp}.\end{aligned}$$

For the remaining integers from 1 through n , let $w_{pq} = e_{pq}$.

The w 's constitute another set of matrix units in Z_n . Each old matrix unit e_{pq} is a linear combination of the w 's with integral coefficients. Replace all the e 's by w 's. The conclusion follows by the linearity of the standard polynomial and by Case 1.

DEFINITION. The *unitary identity of degree k* is

$$\sum_{\pi} x_{\pi(1)} \cdots x_{\pi(k)} = 0,$$

where the sum is over all permutations π of the integers 1 through k .

THEOREM 2. *If A is a ring satisfying the unitary identity of degree k , then A_n satisfies the unitary identity of degree kn .*

Proof. The proof uses Lemma 1 (i) and is similar to Theorem 1 of [4].

THEOREM 3. *If A is an algebra over a field with at least k elements, and A satisfies $x^k = 0$, then A_n satisfies $x^{kn} = 0$.*

Proof. The proof uses Lemma 1(i) and is similar to Theorem 1.2 of [3]. Note: That paper uses without definition the term "homogeneous component" of a polynomial. If $f(x_1, \dots, x_j)$ is a polynomial, the homogeneous component of degree n_1 in x_1 , degree n_2 in x_2 , etc., is the sum of all terms with degree n_1 in x_1 , etc.

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ROCKHURST COLLEGE
KANSAS CITY, MISSOURI

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