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**DOUBLY STOCHASTIC MATRICES WITH MINIMAL
PERMANENTS**

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A simple elementary proof is given for a result of D. London on permanent minors of doubly stochastic matrices with minimal permanents.

A matrix with nonnegative entries is called *doubly stochastic* if all its row sums and column sums are equal to 1. A well-known conjecture of van der Waerden [3] asserts that the permanent function attains its minimum in Ω_n , the set of $n \times n$ doubly stochastic matrices, uniquely for the matrix all of whose entries are $1/n$. The conjecture is still unresolved.

A matrix A in Ω_n is said to be *minimizing* if

$$\text{per}(A) = \min_{S \in \Omega_n} \text{per}(S).$$

The properties of minimizing matrices have been studied extensively in the hope of finding a lead to a proof of the van der Waerden conjecture.

Let $A(i|j)$ denote the submatrix obtained from A by deleting its i th row and its j th column. Marcus and Newman [3] have obtained inter alia the following two results.

THEOREM 1. *A minimizing matrix A is fully indecomposable, i.e.,*

$$\text{per}(A(i|j)) > 0$$

for all i and j .

In other words, if A is a minimizing $n \times n$ matrix then for any (i, j) there exists a permutation σ such that $j = \sigma(i)$ and $a_{s, \sigma(s)} > 0$ for $s = 1, \dots, i-1, i+1, \dots, n$.

THEOREM 2. *If $A = (a_{ij})$ is a minimizing matrix then*

$$(1) \quad \text{per}(A(i|j)) = \text{per}(A)$$

for any (i, j) for which $a_{ij} > 0$.

The result in Theorem 2 is of considerable interest. For, if it could be shown that (1) holds for all permanent minors of A , the van der Waerden conjecture would follow. London [2] obtained the following result.

THEOREM 3. *If A is a minimizing matrix, then*

$$(2) \quad \text{per}(A(i|j)) \geq \text{per}(A)$$

for all i and j .

London's proof of Theorem 3 depends on the theory of linear inequalities. Another proof of London's result is due to Hedrick [1]. In this paper I give an elementary proof of the result that is considerably simpler than either of the above noted proofs.

Proof of Theorem 3. Let $A = (a_{ij})$ be an $n \times n$ minimizing matrix. Let σ be a permutation on $\{1, \dots, n\}$ and $P = (p_{ij})$ be the corresponding permutation matrix. For $0 \leq \theta \leq 1$, define

$$f_P(\theta) = \text{per}((1 - \theta)A + \theta P).$$

Since A is a minimizing matrix, we have

$$f'_P(0) \geq 0$$

for any permutation matrix P . Now

$$\begin{aligned} f'_P(0) &= \sum_{s,t=1}^n (-a_{st} + p_{st}) \text{per}(A(s|t)) \\ &= \sum_{s,t=1}^n p_{st} \text{per}(A(s|t)) - n \text{per}(A) \\ &= \sum_{s=1}^n \text{per}(A(s|\sigma(s))) - n \text{per}(A). \end{aligned}$$

Hence,

$$(3) \quad \sum_{s=1}^n \text{per}(A(s|\sigma(s))) \geq n \text{per}(A)$$

$$\sum_{s=1}^n \text{per}(A(s|\sigma(s))) \geq n \text{per}(A)$$

for any permutation σ . Since A is a minimizing matrix and thus, by Theorem 1, fully indecomposable, we can find for any given (i, j) a permutation σ such that $j = \sigma(i)$ and $a_{s, \sigma(s)} > 0$ for $s = 1, \dots, i-1, i+1, \dots, n$. But then by Theorem 2,

$$\text{per}(A(s | \sigma(s))) = \text{per}(A)$$

for $s = 1, \dots, i-1, i+1, \dots, n$, and it follows from (3) that

$$\text{per}(A(i | j)) \cong \text{per}(A).$$

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