

Pacific Journal of Mathematics

MATRIX RINGS OVER POLYNOMIAL IDENTITY RINGS. II

ELIZABETH BERMAN

MATRIX RINGS OVER POLYNOMIAL IDENTITY RINGS II

ELIZABETH BERMAN

If A is a ring satisfying a polynomial identity, what identity is satisfied by the matrix ring A_n ? Theorem: If A satisfies the standard identity of degree k , then A_n satisfies the standard identity of degree $2kn^2 - n^2 + 1$.

Definition: Suppose that $\{r_1, \dots, r_q\}$ is a sequence of elements of a ring. To *parenthesize the sequence into j clumps* is to insert j pairs of adjacent, nonoverlapping parentheses. The subsequence within one pair of parentheses constitutes a *clump*. It is odd or even, depending on the number of entries. The *value* of the clump is the product of the entries. If the value is zero, the clump *vanishes*.

In the following let Z represent the integers.

LEMMA 1. *Let k, m , and n be positive integers. Let $\{u_1, \dots, u_m\}$ be a nonvanishing sequence of matrix units e_{ij} in Z_n .*

(i) *If $m = kn$, there exists i such that the sequence can be parenthesized into k clumps, each of value e_{ii} .*

(ii) *If $m = (kn - n + 1)n$, there exist i and j such that the sequence can be parenthesized into k clumps, each of value e_{ii} , and each beginning with e_{ij} .*

Proof of (i). Case 1. Suppose there exists i such that at least $k + 1$ of the entries in the sequence have i as initial subscript. Call the first $k + 1$ such entries y_1, y_2, \dots, y_{k+1} . Then parenthesize the sequence as follows: start with y_1 . Enclose it in parentheses, together with all entries to the right, if any, up to y_2 . Next parenthesize y_2 with all entries up to y_3 , etc. We form k clumps, each beginning with a y . Since each clump has to the right an entry with i as initial subscript, and the sequence is nonvanishing, each clump has value e_{ii} .

Case 2. Suppose that for all i , at most k of the entries have i as initial subscript. Since the sequence has kn entries, every i from 1 through n occurs exactly k times as an initial subscript.

Case 2a. The last entry is an idempotent e_{ii} . There are previous entries y_1, \dots, y_{k-1} , each with i as initial subscript. Start with y_1 and

enclose it in parentheses with all entries to the right, up to y_2 . Continue, forming $k - 1$ clumps, each of value e_{ii} . Then form a final clump consisting of the single e_{ii} at the end.

Case 2b. The last entry is e_{ij} , with $i \neq j$. Then there are k previous entries y_1, \dots, y_k with j as initial subscript. Parenthesize, forming $k - 1$ clumps, beginning with y_1, y_2, \dots, y_{k-1} , respectively. Then form a final clump, beginning with y_k and ending with the last e_{ij} . The result is k clumps, each of value e_{ij} .

Proof of (ii). Let $m = (kn - n + 1)n$. Let $\{u_1, \dots, u_m\}$ be a nonvanishing sequence of matrix units. Let $t = kn - n + 1$. By (i) there exists i such that the sequence can be parenthesized into t clumps, each of value e_{ii} . Let y_1, \dots, y_t be the first entries in these clumps. Each y_i has i as initial subscript. The second subscript can be any integer from 1 through n . Now

$$t = kn - n + 1 = (k - 1)n + 1.$$

Thus for some j , at least k of the y 's have j as second subscript. Suppose that $y_{f(1)}, \dots, y_{f(k)}$ are all e_{ij} . Make new clumps as follows: start with $y_{f(1)}$ and enclose it in parentheses together with all entries to the right, up to $y_{f(2)}$. Continue, forming $k - 1$ clumps. In the old parenthesizing $y_{f(k)}$ was the initial entry in a clump of value e_{ij} . Let this old clump be the k th clump in the new parenthesizing. The result is k clumps, each of value e_{ij} , and each beginning with e_{ij} .

Theorem 3.2 of [2] established that if A is an algebra satisfying a standard identity, so is A_n . The following theorem improves this result in three ways: (1) the degree of the identity satisfied by A_n is much lower. (2) The theorem holds for rings, not just algebras over fields. (3) The proof is simpler.

THEOREM 1. *If A is a ring satisfying the standard identity of degree k , then A_n satisfies the standard identity of degree $2kn^2 - n^2 + 1$.*

Proof. Let

$$t = 2kn^2 - n^2 + 1 = (2k - 1)n^2 + 1.$$

Choose t simple tensors in $A \otimes Z_n$ of form $a \otimes e_{ij}$, where $a \in A$, and e_{ij} is a matrix unit. Evaluate on these simple tensors the standard polynomial of degree t . Consider only nonvanishing terms.

Case 1. Suppose that for some i , at least k simple tensors have form

$$a_1 \otimes e_{ii}, \dots, a_k \otimes e_{ii}.$$

Let $y = e_{ii}$. Call the remaining elements

$$b_1 \otimes z_1, b_2 \otimes z_2, \dots.$$

Insert parentheses on the right side of each term: start with the first y and enclose it with all z 's to the right, if any. Similarly parenthesize the next y with its z 's, etc. The last y forms a singleton clump. Thus k clumps are created, each beginning with e_{ii} , and each of value e_{ii} . If there are any z 's in the clump, call them the z *sub-clump*. It also has value e_{ii} .

Let V be the number of even clumps, and let D be the number of odd clumps. Then $V + D = k$. Each even clump yields two new odd clumps: the initial y and the z sub-clump. The result is $2V + D$ adjacent odd clumps, each of value e_{ii} . Note that $2V + D \geq V + D = k$.

In each term find the first set of k adjacent odd clumps of value e_{ii} . Create a corresponding set of clumps on the left side. Call two terms equivalent if the following conditions hold on their left sides:

1. The elements to the left of the clumps are the same elements in the same order.
2. The k clumps are the same, but in any order.
3. The elements to the right of the clumps are the same elements in the same order.

Consider a fixed equivalence class. The sum of the terms in the class is a simple tensor whose right side has the common value for the class. The left side is the product of the following:

1. The product of all elements left of the clumps.
2. The standard polynomial of degree k , evaluated on the values of the k clumps, in some order.
3. The product of all the elements right of the clumps. (Because all these clumps are odd, Corollary to Lemma 4 of [4] ensures correctness of signs of terms.) Since the second factor vanishes, the conclusion follows.

Case 2. Suppose that Case 1 does not hold. Since there are $(2k - 1)n^2 + 1$ simple tensors, by Lemma 1 (ii) there exist i and j such that at least $2k$ simple tensors have form $a \otimes e_{ij}$. Evidently, $i \neq j$. Let $w_{ii} = e_{ii} + e_{ij}$. Then w_{ii} is idempotent, and

$$e_{ij} = e_{ii} + e_{ij} - e_{ii} = w_{ii} - e_{ii}.$$

In each term replace e_{ij} by $w_{ii} - e_{ii}$. Let N be the original number of e_{ij} 's. Each old term, upon expansion, yields 2^N new terms. Every new term has on the right a monomial in w 's and e 's. If there are at least k of the e_{ii} 's in the term, it is suitable for Case 1. Otherwise there are at least k of the w_{ii} 's. In this case, define new elements as follows:

$$w_{ij} = -e_{ij} + e_{jj}$$

$$w_{ji} = -e_{ii} - e_{ij} + e_{ji} + e_{jj}.$$

If $i \neq p \neq j$, let

$$w_{pi} = e_{pi} + e_{pj}$$

$$w_{jp} = -e_{ip} + e_{jp}.$$

For the remaining integers from 1 through n , let $w_{pq} = e_{pq}$.

The w 's constitute another set of matrix units in Z_n . Each old matrix unit e_{pq} is a linear combination of the w 's with integral coefficients. Replace all the e 's by w 's. The conclusion follows by the linearity of the standard polynomial and by Case 1.

DEFINITION. The *unitary identity of degree k* is

$$\sum_{\pi} x_{\pi(1)} \cdots x_{\pi(k)} = 0,$$

where the sum is over all permutations π of the integers 1 through k .

THEOREM 2. *If A is a ring satisfying the unitary identity of degree k , then A_n satisfies the unitary identity of degree kn .*

Proof. The proof uses Lemma 1 (i) and is similar to Theorem 1 of [4].

THEOREM 3. *If A is an algebra over a field with at least k elements, and A satisfies $x^k = 0$, then A_n satisfies $x^{kn} = 0$.*

Proof. The proof uses Lemma 1(i) and is similar to Theorem 1.2 of [3]. Note: That paper uses without definition the term "homogeneous component" of a polynomial. If $f(x_1, \dots, x_r)$ is a polynomial, the homogeneous component of degree n_1 in x_1 , degree n_2 in x_2 , etc., is the sum of all terms with degree n_1 in x_1 , etc.

REFERENCES

1. S. A. Amitsur and J. Levitzki, *Minimal identities for algebras*, Proc. Amer. Math. Soc., **1** (1950), 449–463.
2. Elizabeth Berman, *Group rings, matrix rings, and polynomial identities*, Trans. Amer. Math. Soc., **172** (1972), 241–248.
3. ———, *Matrix rings over polynomial identity rings*, Trans. Amer. Math. Soc., **172** (1972), 231–239.
4. ———, *Tensor products of polynomial identity algebras*, Trans. Amer. Math. Soc., **156** (1971), 259–271. MR 43 # 278.
5. Nathan Jacobson, *Structure of rings*, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 37, Amer. Math. Soc., Providence, R. I., 1964. MR 36 # 5158.
6. Thomas P. Kezlan, *Rings in which certain subsets satisfy polynomial identities*, Trans. Amer. Math. Soc., **125** (1966), 414–421.
7. Uri Leron and Amitai Vapne, *Polynomial identities of related rings*, Israel J. Math., **8** (1970), 127–137.
8. D. S. Passman, *Infinite group rings*, Marcel Dekker Inc., New York, 1971.
9. C. P. Procesi and L. Small, *Endomorphism rings of modules over PI-algebras*, Math. Z., **106** (1968), 178–180.

Received July 1, 1974. Presented to the Society, January 17, 1974. The writer is grateful to Professor Uri Leron for advice on this paper.

ROCKHURST COLLEGE
KANSAS CITY, MISSOURI

CONTENTS

Zvi Artstein and John A. Burns, <i>Integration of compact set-valued functions</i>	297
J. A. Beachy and W. D. Blair, <i>Rings whose faithful left ideals are cofaithful</i>	1
Mark Benard, <i>Characters and Schur indices of the unitary reflection group [321]³</i>	309
H. L. Bentley and B. J. Taylor, <i>Wallman rings</i>	15
E. Berman, <i>Matrix rings over polynomial identity rings II</i>	37
Simeon M. Berman, <i>A new characterization of characteristic functions of absolutely continuous distributions</i>	323
Monte B. Boisen, Jr. and Philip B. Sheldon, <i>Pre-Prüfer rings</i>	331
A. K. Boyle and K. R. Goodearl, <i>Rings over which certain modules are injective</i>	43
J. L. Brenner, R. M. Crabwell and J. Riddell, <i>Covering theorems for finite nonabelian simple groups. V</i>	55
H. H. Brungs, <i>Three questions on duo rings</i>	345
Iracema M. Bund, <i>Birnbaum-Orlicz spaces of functions on groups</i>	351
John D. Elwin and Donald R. Short, <i>Branched immersions between 2-manifolds of higher topological type</i>	361
J. K. Finch, <i>The single valued extension property on a Banach space</i>	61
J. R. Fisher, <i>A Goldie theorem for differentially prime rings</i>	71
Eric M. Friedlander, <i>Extension functions for rank 2, torsion free abelian groups</i>	371
J. Froemke and R. Quackenbusch, <i>The spectrum of an equational class of groupoids</i>	381
B. J. Gardner, <i>Radicals of supplementary semilattice sums of associative rings</i>	387
Shmuel Glasner, <i>Relatively invariant measures</i>	393
G. R. Gordh, Jr. and Sibe Mardešić, <i>Characterizing local connectedness in inverse limits</i>	411
S. Graf, <i>On the existence of strong liftings in second countable topological spaces</i>	419
S. Gudder and D. Strawther, <i>Orthogonally additive and orthogonally increasing functions on vector spaces</i>	427
F. Hansen, <i>On one-sided prime ideals</i>	79
D. J. Hartfiel and C. J. Maxson, <i>A characterization of the maximal monoids and maximal groups in βx</i>	437
Robert E. Hartwig and S. Brent Morris, <i>The universal flip matrix and the generalized faro-shuffle</i>	445

John Allen Beachy and William David Blair, <i>Rings whose faithful left ideals are cofaithful</i>	1
Herschel Lamar Bentley and Barbara June Taylor, <i>Wallman rings</i>	15
Elizabeth Berman, <i>Matrix rings over polynomial identity rings. II</i>	37
Ann K. Boyle and Kenneth R. Goodearl, <i>Rings over which certain modules are injective</i>	43
J. L. Brenner, Robert Myrl Cranwell and James Riddell, <i>Covering theorems for finite nonabelian simple groups. V</i>	55
James Kenneth Finch, <i>The single valued extension property on a Banach space</i>	61
John Robert Fisher, <i>A Goldie theorem for differentially prime rings</i>	71
Friedhelm Hansen, <i>On one-sided prime ideals</i>	79
Jon Craig Helton, <i>Product integrals and the solution of integral equations</i>	87
Barry E. Johnson and James Patrick Williams, <i>The range of a normal derivation</i>	105
Kurt Kreith, <i>A dynamical criterion for conjugate points</i>	123
Robert Allen McCoy, <i>Baire spaces and hyperspaces</i>	133
John McDonald, <i>Isometries of the disk algebra</i>	143
H. Minc, <i>Doubly stochastic matrices with minimal permanents</i>	155
Shahbaz Noorvash, <i>Covering the vertices of a graph by vertex-disjoint paths</i>	159
Theodore Windle Palmer, <i>Jordan *-homomorphisms between reduced Banach *-algebras</i>	169
Donald Steven Passman, <i>On the semisimplicity of group rings of some locally finite groups</i>	179
Mario Petrich, <i>Varieties of orthodox bands of groups</i>	209
Robert Horace Redfield, <i>The generalized interval topology on distributive lattices</i>	219
James Wilson Stepp, <i>Algebraic maximal semilattices</i>	243
Patrick Noble Stewart, <i>A sheaf theoretic representation of rings with Boolean orthogonalities</i>	249
Ting-On To and Kai Wing Yip, <i>A generalized Jensen's inequality</i>	255
Arnold Lewis Villone, <i>Second order differential operators with self-adjoint extensions</i>	261
Martin E. Walter, <i>On the structure of the Fourier-Stieltjes algebra</i>	267
John Wermer, <i>Subharmonicity and hulls</i>	283
Edythe Parker Woodruff, <i>A map of E^3 onto E^3 taking no disk onto a disk</i>	291