DOUBLY STOCHASTIC MATRICES WITH MINIMAL PERMANENTS

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A simple elementary proof is given for a result of D. London on permanental minors of doubly stochastic matrices with minimal permanents.

A matrix with nonnegative entries is called *doubly stochastic* if all its row sums and column sums are equal to 1. A well-known conjecture of van der Waerden [3] asserts that the permanent function attains its minimum in $\Omega_n$, the set of $n \times n$ doubly stochastic matrices, uniquely for the matrix all of whose entries are $1/n$. The conjecture is still unresolved.

A matrix $A$ in $\Omega_n$ is said to be *minimizing* if

$$\text{per}(A) = \min_{S \in \Omega_n} \text{per}(S).$$

The properties of minimizing matrices have been studied extensively in the hope of finding a lead to a proof of the van der Waerden conjecture.

Let $A(i \mid j)$ denote the submatrix obtained from $A$ by deleting its $i$th row and its $j$th column. Marcus and Newman [3] have obtained inter alia the following two results.

**Theorem 1.** A minimizing matrix $A$ is fully indecomposable, i.e.,

$$\text{per}(A(i \mid j)) > 0$$

for all $i$ and $j$.

In other words, if $A$ is a minimizing $n \times n$ matrix then for any $(i, j)$ there exists a permutation $\sigma$ such that $j = \sigma(i)$ and $a_{s, \sigma(s)} > 0$ for $s = 1, \cdots, i - 1, i + 1, \cdots, n$.

**Theorem 2.** If $A = (a_{ij})$ is a minimizing matrix then

(1) $$\text{per}(A(i \mid j)) = \text{per}(A)$$

for any $(i, j)$ for which $a_{ij} > 0$. 

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The result in Theorem 2 is of considerable interest. For, if it could be shown that (1) holds for all permanental minors of $A$, the van der Waerden conjecture would follow. London [2] obtained the following result.

**Theorem 3.** *If $A$ is a minimizing matrix, then*

$$
\text{per}(A(i \mid j)) \geq \text{per}(A)
$$

*for all $i$ and $j$.*

London's proof of Theorem 3 depends on the theory of linear inequalities. Another proof of London's result is due to Hedrick [1]. In this paper I give an elementary proof of the result that is considerably simpler than either of the above noted proofs.

**Proof of Theorem 3.** Let $A = (a_{ij})$ be an $n \times n$ minimizing matrix. Let $\sigma$ be a permutation on $\{1, \ldots, n\}$ and $P = (p_{ij})$ be the corresponding permutation matrix. For $0 \leq \theta \leq 1$, define

$$
f_{\theta}(\theta) = \text{per}((1 - \theta)A + \theta P).
$$

Since $A$ is a minimizing matrix, we have

$$
f'_{\theta}(0) \geq 0
$$

for any permutation matrix $P$. Now

$$
f'_{\theta}(0) = \sum_{s,t=1}^{n} (-a_{st} + p_{st}) \text{per}(A(s \mid t))
$$

$$
= \sum_{s,t=1}^{n} p_{st} \text{per}(A(s \mid t)) - n \text{ per } (A)
$$

$$
= \sum_{s=1}^{n} \text{per}(A(s \mid \sigma(s))) - n \text{ per } (A).
$$

Hence,

$$
\sum_{s=1}^{n} \text{per}(A(s \mid \sigma(s))) \geq n \text{ per } (A)
$$

$$
\sum_{s=1}^{n} \text{per}(A(s \mid \sigma(s))) \geq n \text{ per } (A)
$$
for any permutation $\sigma$. Since $A$ is a minimizing matrix and thus, by
Theorem 1, fully indecomposable, we can find for any given $(i,j)$ a
permutation $\sigma$ such that $j = \sigma(i)$ and $a_{s,\sigma(s)} > 0$ for $s = 1, \cdots, i-1, i+1, \cdots, n$. But then by Theorem 2,

$$\text{per}(A(s | \sigma(s))) = \text{per}(A)$$

for $s = 1, \cdots, i-1, i+1, \cdots, n$, and it follows from (3) that

$$\text{per}(A(i | j)) \geq \text{per}(A).$$

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Received March 15, 1974. This research was supported by the Air Force Office of Scientific
Research under Grant No. 72–2164C.

UNIVERSITY OF CALIFORNIA, SANTA BARBARA
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