

# Pacific Journal of Mathematics

**DOUBLY STOCHASTIC MATRICES WITH MINIMAL  
PERMANENTS**

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## DOUBLY STOCHASTIC MATRICES WITH MINIMAL PERMANENTS

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**A simple elementary proof is given for a result of D. London on permanental minors of doubly stochastic matrices with minimal permanents.**

A matrix with nonnegative entries is called *doubly stochastic* if all its row sums and column sums are equal to 1. A well-known conjecture of van der Waerden [3] asserts that the permanent function attains its minimum in  $\Omega_n$ , the set of  $n \times n$  doubly stochastic matrices, uniquely for the matrix all of whose entries are  $1/n$ . The conjecture is still unresolved.

A matrix  $A$  in  $\Omega_n$  is said to be *minimizing* if

$$\text{per}(A) = \min_{S \in \Omega_n} \text{per}(S).$$

The properties of minimizing matrices have been studied extensively in the hope of finding a lead to a proof of the van der Waerden conjecture.

Let  $A(i|j)$  denote the submatrix obtained from  $A$  by deleting its  $i$ th row and its  $j$ th column. Marcus and Newman [3] have obtained inter alia the following two results.

**THEOREM 1.** *A minimizing matrix  $A$  is fully indecomposable, i.e.,*

$$\text{per}(A(i|j)) > 0$$

*for all  $i$  and  $j$ .*

In other words, if  $A$  is a minimizing  $n \times n$  matrix then for any  $(i, j)$  there exists a permutation  $\sigma$  such that  $j = \sigma(i)$  and  $a_{s, \sigma(s)} > 0$  for  $s = 1, \dots, i-1, i+1, \dots, n$ .

**THEOREM 2.** *If  $A = (a_{ij})$  is a minimizing matrix then*

$$(1) \quad \text{per}(A(i|j)) = \text{per}(A)$$

*for any  $(i, j)$  for which  $a_{ij} > 0$ .*

The result in Theorem 2 is of considerable interest. For, if it could be shown that (1) holds for all permanent minors of  $A$ , the van der Waerden conjecture would follow. London [2] obtained the following result.

**THEOREM 3.** *If  $A$  is a minimizing matrix, then*

$$(2) \quad \text{per}(A(i|j)) \geq \text{per}(A)$$

for all  $i$  and  $j$ .

London's proof of Theorem 3 depends on the theory of linear inequalities. Another proof of London's result is due to Hedrick [1]. In this paper I give an elementary proof of the result that is considerably simpler than either of the above noted proofs.

*Proof of Theorem 3.* Let  $A = (a_{ij})$  be an  $n \times n$  minimizing matrix. Let  $\sigma$  be a permutation on  $\{1, \dots, n\}$  and  $P = (p_{ij})$  be the corresponding permutation matrix. For  $0 \leq \theta \leq 1$ , define

$$f_P(\theta) = \text{per}((1 - \theta)A + \theta P).$$

Since  $A$  is a minimizing matrix, we have

$$f'_P(0) \geq 0$$

for any permutation matrix  $P$ . Now

$$\begin{aligned} f'_P(0) &= \sum_{s,t=1}^n (-a_{st} + p_{st}) \text{per}(A(s|t)) \\ &= \sum_{s,t=1}^n p_{st} \text{per}(A(s|t)) - n \text{per}(A) \\ &= \sum_{s=1}^n \text{per}(A(s|\sigma(s))) - n \text{per}(A). \end{aligned}$$

Hence,

$$(3) \quad \sum_{s=1}^n \text{per}(A(s|\sigma(s))) \geq n \text{per}(A)$$

$$\sum_{s=1}^n \text{per}(A(s|\sigma(s))) \geq n \text{per}(A)$$

for any permutation  $\sigma$ . Since  $A$  is a minimizing matrix and thus, by Theorem 1, fully indecomposable, we can find for any given  $(i, j)$  a permutation  $\sigma$  such that  $j = \sigma(i)$  and  $a_{s, \sigma(s)} > 0$  for  $s = 1, \dots, i-1, i+1, \dots, n$ . But then by Theorem 2,

$$\text{per}(A(s | \sigma(s))) = \text{per}(A)$$

for  $s = 1, \dots, i-1, i+1, \dots, n$ , and it follows from (3) that

$$\text{per}(A(i | j)) \geq \text{per}(A).$$

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