COVERING THEOREMS FOR FINITE NONABELIAN SIMPLE GROUPS. V

J. L. Brenner, Robert Myrl Cranwell and James Riddell
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In the alternating group $A_n$, $n = 4k + 1 > 5$, the class $C$ of
the cycle $(12\cdots n)$ has the property that $CC$ covers the
group. For $n = 16k$ there is a class $C$ of period $n/4$ in $A_n$ such
that $CC$ covers $A_n$; $C$ is the class of type $(4k)^4$.

1. Introduction. It was shown by E. Bertram [1] that for
$n \geq 5$ every permutation in $A_n$ is the product of two $l$-cycles, for any $l$
satisfying $[3n/4] \leq l \leq n$. Hence $A_n$ can be covered by products of two
$n$-cycles and also by products of two $(n-1)$-cycles. But if $n$ is odd the
$n$-cycles in $A_n$ fall into two conjugate classes $C, C'$, and similarly for the
$(n-1)$-cycles if $n$ is even, so that the quoted result does not decide
whether

$$CC = A_n.$$ 

The question was decided affirmatively for $n = 4k + 2$ and negatively
for $n = 4k, 4k - 1$ in [2]. The question is now decided affirmatively in
the remaining case $n = 4k + 1, n \neq 5$.

**Theorem 1.** For $n = 4k + 1 > 5$, the class $C$ of the cycle $(12\cdots n)$
has property (1).

The proof is in §§2–4.

Regarding the product $CC'$, it was shown in [2] that $CC'$ covers
$A_n (n \geq 5)$ if $n = 4k, 4k - 1$, while if $n = 4k + 1, 4k + 2$, $CC'$ contains all
of $A_n$ but the identity.

By an argument quite similar to the proof of Theorem 1, we have proved

**Theorem 2.** For $n = 16k$, the class $C$ of type $(4k)^4$ in $A_n$ has
property (1).

The proof and some related matters are discussed in §5. Note that
the class in Theorem 2 has period $n/4$.

2. The case $n = 9$. Let $a = (123456789)$. For every class in
$A_9$, a conjugate $b$ of $a$ can be found such that $ab$ represents (lies in) that
class. This assertion is the substance of the table below.
3. A lemma. In §3 and §4, $C$ will denote the class of the cycle $a = (12\cdots n)$ in $A_n$.

**Lemma.** If $n = 4k + 1 > 5$, then $CC$ contains the type $2^{2k}1^1$.

**Proof.** If $n = 1 \pmod{8}$, then $x = (nn-3 n-2 n-1, n-4 n-7 n-6 n-5; \cdots; 9678, 5234; 1)$
is conjugate to $a$ and

$$ax = (1 3)(2 4)(5 7)(6 8) \cdots (n-4 n-2)(n-3 n-1).$$

If $n = 5 \pmod{8}$, $n > 13$, then $y = (nn-3 n-2 n-1, n-4 n-7 n-6 n-5; \cdots; 21 18 19 20, 17 14 15 16; 13 96 10, 12 78 11; 5 23 4, 1)$
is conjugate to $a$ and

If \( n = 13 \) use the last 13 letters of the above \( y \). (The pattern of \( y \) differs from that of \( x \) only in the last block of 8 letters between semi-colons, 13 9 \( \cdots \) 11, in which the number of reversals is odd, whereas in every other such block of 8 letters in either \( x \) or \( y \), the number of reversals is even.)

4. The induction. The induction proceeds from \( n - 4 \) to \( n = 4k + 1 \). The induction hypothesis is: For every permutation \( T \) in \( A_{n-4} \), there are two \((n-4)\)-cycles \( d_1 \) and \( d_2 \), both in the class of the \((n-4)\)-cycle \((1 2 \cdots n - 6 n - 5 n - 4)\), and also two other \((n-4)\)-cycles \( d'_1 \) and \( d'_2 \), both in the class of \((1 2 \cdots n - 6 n - 4 n - 5)\), such that \( T = d_1d_2 = d'_1d'_2 \).

Let \( S \neq 1 \) be a permutation in \( A_n \). To show that \( CC \) contains \( S \) we consider several cases. In each case we find a conjugate \( S_i \) of \( S \), and a certain permutation \( g \) in \( A_n \), such that \( T = S_i g^{-1} \) fixes the letters \( n, n - 1, n - 2, n - 3 \) and thus its restriction to \( 1, 2, \cdots, n - 4 \) lies in \( A_{n-4} \).

**Case 1.** \( S \) contains a cycle with 5 or more letters: take

\[
g = (n n - 1 n - 2 n - 3 n - 4).
\]

**Case 2.** \( S \) contains no cycle with 5 or more letters, but \( S \) contains at least one cycle with 4 letters: take

\[
g = (n n - 1 n - 2 n - 3)(n - 4 n - 5).
\]

**Case 3.** \( S \) contains no cycle with more than 3 letters, but \( S \) does contain two 3-cycles: take

\[
g = (n n - 1 n - 2)(n - 3 n - 4 n - 5).
\]

**Case 4.** \( S \) is of type \( 3^1 2^{2k-2} 1^2 \): take

\[
g = (n n - 1 n - 2).
\]

Now, if \( S \) contains no cycle longer than a transposition, either \( S \) is of type \( 2^{2k} 1^1 \), whence \( CC \) contains \( S \) by the lemma, or we have

**Case 5.** \( S \) fixes 5 or more letters: take \( g = 1 \).

The argument in Case 5 is quite simple. Since \( S \) fixes 5 or more letters, \( S \) has a conjugate \( S_i \) that fixes \( n, n - 1, n - 2, n - 3 \). Hence by the induction hypothesis \( S_i = d_1d_2 \), where \( d_1 \) and \( d_2 \) both fix \( n, n - 1, n - 2, n - 3 \), and can be expressed...
where the permutation $a_i \rightarrow b_i$ is an even permutation of the letters $1, 2, \ldots, n - 5$. Then $S_1 = d_3 d_4$, with

$$d_3 = (a_1 a_2 \cdots a_{n-5} n n - 1 n - 2 n - 3 n - 4),$$
$$d_4 = (b_1 b_2 \cdots b_{n-5} n - 4 n - 3 n - 2 n - 1 n),$$

and $d_3, d_4$ belong to the same class, be it $C$ or $C'$. If the other part of the induction hypothesis is used in a similar fashion, the assertion that $CC$ contains $S$ follows.

The details for Case 1 are as follows. Since $T = S_1 g^{-1}$ moves at most the first $n - 4$ letters, we have by the induction hypothesis $T = d_1 d_2 = d'_1 d'_2$ where $d_1, d_2 \ [d'_1, d'_2]$ are from the same class in $A_{n-4}$. Writing

$$d_1 = (a_1 a_2 \cdots a_{n-5} n - 4), \quad d_2 = (b_1 b_2 \cdots b_{n-5} n - 4),$$

the permutation $a_i \rightarrow b_i$ is an even permutation of $1, 2, \ldots, n - 5$. Now $S_1 = T g = d_3 d_4$, with $g = (n n - 1 n - 2 n - 3 n - 4)$ and

$$d_3 = (a_1 \cdots a_{n-5} n - 2 n n - 3 n - 1 n - 4),$$
$$d_4 = (b_1 \cdots b_{n-5} n - 3 n - 1 n - 4 n - 2).$$

Note that $d_3$ and $d_4$ are in the same class, be it $C$ or $C'$, in $A_n$. By again using $d'_1$ and $d'_2$ in place of $d_1$ and $d_3$, the proof is completed in this case.

In Case 2, $S$ has a conjugate $S_1$ such that $T = S_1 g^{-1}$ fixes at least 5 letters. Hence without loss of generality the factors $d_1, d_2 \ [d'_1, d'_2]$ can be chosen so that $T = d_1 d_2 = d'_1 d'_2$ with

$$d_1 = (a_1 \cdots a_{n-6} n - 5 n - 4), \quad d'_1 = (a'_1 \cdots a'_{n-6} n - 5 n - 4)$$
$$d_2 = (b_1 \cdots b_{n-6} n - 4 n - 5), \quad d'_2 = (b'_1 \cdots b'_{n-6} n - 4 n - 5)$$

and where $a_i \rightarrow b_i \ [a'_i \rightarrow b'_i]$ is an odd permutation of the letters $1, 2, \ldots, n - 6$. Now $S_1 = T g = d_3 d_4$, where

$$d_3 = (a_1 \cdots a_{n-6} n - 1 n - 5 n - 3 n - 2 n n - 4),$$
$$d_4 = (b_1 \cdots b_{n-6} n - 5 n - 2 n n - 3 n - 4 n - 1).$$

The permutations $d_3$ and $d_4$ belong to the same class in $A_n$. Priming the $a_i$ and $b_i$ completes the proof in this case.
In Case 3, $S$ has at least two 3-cycles, and has a conjugate $S_1$ such that $T = S_1g^{-1}$ fixes the letters $n, n-1, n-2, n-3, n-4, n-5$. By the induction hypothesis permutations $d_1$ and $d_2$ exist such that $T = d_1 d_2$ with

$$d_1 = (n - 4 \ a_1 \cdots a_k n - 5 a_{k+1} \cdots a_{n-6}),$$
$$d_2 = (n - 4 \ b_1 \cdots b_l n - 5 b_{l+1} \cdots b_{n-6}),$$

and where $d_1$ and $d_2$ are in the same class in $A_n$. (We cannot assume that $n - 4$ and $n - 5$, which are fixed by $T$, are neighbors in $d_1$ and $d_2$, but it is possible that $k = 0$ and $l = n - 6$ or that $k = n - 6$ and $l = 0$.) Now $S_1 = Tg = d_3 d_4$, where

$$d_3 = d_1 h, \quad d_4 = h^{-1} d_2 g,$$

with $h = (n - 5 \ n - 3 \ n - 2)(n - 4 \ n - 1 \ n)$. Then $d_3$ and $d_4$ are both $n$-cycles. It has only to be checked that they are in the same class in $A_n$; to do this is tedious, but straightforward. To complete the proof in this case we observe that since $S$ contains two 3-cycles and $S_1 = d_3 d_4$, the decomposition $S_1 = d_1 d_2$ can be obtained by applying a certain outer automorphism of $A_n$.

In the only remaining case, $S$ fixes 2 letters, and therefore has a conjugate $S_1$ such that $T = S_1g^{-1}$ fixes

$$n, n - 1, n - 2, n - 3, n - 4.$$

Again we have $T = d_1 d_2$, where we can write

$$d_1 = (a_1 \cdots a_{n-6} n - 4 n - 5), \quad d_2 = (b_1 \cdots b_{n-6} n - 5 n - 4),$$

and where the permutation $a_i \to b_i$ is an odd permutation of the letters $1, 2, \cdots, n - 6$. Then $S_1 = Tg = d_3 d_4$, with

$$d_3 = (a_1 \cdots a_{n-6} n - 1 n n - 3 n - 2 n - 4 n - 5),$$
$$d_4 = (b_1 \cdots b_{n-6} n - 5 n - 4 n n - 2 n - 3 n - 1),$$

and these belong to the same class. By priming we again conclude $CC$ contains $S$, and the proof is complete in all cases. Hence Theorem 1.

5. Covering $A_{16k}$. By means of an almost identical argument we have shown that the class $C$ of type $4l_1 \ 4l_2 \ 4l_3 \ 4l_4$ ($l_i \geq 1$) in $A_n$ ($n = 4 \Sigma l_i$) has the covering property (1). The lemma required is simpler: Let $m = 4l, b = (12 \cdots m)$. Taking $x =$
\[(m m - 3 m - 2 m - 1, m - 4 m - 7 m - 6 m - 5, \ldots, 8567, 4123)\]
gives
\[bx = (1 3)(2 m)(4 6)(5 7) \cdots (m - 4 m - 2)(m - 3 m - 1).\]

Hence if \(D\) is the class of type \(4l_1, 4l_2 \cdots 4l_r\) (\(r\) even) in \(A_n\), then \(DD\) contains the type \(2^{n/2}\).

In order to start the induction we had to prove that the class \(C\) of type \(4^r\) has the property \(CC = A_{16}\). The calculations are too lengthy to be included. (A copy can be had from any of the authors.) This yields Theorem 2.

One can ask how small a period is possible for a class \(C\) with property (1). The first result in this direction was that of Xu [4] who found such a class with period \(n - 3\) if \(n\) is odd and period \(n - 2\) if \(n\) is even. From the result of Bertram quoted in the introduction, it follows that the smallest period of such \(C\) is \(\leq \frac{3n}{4}\). While Theorem 2 does not give covering for all \(n\), it nevertheless yields, among classes \(C\) in \(A_n\) satisfying (1),

\[
\lim \inf_{n \to \infty} \frac{\text{period of } C}{n} \leq \frac{1}{4}
\]
as opposed to Bertram's \(3/4\).

From the other direction we have shown [3] that for \(n > 6\) there is no class \(C\) in \(A_n\) having property (1) and period 2, and if \(n = 12k + 10\) there is no such class of period 3. There may be such a class of period 4, however. More precisely, we conjecture that for \(n = 8k\), the class \(C = 4^{2k}\) has the covering property (1).

REFERENCES


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