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JOHN MCDONALD

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J. N. McDONALD

In this paper we are concerned with the problem, posed by R. R. Phelps, of describing the into isometries of the disk algebra. We show that, in a certain sense, every isometry can be approximated by convex combinations of isometries of the form $f \to k(f \circ \phi)$. We also give some sufficient conditions for an isometry to be of the form $f \to k(f \circ \phi)$.

Let D and Γ denote, respectively, the open unit disk and the unit circle. The disk algebra, i.e., the algebra of all complex valued functions which are continuous on $D \cup \Gamma$ and analytic on D, will be denoted by A. It will be assumed that A is equipped with the sup-norm.

Operators of the form

$$Tf = k(f \circ \phi) \tag{1}$$

are isometries of A: if $k \in A$, if $\|k\| = 1$, and if $\phi: D \cup \Gamma \to D \cup \Gamma$ is analytic on D, continuous on $D \cup \Gamma \sim k^{-1}(0)$, and satisfies $\phi(k^{-1}(\Gamma)) \supset \Gamma$. In fact, if T is a surjective linear isometry of A, then it must be of the form (1) with k being a constant, and ϕ being a Mobius transformation. (See [**3**, pp. 142–148].) Rochberg [**8**] has shown that if T is an isometry such that $T1 = 1$, and $T(A)$ is a sub-algebra of A, then T is of the form (1) with $k \equiv 1$.

Note that any bounded linear operator $T: A \to A$ which satisfies (1) also satisfies.

$$T1T(fg) = TfTg \tag{2}$$

for all f and g in A. Moreover, we have the following.

PROPOSITION 1.1. *A bounded linear operator $T: A \to A$ satisfies* (2) *for all $f, g \in A$ iff it is of the form* (1).

Proof. It is only necessary to show that, if T satisfies (2) for all $f, g \in A$, then it satisfies (1).

Suppose that w is a point of D where $T1$ is not 0. Consider the linear functional defined on A by

$$L_w(f) = (T1(w))^{-1}Tf(w).$$

By (2), L_w is a multiplicative. Hence, there is a v in $D \cup \Gamma$ such that

$L_w(f) = f(v)$. Since $v = (T1(w))^{-1}TZ(w)$, where Z is the identity function on $D \cup \Gamma$, it follows that the function $\phi = (T1)^{-1}TZ$ is bounded on D. Thus, the singularities of ϕ in D are removable. Let S be the operator defined on A by

$$Sf = T1(f \circ \phi).$$

It follows easily from (2) that $SZ^n = TZ^n$ for $n = 0, 1, \cdots$. Since the polynomials in Z are dense in A, the operators T and S are the same. If $T1 \equiv 0$, then, by (2), $(Tf)^2 = T1Tf^2 \equiv 0$. It follows that T is of the form (1) with $k \equiv 0$.

For an example of an isometry which fixes 1 but is not multiplicative, see **[8]**.

For the remainder of this section, T will denote an arbitrary isometry of A. Consider the closed set $\Gamma(T) = \{z \in \Gamma \,|\, |Tl(z)| = 1$ and there is a point $\hat{T}(z)$ in Γ such that $Tf(z) = T1(z)f(\hat{T}(z))$ for all $f \in A\}$. Since A separates the points of Γ, it follows that the mapping $z \to \hat{T}(z)$, denoted by $\hat{T}$, is well defined and continuous on $\Gamma(T)$. In **[5]**, we showed that $\hat{T}$ maps $\Gamma(T)$ onto Γ. The following proposition gives a simple description of $\Gamma(T)$.

PROPOSITION 1.2.

$$\Gamma(T) = \{w \in \Gamma \,|\, |T1(w)| = 1 \quad \textit{and} \quad |TZ(w)| = 1\}.$$

Proof. It is enough to show that if $|T1(z_1)| = |TZ(z_1)| = 1$, then $z_1 \in \Gamma(T)$. By the Hahn-Banach theorem, there is a measure μ on Γ having total variation $\leqq 1$ such that $Tf(z_1) = \int f d\mu$ for all $f \in A$. Let $a = \int 1 d\mu$ and $b = \int Z d\mu$, where Z is the identity on $D \cup \Gamma$. Since $\bar{a}\mu$ has total variation $\leqq 1$ and $\int \bar{a} d\mu = 1$, it follows that $\bar{a} d\mu$ is nonnegative. Note that $\int \operatorname{Re}(1 - a\bar{b}Z)\bar{a} d\mu = 0$. Thus, $\operatorname{Re}(1 - a\bar{b}Z)$ is 0 on the support of μ. Hence the support of μ consists of a single point, i.e., $\hat{T}(z_1)$.

THEOREM 1.1. *Suppose $m(\Gamma(T)) > 0$, where m denotes Lesbegue measure on Γ. Then T is of the form* (1).

Proof. For $f, g \in A$, we have

$$T1(z)\, T(fg)(z) = Tf(z)\, Tg(z)$$

for every $z \in \Gamma(T)$. Any two functions in A which agree on a subset of Γ having positive Lesbeque measure are equal. (See [3, p. 52].) Thus $T1\, T(fg) = Tf\, Tg$. It follows by Proposition 1.1 that T is of the form (1).

THEOREM 1.2. *Assume that $T1$ is an inner function. Suppose that $T(A)$ contains a function G having the following properties: $\|G\| = 1$, $m(G^{-1})(\Gamma)) > 0$, the set of connected components of $G^{-1}(\Gamma)$ is countab e, and G is not a constant multiple of $T1$. Then T is of the form* (1).

Proof. Let $H = \overline{T1}G$. Note that $H^{-1}(\Gamma) = G^{-1}(\Gamma)$. Let $\{J_1, J_2, \cdots\}$ denote the collection of connected components of $H^{-1}(\Gamma)$. Suppose it can be shown that, for some q, $m(H(J_q \cap \Gamma(T))) > 0$. Then J_q is necessarily a nontrivial sub-arc of Γ. By a form of the Schwartz reflection principle (See, e.g. [2, p. 187].), G can be continued analytically across the interior of J_q. It follows that the restriction of H to the interior of J_q is continuously differentiable. If H were constant on J_q, then we would have $G = cT1$ where c is a constant. Thus, H is not constant and, hence, $m(J_q \cap \Gamma(T)) > 0$. It now follows by Theorem 1.1 that T is of the form (1).

It remains to be shown that $m(H(J_q \cap \Gamma(T))) > 0$ for some q. It is claimed that

$$H(H^{-1}(\Gamma)) = H(H^{-1}(\Gamma) \cap \Gamma(T)).$$

For each $z \in \Gamma$, there exists a measure μ_z, having total variation $\leqq 1$, such that $\int f d\mu_z = Tf(z)$ for each $f \in A$. In particular, we have $1 = \int \overline{T1(z)} d\mu_z$. It follows that the measure $\overline{T1(z)\mu_z}$ is nonnegative. Suppose that z is choosen so that $|G(z)| = |H(z)| = 1$. Let F be the unique function in A such that $G = TF$. Then

$$\int \mathrm{Re}(1 - \overline{H(z)}F)\overline{T1(z)} d\mu = 0.$$

It follows that $H(z) = F(w)$ for each w in the support of μ_z. Since the mapping $\hat{T}$ is onto, there exists a $z_1 \in \Gamma(T)$ such that

$$\begin{aligned} H(z) &= F(\hat{T}(z_1)) \\ &= \overline{T1(z_1)}T1(z_1)F(\hat{T}(z_1)) \\ &= H(z_1). \end{aligned}$$

Next it is claimed that $m(H(H^{-1}(\Gamma)))>0$. If $m(H(H^{-1}(\Gamma)))=0$, then H is constant on all of the J_n's. Since at least one of the J_n's is a nontrivial sub-arc of Γ, it follows that $G=cT1$ for some constant c – a contradiction to the hypothesis that G not be a scalar multiple of $T1$. Finally, we have

$$0<m(H(H^{-1}(\Gamma)))\leqq \Sigma m(H(J_n\cap\Gamma(T))).$$

It follows that $m(H(J_q\cap\Gamma(T)))>0$ for some q.

COROLLARY. *Suppose that* $T1$ *is an inner function. If* TA *contains an inner function which is not a scalar multiple of* Tl *then* T *is of the form* (1).

REMARK. Let $\mathscr{A}$ denote the sub-algebra of A consisting of functions which are analytic in a neighborhood of $D\cup\Gamma$. By arguments similar to those used to prove Theorem 1.2, one can show that every isometry of $\mathscr{A}$ must be of the form (1).

2. Approximation of arbitrary isometries. As in the previous section, T will denote an arbitrary isometry of A. Let B denote the space of bounded linear operators: $A\to A$ and let B_1 denote the set of members of B having norm $\leqq 1$. As in **[5]**, we define $E(T)=\{U\in B_1 \mid Uf(z)=Tf(z)$ for every $z\in\Gamma(T)$ and every $f\in A\}$. In **[5]** we showed that $E(T)$ is a face of B_1, that $E(T)$ is closed in the weak operator topology, and that each member of $E(T)$ is an isometry. Thus, the set of isometries of A is the union of weak operator-closed faces of B_1. It follows from Proposition 1.2, that

$$E(T)=\{U\in B_1 \mid UZ\,|\,\Gamma(T)=TZ\,|\,\Gamma(T) \quad\text{and}\quad U1\,|\,\Gamma(T)=T1\,|\,\Gamma(T)\},$$

where Z denotes the identity function on $D\cup\Gamma$. If $m(\Gamma(T))>0$, it follows that $E(T)=\{T\}$. Suppose that $m(\Gamma(T))=0$. Let A_1 denote the unit ball in A, let $S_1=\{f\in A_1 \mid f\,|\,\Gamma(T)=\hat{T}\}$, and let $S_2=\{g\in A_1 \mid g\,|\,\Gamma(T)=T1\,|\,\Gamma(T)\}$. By a result due to Rudin **[9]**, both S_1 and S_2 have infinitely many members. Let $h\in S_1$ and $k\in S_2$. The operator U defined by $Uf=k(f\circ h)$ is in $E(T)$. Thus, $E(T)$ contains infinitely many elements iff $m(\Gamma(T))=0$. For the remainder of the paper, we will consider only isometries T for which $m(\Gamma(T))=0$.

Let $F(T)=\{U\in E(T) \mid U$ is of the form (1)$\}$. In view of **[5**, Th. 3**]**, it is natural to ask whether $E(T)$ is the closed convex hull of $F(T)$, where the closure is taken in the weak operator topology? Although we are unable to answer this question, we will show that there is a family $\mathfrak{S}$

of locally convex Hausdorff topologies on B with the following properties: for each $\mathcal{T} \in \mathfrak{S}$, $E(T)$ is the $\mathcal{T}$-closed convex hull of $F(T)$, and the weakest topology containing all the members of $\mathfrak{S}$ is the weak operator topology.

The weak operator topology on B is the weakest topology in which all linear functionals of the form $H \to \int Hf\,d\mu$, where f is in A and μ is a Baire measure on Γ, are continuous. It follows that the space B^* of weak operator continuous linear functionals on B is the direct sum of sub-spaces $\mathcal{A}$ and $\mathcal{S}$, where $\mathcal{A}$ is the sub-space of B^* spanned by linear functionals of the form $H \to \int (Hf)gdm$ with $g \in L_1(m)$, and where $\mathcal{S}$ is the sub-space of B^* spanned by functionals of the form $H \to \int Hf\,d\nu$ with ν being singular with respect to m. (See [1, p. 421]). Let $L_1^+(m)$ denote $\{g \in L_1(m) | g \geqq 0 \text{ a.e.}\}$. For each $g \in L_1^+(m)$ we define the *$\mathcal{S}g$-topology* on B to be the weakest topology in which the linear functionals of the form $H \to \int (Hf)gdm$ with f in A, and the linear functionals in $\mathcal{S}$, are continuous. Set $\mathfrak{S} = \{\mathcal{S}g \,|\, g \in L_1^+(m)\}$. Let $\mathcal{W}$ denote the weak operator topology on B. Note that $\mathcal{S}g \subseteq \mathcal{W}$ for each $g \in L_1^+(m)$. By [1, p. 421], the $\mathcal{S}g$-continuous linear functionals on B are those of the form $l(H) = \int Hfgdm + \Sigma_{i=1}^n \int Hf_i d\mu_i$, where the measures μ_i, $i = 1, 2, \cdots, n$, are singular with respect to m and $f, f_1, f_2 \cdots f_n \in A$. Let $\mathcal{U}$ denote the smallest locally convex topology on B which contains all members of $\mathfrak{S}$. Any functional of the form $L(H) = \int Hf\,d\nu$, where $f \in A$, and ν is a regular Baire measure, can be written in the form

$$L(H) = \int Hf\,d\mu + \sum_{n=1}^{4} \int H(i^n f)g_n dm,$$

where μ is singular with respect to m, and $g_1, g_2, g_3, g_4 \in L_1^+(m)$. It follows that L is $\mathcal{U}$-continuous. Hence, by the definition of $\mathcal{W}$, we have $\mathcal{U} \subseteq \mathcal{W}$.

THEOREM 2.1. *For each $g \in L_1^+(m)$, $E(T)$ is the $\mathcal{S}g$-closed convex hull of $F(T)$.*

REMARK. It is not possible to prove Theorem 2.1 by using arguments based on the Krein-Milman theorem. For in order for the Krein-Milman theorem to apply to $E(T)$ it would be necessary for

$E(T)$ to be compact in the $\mathcal{S}g$-topology, but the following argument shows that $E(T)$ is not $\mathcal{S}g$-compact for any $g \in L_1^+(m)$: Let K be a "Cantor" subset of Γ which is disjoint from $\Gamma(T)$. Let $C_1(K)$ denote the set of continuous complex valued functions on K having absolute value $\leqq 1$. Define $j: E(T) \to C_1(K)$ by $j(U) = UZ|K$. If $C_1(K)$ is equipped with the topology of pointwise convergence, then j is $\mathcal{S}g$-continuous for each $g \in L_1^+(m)$. By **[9]**, the map j is onto. Since $C_1(K)$ is not compact in the topology of pointwise convergence, it follows that $E(T)$ is not compact in the $\mathcal{S}g$-topology.

Our proof of Theorem 3 will depend on the following two lemmas:

LEMMA 2.1. *For z in D, and $t \in [-\pi, \pi]$ let*

$$P_z(e^{it}) = \mathrm{Re}((z + e^{it})\ (z - e^{it})^{-1}),$$

i.e., $P_z(e^{it})$ is the Poisson kernel. Consider the set $V = \{\Sigma_{i=1}^n c_i P_{z_i} \mid c_i \geqq 0$ and $z_i \in D$ for $i = 1, 2, \cdots, n\}$. Then the L_1-closure of V is $L_1^+(m)$.

Proof. Suppose that $g_1 \in L_1^+(m)$, but g_1 is not in the closure of V. Then there exists an h in $L_\infty(m)$, such that $\int g_1 h dm > 0$ and $\int v h dm \leqq 0$ for every $v \in V$. In particular

$$\int P_z(e^{it}) h(e^{it}) dm(e^{it}) \leqq 0.$$

for all z in D. Fatou's Theorem **[3**, p. 34**]** implies that $h \leqq 0$ almost everywhere with respect to m. Hence, $\int g h dm \leqq 0$. Thus, we have reached a contradiction.

LEMMA 2.2. *Let E be a closed subset of Γ such that $m(E) = 0$. Let $\phi_0: E \to D \cup \Gamma$ be continuous. Consider $z_1, z_2, \cdots, z_n$, $w \in D$. There is a function ϕ in the unit ball A_1 of A which extends ϕ_0 and satisfies $\phi(z_i) = w$ for $i = 1, 2, \cdots, n$.*

Proof. Suppose that $w = 0$. Let

$$B(Z) = \Pi_{i=1}^n [(z - \bar{z}_i)\ (1 - \bar{z}_i z)^{-1}].$$

For each u in Γ, we have $|B(u)| = 1$. Define β_0 on E by $\beta_0(u) = \overline{B(u)}\phi_0(u)$. The function β_0 has an extension β in A_1. It follows that

$B\beta$ satisfies the assertion of the lemma in the case where $w = 0$. If $w \neq 0$, we choose a Mobius transformation τ of D such that $\tau(0) = w$, and apply the preceeding argument to obtain a function ϕ_1 in A_1 which extends $\tau^{-1} \circ \phi_0$ and maps $z_1, \cdots, z_n$ into 0. Thus, the function $\phi = \tau \circ \phi_1$, extends ϕ_0, lies in A_1, and statisfies $\phi(z_i) = w$ for $i = 1, 2, \cdots n$.

Proof of Theorem 2.1. (The argument used here is an adaptation of one due to Morris and Phelps [**6**, Th. 2.1].)

Suppose $U \in E(T)$ but it is not in the $\mathcal{S}g$-closed convex hull of $F(T)$. By [**1**, Th. 9, p. 421], there are functions $f, f_1, f_2, \cdots, f_n \in A$, measures $\mu_1, \mu_2, \cdots, \mu_n$ on Γ which are singular with respect to m, and a real number $r > 0$ such that

$$\mathrm{Re}\left(\int (Uf)\, g dm + \sum_{i=1}^{n} \int Uf_i d\mu_i\right) \geqq \mathrm{Re}\left(\int (Ff)\, g dm + \sum_{i=1}^{n} \int Ff_i d\mu_i\right) + r,$$

for every F in $E(T)$.

By Lemma 2.1, there are points $z_1, z_2, \cdots, z_p \in D$ and nonnegative real numbers $c_1, c_2, \cdots, c_p$ such that

$$\begin{aligned}(3) \qquad &\mathrm{Re}\left(\sum_{j=1}^{p} c_j Uf(z_j) + \sum_{i=1}^{n} \int Uf_i d\mu_i\right) \\ &\geq \mathrm{Re}\left(\sum_{j=1}^{p} c_j Ff(z_j) + \sum_{i=1}^{n} \int Ff_i d\mu \hat{i}\right) + \frac{r}{2},\end{aligned}$$

for every F in $F(T)$. We can assume without loss of generality that $\mu_i \geqq 0$ for $i = 1, 2, , \cdots, n$. Since $Uf_i = Ff_i$ on $\Gamma(T)$ for $i = 1, 2, \cdots, n$ and $F \in F(T)$, we can also assume that $\mu_i(\Gamma(T)) = 0$ for $i = 1, 2, \cdots, n$. Let $\nu = \Sigma_{i=1}^{n} \mu_i$. Given $\epsilon > 0$, there is a closed subset Y of $\Gamma \sim \Gamma(T)$ such that $m(Y) = 0$ and $\nu(\Gamma \sim Y) < \epsilon$. Let h_i denote the Radon-Nikodym derivative of μ_i with respect to ν for $i = 1, 2, \cdots, n$. Choose continuous functions h'_i on Γ such that $0 \leqq h'_i \leqq 1$ and $\int |h_i - h'_i| d\nu < \epsilon$ for $i = 1, 2, \cdots, n$.

Let $g = \Sigma_{i=1}^{n} h'_i Uf_i$. For each $y \in \Gamma$, define $k_y = \Sigma_{i=1}^{n} h'_i(y) f_i$. Then $g(y) = Uk_y(y)$. $g(y)$ is also equal to $(U^* e_y)(k_y)$, where U^* is the adjoint of U and e_y represents the "evaluation at y" functional on A. Let S denote the unit ball in the dual space of A. Since U^* maps S into S, it follows that $U^* e_y \in S$. The function $W(\rho) = \mathrm{Re}\, \rho(k_y)$ is weak* continuous on S and $\sup W(S) \geqq \mathrm{Re}\, U^* e_y(k_y) = \mathrm{Re}\, g(y)$. The extreme points of S are exactly the functionals ce_y, where $c, y \in \Gamma$. It follows from the Krein-Milman theorem that, for each $y \in \Gamma$, there exist $\phi(y)$, $c(y) \in \Gamma$, such that

$$\operatorname{Re}\left(\sum_{i=1}^{n} c(y)h_i'(y)f_i(\phi(y))\right) > \operatorname{Re} g(y) - \epsilon.$$

For each $y \in Y$ choose an open neighborhood V_y of y such that $V_y \cap \Gamma(T) = \phi$ and

$$\operatorname{Re}\left(\sum_{i=1}^{n} c(y)h_i'(w)f_i(\phi(y))\right) > \operatorname{Re} g(w) - 2\epsilon$$

for every $w \in V_y$. Let $\{V_{y_1}, \cdots, V_{y_p}\}$ be a finite collection of V_y's which covers y. We can easily find another open cover $\{U_1, \cdots, U_p\}$ of Y such that $U_i \subset V_{y_i}$ and $\nu(\{y \mid y \text{ is in more than one } U_j\}) < \epsilon$. Consider the sets

$$H_j = (Y \cap U_j) \sim \cup \{U_i \mid i \neq j\}, \quad j = 1, 2, \cdots, p.$$

Then H_j's are closed and disjoint and $\nu(Y \sim \cup_{j=1}^{p} H_j) < \epsilon$.

Define mappings θ_0, k_0: $\Gamma(T) \cup [\cup_{i=1}^{p} H_i] \rightarrow \Gamma$ by

$$\theta_0(y) = \begin{cases} \phi(y_j) & \text{if} \quad y \in H_j \\ \hat{T}(y) & \text{if} \quad y \in \Gamma(T). \end{cases}$$

$$k_0(y) = \begin{cases} c(u_j) & \text{if} \quad y \in H_j \\ T1(y) & \text{if} \quad y \in \Gamma(T). \end{cases}$$

Note that $m(\Gamma(T) \cup [\cup_{i=1}^{p} H_j]) = 0$. Since U is an isometry, there are points w_0 and w_1 in D such that $\operatorname{Re} w_0 f(w_1) \geqq \operatorname{Re} Uf(z_i) - \epsilon$ for $i = 1, 2, \cdots, p$. By Lemma 2.2, there are extensions θ and k, of θ_0 and k_0 respectively, which lie in A_1 and satisfy $\theta(z_j) = w_1$ and $k(z_j) = w_0$ for $j = 1, 2, \cdots, n$. Define the linear operator F_1: $A \rightarrow A$ by $F_1 h = k(h \circ \theta)$. Clearly, we have $F_1 \in F(T)$. By a straightforward argument, we can find a constant $M > 0$ independent of ϵ such that

$$\operatorname{Re}\left(\sum_{j=1}^{p} c_j F_1 f(z_j) + \sum_{i=1}^{n} \int F_1 f_i d\mu_i\right)$$
$$> \operatorname{Re}\left(\sum_{j=1}^{p} c_j Uf(z_j) + \sum_{i=1}^{n} \int Uf_i d\mu_i\right) - M\epsilon.$$

We can obtain a contradiction to (3) by taking ϵ to be sufficient small.

COROLLARY 2.1. *Suppose* $T1 \equiv 1$. *Let* $E_1(T) = \{U \mid U \in E(T)$ *and* $U1 = 1\}$, *and let* $F_1(T) = E_1(T) \cap F(T)$. *Then for each* $g \in L_1^+(m)$, *the set* $E_1(T)$ *is the closed convex hull of* $F_1(T)$ *where the closure is taken in the* $\mathcal{S}g$-*topology.*

Proof. Let $S_1 = \{L \in S \mid L(1) = 1\}$. The adjoint T^* of T maps S_1 to S_1. The extreme points of S_1 are the functionals of the form e_y with $y \in \Gamma$. Thus, in the proof of Theorem 2.1 we may take $c(y) = 1$. Also, it is clear that, in this case, we may take $w_0 = 1$. Consequently, it can be asumed that the function k is identically 1.

3. The case $T1 = 1$.

In this section it will be assumed that $T1 = 1$. We will investigate the closure in the weak operator topology of the set $\operatorname{cov} F_1(T)$.

Let H_∞ denote the space of bounded analytic functions on D and let $B(H_\infty)$ denote the space of bounded linear operators on H_∞. Denote by $\mathcal{P}$ the weakest topology on $B(H_\infty)$ such that all linear functionals of the form $M \to Mg(z)$, where $g \in H_\infty$ and $z \in D$, are continuous. The following property of $B(H_\infty)$ will be very useful in this section: *The unit ball of* $B(H_\infty)$ *is* $\mathcal{P}$-*compact.* To verify this property it sufficies to use a result due to Kadison [4] together with the fact that the unit ball H_∞^1 of H_∞ is compact in the topology of pointwise convergence.

Let $A_T = \{\phi \in A_1 \mid \phi \mid \Gamma(T) = \hat{T}\}$. Let H_T denote the closure of A_T in the topology of pointwise convergence on D. Since $H_T \subseteq H_\infty^1$, it follows that H_T is compact in the topology of pointwise convergence on D. Each $F \in F_1(T)$ is of the form $Ff = f \circ \phi$ for all $f \in A$, where $\phi \in A_T$. Thus, F has an extension to H_∞ denoted F^* which is defined by $F^*g = g \circ \phi$ for every $g \in H_\infty$. Similarly, each $V \in \operatorname{cov} F_1(T)$ has an extension V^* lying in $\operatorname{cov} F_1^*(T)$, where $F_1^*(T) = \{F^* \mid F \in F_1(T)\}$. Since $F_1^*(T)$ is contained in the unit ball of $B(H_\infty)$, it follows that the $\mathcal{P}$-closed convex hull of $F_1^*(T)$, denoted by R, is compact in the $\mathcal{P}$-topology. Let Q denote the $\mathcal{P}$-closure of $F_1^*(T)$. Suppose that $W \in R$. By the integral form of the Krein-Milman Theorem [7, p. 6], there is a probability measure μ_W supported by Q such that

$$Wg(z) = \int_Q W'g(z)d\mu_W(W')$$

for every $g \in H_\infty$ and every $z \in D$. Note that $Q = \{W \mid Wg = g \circ \phi,$ where $\phi \in H_T\}$. Thus, Q may be identified with H_T. Consequently, we can write

$$Wg(z) = \int_{H_T} g \circ \phi(z)d\mu_W(\phi)$$

for all $g \in H_\infty$ and all $z \in D$. Suppose now that U is in the weak operator closed convex hull of $F_1(T)$. Then there exists a net $\{V_\alpha\}$ in cov $F_1(T)$ which converges in the weak operator topology to U. In particular, $Uf(z) = \lim V_\alpha f(z)$ for each $f \in A$ and each $z \in D$. The net V_α^* has a subnet V_β^* which converges to some $U^* \in R$. By the definition of the $\mathcal{P}$-topology, we have $U^*f(z) = Uf(z)$ for $f \in A$ and $z \in D$. Thus, we have proved the following:

THEOREM 3.1. *Let U be in the closure of* $\operatorname{cov} F_1(T)$ *in the weak operator topology. Then there exists a probability measure μ on H_T such that*

$$Uf(z) = \int_{H_T} f \circ \phi(z) d\mu(\phi)$$

for each $f \in A$ and each $z \in D$.

We will now use Theorem 3.1 to derive another sufficient condition for an isometry to be of the form (1).

THEOREM 3.2. *Suppose U is in the weak operator closure of* $\operatorname{cov} F_1(T)$. *If there is a nonconstant inner function G such that UG is an extreme point of A_1, then U is of the form* (1).

Our proof of Theorem 3.2 depends upon the following technical lemma.

LEMMA 3.1. *Let G be a nonconstant inner function in A.* (a) *Suppose that $k \in A$ is of the form $k = G \circ h$ on D, where $h \in H_\infty^1$. Then h has an extension to $D \cup \Gamma$ which is continuous.* (b) *Let $h_1, h_2 \in A_1$. Consider the set*

$$S = \{z \in D \cup \Gamma \mid h_1(z) = h_2(z)\}.$$

Suppose that $h_1(S)$ is infinite. Suppose also that $G \circ h_1 = G \circ h_2$. Then $h_1 = h_2$.

Proof. Since G is an inner function and is a member of A, it follows that G is of the form

$$G(z) = e^{ix} \prod_{n-1}^{N} (z - z_n)/(1 - \bar{z}_n z),$$

where the z_n's are (not necessarily distinct) points of D. It follows that, given any point $u_0 \in \Gamma \cup D$, there exists a disk D_0 about u_0 and analytic functions $g_1, g_2, \cdots, g_n$ defined on D_0, such that if $G(w) = u \in D_0$ then $w = g_j(u)$ for some j. Suppose that $u_0 = k(z_0)$, where $z_0 \in \Gamma$. Choose a set W containing z_0 which is open relative to $\Gamma \cup D$ and satisfies $k(W) \subseteq D_0$. On $W \cap D$, we have $k = G \circ h$. It follows that for some j, $h(z) = g_j \circ k(z)$ for all $z \in W$. Thus, h can be extended continuously to $W \cap \Gamma$. A simple compactness argument now shows that h can be extended continuously to all of Γ.

Consider the set $Y = \{z \in D \cup \Gamma \mid G'(h_1(z)) \neq 0$ and $h_1(z) = h_2(z)\}$. We will show that Y is open relative to $D \cup \Gamma$. Since Y is nonempty, it will follow that $h_1 = h_2$. Let $z_0 \in Y$. Since $G'(h_1(z_0)) \neq 0$, there exists an open disk D_0 about $h_1(z_0)$ such that G is one-to-one on D_0. Choose a set N, which is open relative to $D \cup \Gamma$, such that $h_1(N) \subseteq D_0$ and $h_2(N) \subseteq D_0$. Then, for $z \in N$, $G(h_1(z)) = G(h_2(z))$. It follows that $h_1 = h_2$ on N.

Proof of Theorem 3.2: By Theorem 3.1, we may write

$$Uf(z) = \int_{H_T} f \circ \phi(z) d\mu(\phi)$$

for all $f \in A$ and all $z \in D$. For each $z \in D$, let

$$J_z = \{\phi \in H_T \mid \operatorname{Re} UG(z) < \operatorname{Re} G \circ \phi(z)\}.$$

Suppose that for some $u \in D$, we have $c = \mu(J_u) > 0$. Define measures μ_1 and μ_2 on H_T by

$$\mu_1(K) = c^{-1}\mu(K \cap J_u)$$
$$\mu_2(K) = (1-c)^{-1}\mu(K \cap (H_T \sim J_u)).$$

By [7, Prop. 1.1], there are operators U_1 and U_2 in R such that

$$U_i f(z) = \int_{H_T} f \circ \phi(z) d\mu_i \qquad i = 1,2,$$

for each $f \in A$ and each $z \in D$. (Note that for $f \in A$, $U_i f$ is not necessarily in A.) It follows that

$$Uf(z) = cU_1 f(z) + (1-c) U_2 f(z)$$

for $f \in A$ and $z \in D$. Since UG is an extreme point of A_1, it is also extreme point of H^1_∞. (See [3, p. 139].) Thus, we have $UG = U_1 G = U_2 G$ on D, but

$$\begin{aligned}\operatorname{Re} UG(u) &= \int_{H_T} \operatorname{Re} UG(u)\, d\mu_1(\phi) \\ &< \int_{H_T} \operatorname{Re} G \circ \phi(u)\, d\mu_1(\phi) = \operatorname{Re} U_1 G(u),\end{aligned}$$

a contradiction. It follows that for each $z \in D$ we have $\mu\{\phi \mid \operatorname{Re} UG(z) < \operatorname{Re} G \circ \phi(z)\}) = 0$. Similarly, we can show that

$$\mu(\{\phi \mid \operatorname{Re} UG(z) > \operatorname{Re} G \circ \phi(z)\}) = \mu(\{\phi \mid \operatorname{Im} UG(z) \neq \operatorname{Im} G \circ \phi(z)\}) = 0.$$

Thus, $UG(z) = G \circ \phi(z)$ for all ϕ in the support of μ. It follows that the support of μ consists of finitely many functions $\phi_1, \cdots, \phi_m \in H_T$, where each ϕ_i satisfies $G \circ \phi_i = UG$ on D. By Lemma 3.1, each ϕ_i is continuous on $D \cup \Gamma$. Thus, there exist positive numbers $c_1, \cdots, c_m$ such that $\Sigma c_i = 1$ and

$$Uf(z) = \sum c_i f \circ \phi_i(z)$$

for each $f \in A$ and each $z \in D \cup \Gamma$. For $z \in \Gamma(T)$, we have $\phi_i(z) = \hat{T}(z)$ for $i = 1, 2, \cdots, m$. It follows by Lemma 3.1, that $\phi_1 = \phi_2 = \cdots = \phi_m$. Hence $U \in F_1(T)$.

REMARK. Theorem 3.2 provides a possible approach to the problem of finding an isometry T such that $T1 = 1$ and $E_1(T)$ is not the weak operator closure of $\operatorname{cov} F_1(T)$. If an isometry T can be found such that: $T1 = 1$, T is not of the form (1), and TG is an extreme point of A_1 for some nonconstant inner function $G \in A$, then it will follow from Theorem 3.2 that $T \notin$ weak operator closure of $\operatorname{cov} F_1(T)$.

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ARIZONA STATE UNIVERSITY

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