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COVERING THEOREMS FOR FINITE NONABELIAN SIMPLE GROUPS. V

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COVERING THEOREMS FOR FINITE NONABELIAN SIMPLE GROUPS. V.

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In the alternating group A_n , $n = 4k + 1 > 5$, the class C of the cycle $(12 \cdots n)$ has the property that CC covers the group. For $n = 16k$ there is a class C of period $n/4$ in A_n such that CC covers A_n ; C is the class of type $(4k)^4$.

1. Introduction. It was shown by E. Bertram [1] that for $n \geq 5$ every permutation in A_n is the product of two l -cycles, for any l satisfying $[3n/4] \leq l \leq n$. Hence A_n can be covered by products of two n -cycles and also by products of two $(n - 1)$ -cycles. But if n is odd the n -cycles in A_n fall into two conjugate classes C, C' , and similarly for the $(n - 1)$ -cycles if n is even, so that the quoted result does not decide whether

$$(1) \quad CC = A_n.$$

The question was decided affirmatively for $n = 4k + 2$ and negatively for $n = 4k, 4k - 1$ in [2]. The question is now decided affirmatively in the remaining case $n = 4k + 1, n \neq 5$.

THEOREM 1. *For $n = 4k + 1 > 5$, the class C of the cycle $(12 \cdots n)$ has property (1).*

The proof is in §§2-4.

Regarding the product CC' , it was shown in [2] that CC' covers A_n ($n \geq 5$) if $n = 4k, 4k - 1$, while if $n = 4k + 1, 4k + 2$, CC' contains all of A_n but the identity.

By an argument quite similar to the proof of Theorem 1, we have proved

THEOREM 2. *For $n = 16k$, the class C of type $(4k)^4$ in A_n has property (1).*

The proof and some related matters are discussed in §5. Note that the class in Theorem 2 has period $n/4$.

2. The case $n = 9$. Let $a = (123456789)$. For every class in A_9 , a conjugate b of a can be found such that ab represents (lies in) that class. This assertion is the substance of the table below.

b	ab
a^{-1}	1
(193248765)	(14) (38)
(176235894)	(13) (25) (48) (79)
(132987654)	(193)
(134765289)	(18) (24) (379)
(132798465)	(174) (369)
(184523796)	(135) (274) (698)
(137259486)	(15) (276) (3849)
(123794865)	(1384) (2769)
(132798654)	(17693)
(189623574)	(13) (25) (47986)
(132869745)	(18764) (359)
(132845697)	(18746) (359)
(159348726)	(162495) (38)
(186974532)	(3598764)
a	(135792468) $\sim a$
(125678934)	(315792468)

3. A lemma. In §3 and §4, C will denote the class of the cycle $a = (12 \cdots n)$ in A_n .

LEMMA. If $n = 4k + 1 > 5$, then CC contains the type $2^{2k} 1^1$.

Proof. If $n \equiv 1 \pmod{8}$, then $x =$

$$(n \ n-3 \ n-2 \ n-1, \ n-4 \ n-7 \ n-6 \ n-5; \cdots; 9678, 5234; 1)$$

is conjugate to a and

$$ax = (1\ 3)(2\ 4)(5\ 7)(6\ 8) \cdots (n-4 \ n-2)(n-3 \ n-1).$$

If $n \equiv 5 \pmod{8}$, $n > 13$, then $y =$

$$(n \ n-3 \ n-2 \ n-1, \ n-4 \ n-7 \ n-6 \ n-5; \cdots; 21\ 18\ 19\ 20, \\ 17\ 14\ 15\ 16; 13\ 96\ 10, 12\ 78\ 11; 5234, 1)$$

is conjugate to a and

$$ay = (1\ 3)(2\ 4)(5\ 10)(68)(7\ 11)(9\ 12)(13\ 15)(14\ 16) \cdots \\ (n-4 \ n-2)(n-3 \ n-1).$$

If $n = 13$ use the last 13 letters of the above y . (The pattern of y differs from that of x only in the last block of 8 letters between semi-colons, $13\ 9 \cdots 11$, in which the number of reversals is odd, whereas in every other such block of 8 letters in either x or y , the number of reversals is even.)

4. The induction. The induction proceeds from $n - 4$ to $n = 4k + 1$. The induction hypothesis is: For every permutation T in A_{n-4} , there are two $(n - 4)$ -cycles d_1 and d_2 , both in the class of the $(n - 4)$ -cycle $(1\ 2 \cdots n - 6\ n - 5\ n - 4)$, and also two other $(n - 4)$ -cycles d'_1 and d'_2 , both in the class of $(1\ 2 \cdots n - 6\ n - 4\ n - 5)$, such that $T = d_1 d_2 = d'_1 d'_2$.

Let $S (\neq 1)$ be a permutation in A_n . To show that CC contains S we consider several cases. In each case we find a conjugate S_1 of S , and a certain permutation g in A_n , such that $T = S_1 g^{-1}$ fixes the letters $n, n - 1, n - 2, n - 3$ and thus its restriction to $1, 2, \dots, n - 4$ lies in A_{n-4} .

Case 1. S contains a cycle with 5 or more letters: take

$$g = (n\ n - 1\ n - 2\ n - 3\ n - 4).$$

Case 2. S contains no cycle with 5 or more letters, but S contains at least one cycle with 4 letters: take

$$g = (n\ n - 1\ n - 2\ n - 3)(n - 4\ n - 5).$$

Case 3. S contains no cycle with more than 3 letters, but S does contain two 3-cycles: take

$$g = (n\ n - 1\ n - 2)(n - 3\ n - 4\ n - 5).$$

Case 4. S is of type $3^1 2^{2k-2} 1^2$: take

$$g = (n\ n - 1\ n - 2).$$

Now, if S contains no cycle longer than a transposition, either S is of type $2^{2k} 1^1$, whence CC contains S by the lemma, or we have

Case 5. S fixes 5 or more letters: take $g = 1$.

The argument in Case 5 is quite simple. Since S fixes 5 or more letters, S has a conjugate S_1 that fixes $n, n - 1, n - 2, n - 3$. Hence by the induction hypothesis $S_1 = d_1 d_2$, where d_1 and d_2 both fix $n, n - 1, n - 2, n - 3$, and can be expressed

$$d_1 = (a_1 a_2 \cdots a_{n-5} n - 4), \quad d_2 = (b_1 b_2 \cdots b_{n-5} n - 4),$$

where the permutation $a_i \rightarrow b_i$ is an even permutation of the letters $1, 2, \dots, n - 5$. Then $S_1 = d_3 d_4$, with

$$d_3 = (a_1 a_2 \cdots a_{n-5} n n - 1 n - 2 n - 3 n - 4),$$

$$d_4 = (b_1 b_2 \cdots b_{n-5} n - 4 n - 3 n - 2 n - 1 n),$$

and d_3, d_4 belong to the same class, be it C or C' . If the other part of the induction hypothesis is used in a similar fashion, the assertion that CC contains S follows.

The details for Case 1 are as follows. Since $T = S_1 g^{-1}$ moves at most the first $n - 4$ letters, we have by the induction hypothesis $T = d_1 d_2 = d'_1 d'_2$ where $d_1, d_2 [d'_1, d'_2]$ are from the same class in A_{n-4} . Writing

$$d_1 = (a_1 a_2 \cdots a_{n-5} n - 4), \quad d_2 = (b_1 b_2 \cdots b_{n-5} n - 4),$$

the permutation $a_i \rightarrow b_i$ is an even permutation of $1, 2, \dots, n - 5$. Now $S_1 = Tg = d_3 d_4$, with $g = (n n - 1 n - 2 n - 3 n - 4)$ and

$$d_3 = (a_1 \cdots a_{n-5} n - 2 n n - 3 n - 1 n - 4),$$

$$d_4 = (b_1 \cdots b_{n-5} n n - 3 n - 1 n - 4 n - 2).$$

Note that d_3 and d_4 are in the same class, be it C or C' , in A_n . By again using d'_1 and d'_2 in place of d_1 and d_2 , the proof is completed in this case.

In Case 2, S has a conjugate S_1 such that $T = S_1 g^{-1}$ fixes at least 5 letters. Hence without loss of generality the factors $d_1, d_2 [d'_1, d'_2]$ can be chosen so that $T = d_1 d_2 = d'_1 d'_2$ with

$$d_1 = (a_1 \cdots a_{n-6} n - 5 n - 4), \quad d'_1 = (a'_1 \cdots a'_{n-6} n - 5 n - 4)$$

$$d_2 = (b_1 \cdots b_{n-6} n - 4 n - 5), \quad d'_2 = (b'_1 \cdots b'_{n-6} n - 4 n - 5)$$

and where $a_i \rightarrow b_i [a'_i \rightarrow b'_i]$ is an *odd* permutation of the letters $1, 2, \dots, n - 6$. Now $S_1 = Tg = d_3 d_4$, where

$$d_3 = (a_1 \cdots a_{n-6} n - 1 n - 5 n - 3 n - 2 n n - 4),$$

$$d_4 = (b_1 \cdots b_{n-6} n - 5 n - 2 n n - 3 n - 4 n - 1).$$

The permutations d_3 and d_4 belong to the same class in A_n . Priming the a_i and b_i completes the proof in this case.

In Case 3, S has at least two 3-cycles, and has a conjugate S_1 such that $T = S_1 g^{-1}$ fixes the letters $n, n-1, n-2, n-3, n-4, n-5$. By the induction hypothesis permutations d_1 and d_2 exist such that $T = d_1 d_2$ with

$$\begin{aligned} d_1 &= (n-4 \ a_1 \cdots a_k \ n-5 \ a_{k+1} \cdots a_{n-6}), \\ d_2 &= (n-4 \ b_1 \cdots b_l \ n-5 \ b_{l+1} \cdots b_{n-6}), \end{aligned}$$

and where d_1 and d_2 are in the same class in A_n . (We cannot assume that $n-4$ and $n-5$, which are fixed by T , are neighbors in d_1 and d_2 , but it is possible that $k=0$ and $l=n-6$ or that $k=n-6$ and $l=0$.) Now $S_1 = Tg = d_3 d_4$, where

$$d_3 = d_1 h, \quad d_4 = h^{-1} d_2 g,$$

with $h = (n-5 \ n-3 \ n-2)(n-4 \ n-1 \ n)$. Then d_3 and d_4 are both n -cycles. It has only to be checked that they are in the same class in A_n ; to do this is tedious, but straightforward. To complete the proof in this case we observe that since S contains two 3-cycles and $S_1 = d_3 d_4$, the decomposition $S_1 = d'_3 d'_4$ can be obtained by applying a certain outer automorphism of A_n .

In the only remaining case, S fixes 2 letters, and therefore has a conjugate S_1 such that $T = S_1 g^{-1}$ fixes

$$n, n-1, n-2, n-3, n-4.$$

Again we have $T = d_1 d_2$, where we can write

$$d_1 = (a_1 \cdots a_{n-6} \ n-4 \ n-5), \quad d_2 = (b_1 \cdots b_{n-6} \ n-5 \ n-4),$$

and where the permutation $a_i \rightarrow b_i$ is an odd permutation of the letters $1, 2, \dots, n-6$. Then $S_1 = Tg = d_3 d_4$, with

$$\begin{aligned} d_3 &= (a_1 \cdots a_{n-6} \ n-1 \ n \ n-3 \ n-2 \ n-4 \ n-5), \\ d_4 &= (b_1 \cdots b_{n-6} \ n-5 \ n-4 \ n \ n-2 \ n-3 \ n-1), \end{aligned}$$

and these belong to the same class. By priming we again conclude CC contains S , and the proof is complete in all cases. Hence Theorem 1.

5. Covering A_{16k} . By means of an almost identical argument we have shown that the class C of type $4l_1 \ 4l_2 \ 4l_3 \ 4l_4$ ($l_i \geq 1$) in A_n ($n = 4\sum l_i$) has the covering property (1). The lemma required is simpler: Let $m = 4l$, $b = (12 \cdots m)$. Taking $x =$

$$(m\ m - 3\ m - 2\ m - 1, m - 4\ m - 7\ m - 6\ m - 5, \dots, 8\ 5\ 6\ 7, 4\ 1\ 2\ 3)$$

gives

$$bx = (1\ 3)(2\ m)(4\ 6)(5\ 7) \cdots (m - 4\ m - 2)(m - 3\ m - 1).$$

Hence if D is the class of type $4l_1\ 4l_2 \cdots 4l_r$ (r even) in A_n , then DD contains the type $2^{n/2}$.

In order to start the induction we had to prove that the class C of type 4^4 has the property $CC = A_{16}$. The calculations are too lengthy to be included. (A copy can be had from any of the authors.) This yields Theorem 2.

One can ask how small a period is possible for a class C with property (1). The first result in this direction was that of Xu [4] who found such a class with period $n - 3$ if n is odd and period $n - 2$ if n is even. From the result of Bertram quoted in the introduction, it follows that the smallest period of such C is $\leq 3n/4$. While Theorem 2 does not give covering for all n , it nevertheless yields, among classes C in A_n satisfying (1),

$$\liminf_{n \rightarrow \infty} \frac{\text{period of } C}{n} \leq \frac{1}{4}$$

as opposed to Bertram's $3/4$.

From the other direction we have shown [3] that for $n > 6$ there is no class C in A_n having property (1) and period 2, and if $n = 12k + 10$ there is no such class of period 3. There may be such a class of period 4, however. More precisely, we conjecture that for $n = 8k$, the class $C = 4^{2k}$ has the covering property (1).

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