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**COVERING THE VERTICES OF A GRAPH BY
VERTEX-DISJOINT PATHS**

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Define the path-covering number $\mu(G)$ of a finite graph G to be the minimum number of vertex-disjoint paths required to cover the vertices of G . Let $g(n, k)$ be the minimum integer so that every graph, G , with n vertices and at least $g(n, k)$ edges has $\mu(G) \leq k$. A relationship between $\mu(G)$ and the degree sequence for a graph G is found; this is used to show that

$$\frac{1}{2}(n-k)(n-k-1) + 1 \leq g(n, k) \leq \frac{1}{2}(n-1)(n-k-1) + 1$$

A further extremal problem is solved.

1. Introduction. A graph G is a finite collection $\mathcal{V}(G)$ of vertices (or points) some pairs of which are joined by a single edge; the collection of edges is denoted by $\mathcal{E}(G)$. H is a *subgraph* of G if $\mathcal{V}(H) \subseteq \mathcal{V}(G)$ and $\mathcal{E}(H) \subseteq \mathcal{E}(G)$. If H and K are two vertex-disjoint graphs, $H \cup K$ is the graph with $\mathcal{V}(H \cup K) = \mathcal{V}(H) \cup \mathcal{V}(K)$ and $\mathcal{E}(H \cup K) = \mathcal{E}(H) \cup \mathcal{E}(K)$; $H + K$ is $H \cup K$ together with all $|\mathcal{V}(H)| |\mathcal{V}(K)|$ possible choices of edges joining a vertex of H to a vertex of K . \bar{G} denotes the complement of G ; Γ_n denotes the complete graph with n vertices and $\Gamma_{m,n}$ denotes the complete bipartite graph, $\bar{\Gamma}_m + \bar{\Gamma}_n$.

Let G be a graph. A path of length n in G is an ordered sequence $P = \langle a_1, a_2, \dots, a_n \rangle$ of distinct points, where if $n \geq 2$, a_i is adjacent to a_{i+1} for $1 \leq i \leq n-1$. $\langle a_1, a_2, \dots, a_n \rangle$ is the same path as $\langle a_n, a_{n-1}, \dots, a_1 \rangle$. If P and Q are paths, by $P * Q$ we shall mean that one end-point, a of P , is adjacent to one end-point, b of Q , and that $P * Q$ is formed by joining a to b . More specifically we may write $Pa * bQ$ or $P * bQ$ or $Pa * Q$ to specify, in varying degrees, which end-point of P is joined to which end-point of Q . Also, $\langle a_1, a_2, \dots, a_n \rangle * \langle b_1, b_2, \dots, b_m \rangle = \langle a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_m \rangle$ where a_n must be adjacent to b_1 . A *Hamilton-path* is a path of length $|\mathcal{V}(G)|$. A *path-cover* of G is a collection, \mathcal{S} , of vertex-disjoint paths such that every vertex of G lies on some path in \mathcal{S} . The *path-covering number*, denoted by $\mu(G)$, of G is defined by:

$$\mu(G) = \text{Min} \{ |\mathcal{S}| : \mathcal{S} \text{ is a path-cover of } G \}.$$

A *minimal path-cover* (M.P.C.) of G is a path-cover, \mathcal{S} of G , with $|\mathcal{S}| = \mu(G)$.

We note that $\mu(G)$ is an invariant of G and remark that a graph, G , has a Hamilton-path if and only if $\mu(G) = 1$. It has been shown by Nash-Williams [1] and others that the problem of classifying all Hamiltonian graphs is equivalent to that of classifying all graphs which have a Hamilton-path. Thus a classification of all graphs with $\mu(G) = k$ ($k = 1, 2, 3, \dots$) would also solve the Hamiltonian problem as a special case.

Historically, O. Ore [3] first introduced the graphical invariant μ . In [2] some elementary properties of μ are derived. In §2 we generalize a result of O. Ore (Theorem 2.1 in [3]) and in §3 we consider two extremal problems involving μ .

2. Valency considerations. In this section we derive a connection between the path-covering number and the degree sequence of a graph. We begin with some definitions:

DEFINITION 2.1. Let A be a finite set and f a real-valued function defined on the collection of subsets of A . For $B \subseteq A$ and for any integer i with $1 \leq i \leq |B|$, define the function S_i by:

$$S_i(f, B) = \sum_{\substack{C \subseteq B \\ |C|=i}} f(C).$$

DEFINITION 2.2. If G is a graph, $B \subseteq \mathcal{V}(G)$, and either $H \subseteq \mathcal{V}(G)$ or H is a subgraph of G , then define the generalized valence function, ρ , by

$$\rho_H(B) = \text{the number of vertices of } H \text{ which are adjacent to every member of } B.$$

If x is a vertex of G , then we write $\rho(x)$ for $\rho_G(\{x\})$.

DEFINITION 2.3. Let G be a graph and $X \subseteq \mathcal{V}(G)$ with $|X| = k \geq 2$. Define:

$$D(G, X) = \frac{1}{k} S_1(\rho_G, X) - \sum_{i=1}^k (-1)^i \binom{k-i}{k} S_i(\rho_G, X).$$

The following lemma is easily verified:

LEMMA 2.4. If $X = \{x_1, x_2, \dots, x_k\}$, and $1 \leq m \leq k - 1$, then

$$\sum_{i=1}^k S_m(f, X - \{x_i\}) = (k - m)S_m(f, X).$$

We now state the main result of this section:

THEOREM 2.5. Let G be a graph with $\mu = \mu(G) \geq 2$, $|\mathcal{V}(G)| = n$ and k an integer with $2 \leq k \leq \mu$, then there exists a set X consisting of k mutually non-adjacent vertices of G , satisfying:

$$(2.6) \quad \mu \leq n - D(G, X).$$

Note that the case $k = 2$ reduces to the result of Ore (Theorem 2.1 in [3]):

$$\mu \leq n - \rho(x_1) - \rho(x_2).$$

Proof. Let $\mathcal{S} = \{P_1, P_2, \dots, P_\mu\}$ be a M.P.C. for G . For each $1 \leq i \leq k$, let x_i be an end-vertex of P_i . Since \mathcal{S} is a M.P.C., x_i is not adjacent to x_j for $i \neq j$.

Let $X = \{x_1, x_2, \dots, x_k\}$. We first show that for $1 \leq i \leq k$ and $1 \leq j \leq \mu$, the inequality:

$$(2.7) \quad \rho_{P_j}(\{x_i\}) \leq |P_j| - \left(1 - \sum_{i=1}^{k-1} (-1)^i S_i(\rho_{P_j}, X - \{x_i\})\right)$$

holds. Let P_j be the path $\langle a_1, a_2, \dots, a_t \rangle$, let $1 \leq m \leq k$, $m \neq i$, and consider the following cases:

- (i) $i = j$. In this case assume that $x_i = a_1$.
- (ii) $m = j$. In this case assume that $x_m = a_t$.
- (iii) $m \neq j$ and $i \neq j$.

Let

$$A = \{r: a_r \text{ is adjacent to } x_i\},$$

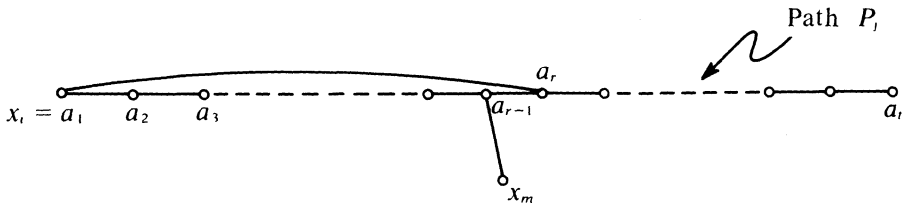
$$B_m = \{r: a_{r-1} \text{ is adjacent to } x_m\}$$

and

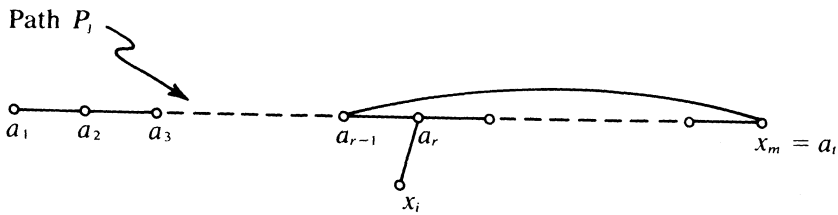
$$B = \bigcup_{\substack{1 \leq m \leq k \\ m \neq i}} B_m.$$

We claim that $A \cap B_m = \phi$, for if $r \in A \cap B_m$, then in each case we can

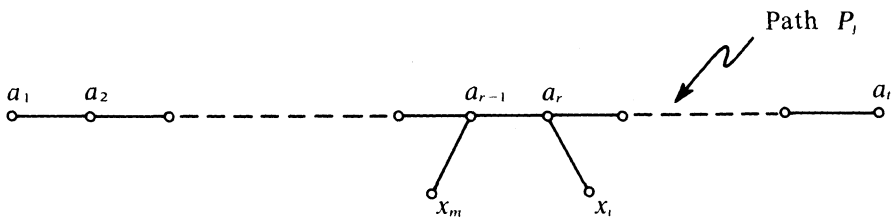
construct a path-cover, \mathcal{T} for G , as follows (see Figure 2.8):



Case (i)



Case (ii)



Case (iii)

FIGURE 2.8

In case (i), let:

$$\mathcal{T} = \mathcal{P} \cup \{ \langle a_t, a_{t-1}, \dots, a_r, x_i, a_2, a_3, \dots, a_{r-1} \rangle * x_m P_m \} - \{ P_i, P_m \}.$$

In case (ii), let:

$$\mathcal{T} = \mathcal{S} \cup \{ \langle a_1, a_2, \dots, a_{r-1}, x_m, a_{t-1}, a_{t-2}, \dots, a_r \rangle * x_i P_i \} - \{ P_i, P_m \}.$$

In case (iii), let:

$$\mathcal{T} = \mathcal{S} \cup \{ \langle a_1, \dots, a_{r-1} \rangle * x_m P_m, \langle a_t, a_{t-1}, \dots, a_r \rangle * x_i P_i \} - \{ P_i, P_j, P_m \}.$$

In either case, $|\mathcal{T}| = |\mathcal{S}| - 1 < |\mathcal{S}|$, contradicting the minimality of \mathcal{S} . Hence $A \cap B_m = \emptyset$. Also, in each case $a_1 \notin A$; so $A \subseteq P_j - B \cup \{a_1\}$. This gives $|A| \leq |P_j| - |B \cup \{a_1\}|$, since $B \cup \{a_1\} \subseteq P_j$. But then, since $a_1 \notin B$, we get:

$$(2.9) \quad |A| \leq |P_j| - (1 + |B|).$$

For $1 \leq m \leq k$, let:

$$C_m = \{r : a_r \text{ is adjacent to } x_m\}.$$

Then since x_m is not adjacent to a_1 , $|C_m| = |B_m|$ and:

$$\begin{aligned} |B| &= \left| \bigcup_{\substack{1 \leq m \leq k \\ m \neq i}} B_m \right| = \left| \bigcup_{\substack{1 \leq m \leq k \\ m \neq i}} C_m \right| \\ &= \sum_{l=1}^{k-1} (-1)^{l+1} \sum_{\substack{1 \leq m_1 < m_2 < \dots < m_l \leq k \\ m_1, m_2, \dots, m_l \neq i}} |C_{m_1} \cap C_{m_2} \cap \dots \cap C_{m_l}| \\ (2.10) \quad &= - \sum_{l=1}^{k-1} (-1)^l S_l(\rho_P, X - \{x_i\}). \end{aligned}$$

So since $|A| = \rho_P(\{x_i\})$, (2.7) follows from (2.9) and (2.10). Summing (2.7) for $1 \leq i \leq k$ and applying Lemma 2.4, we get:

$$(2.11) \quad S_1(\rho_P, X) \leq k|P_j| - \left(k - \sum_{l=1}^{k-1} (-1)^l (k-l) S_l(\rho_P, X) \right).$$

Summing (2.11) for $1 \leq j \leq \mu$, we get:

$$S_i(\rho_G, X) \leq kn - \left(k\mu - \sum_{l=1}^{k-1} (-1)^l (k-l) S_l(\rho_G, X) \right).$$

from which (2.6) follows.

3. Extremal problems.

DEFINITION 3.1. Let k and n be integers with $1 \leq k \leq n$. Define:

$$g(n, k) = \text{Min} \{ m : \text{every graph, } G, \text{ with } |\mathcal{V}(G)| = n \text{ and} \\ |\mathcal{E}(G)| \geq m \text{ has } \mu(G) \leq k \}.$$

In this section we determine bounds for $g(n, k)$. See [4] for techniques in proving the following:

LEMMA 3.2.

$$(3.3) \quad \sum_{i=1}^{k-1} (-1)^i \binom{k-i}{k} \binom{k}{i} = -1 \quad \text{if } k \geq 2,$$

$$(3.4) \quad \sum_{i=2}^k (-1)^i (k-i+1) \binom{k}{i-1} = k \quad \text{if } k \geq 2,$$

$$(3.5) \quad \sum_{i=2}^j (-1)^i (k-i+1) \binom{j-1}{i-1} = k \quad \text{if } 3 \leq j \leq k.$$

LEMMA 3.6. Let K be a graph with $|\mathcal{V}(K)| = s \geq 1$, and let k be an integer with $k \geq 2$, and suppose $H = \bar{\Gamma}_k + K$, then:

$$D(H, \mathcal{V}(\bar{\Gamma}_k)) = 2s.$$

Proof. For $1 \leq i \leq k-1$ and $B \subseteq \mathcal{V}(\bar{\Gamma}_k)$ with $|B| = i$, each member of B is adjacent to every member of $\mathcal{V}(K)$. There are $\binom{k}{i}$ choices for B and $|\mathcal{V}(K)| = s$; thus:

$$S_i(\rho_H, \mathcal{V}(\bar{\Gamma}_k)) = s \binom{k}{i}.$$

This gives:

$$\begin{aligned} D(H, \mathcal{V}(\bar{\Gamma}_k)) &= \frac{s}{k} \binom{k}{1} - \sum_{i=1}^{k-1} (-1)^i s \binom{k-i}{k} \binom{k}{i} \\ &= s \left[1 - \sum_{i=1}^{k-1} (-1)^i \binom{k-i}{k} \binom{k}{i} \right] \\ &= 2s, \quad \text{using (3.3).} \end{aligned}$$

THEOREM 3.7. For $1 \leq k \leq n$,

$$(3.8) \quad g(n, k) \leq \frac{1}{2}(n-1)(n-k-1) + 1.$$

Proof. Let G be a graph with $|\mathcal{V}(G)| = n$, and $|\mathcal{E}(G)| \geq \frac{1}{2}(n-1)(n-k-1) + 1$. Suppose $\mu(G) > k$ and $X = \{x_1, x_2, \dots, x_k, x_{k+1}\}$ is a set of mutually nonadjacent vertices of G .

G may be considered to have been obtained from the complete graph Γ_n through the elimination of at most:

$$\frac{1}{2}n(n-1) - \frac{1}{2}(n-1)(n-k-1) - 1 = \frac{1}{2}(n-1)(k+1) - 1$$

edges. $\frac{1}{2}k(k+1)$ are removed in obtaining, from Γ_n , the graph H in which only members of X are nonadjacent. Thus, to obtain G from H , at most:

$$(3.9) \quad \frac{1}{2}(n-1)(k+1) - 1 - \frac{1}{2}k(k+1) = \frac{1}{2}(n-k-1)(k+1) - 1$$

edges are removed from H .

We wish to compute $D(G, X)$. By Lemma 3.6,

$$(3.10) \quad D(H, X) = 2(n-k-1).$$

Now suppose that at some stage in the transformation from H to G , we have obtained a graph K with $\mathcal{E}(H) \supseteq \mathcal{E}(K) \supseteq \mathcal{E}(G)$ and $\mathcal{V}(K) = \mathcal{V}(H) = \mathcal{V}(G)$. Let $L = K - e$ where $e \in \mathcal{E}(K) - \mathcal{E}(G)$. We wish to know the effect, $f(e) = D(L, X) - D(K, X)$, on D , of removing the edge e . Since e is an edge of H , it cannot join two points of X . If neither end-point of e is in X , then $f(e) = 0$ since $S_i(\rho_K, X) = S_i(\rho_L, X)$ for $1 \leq i \leq k$. Now suppose that one end-point, y_1 , of e is in X and that the other end-point, v , is not in X . Let y_1, y_2, \dots, y_j be the points of X which are adjacent to v in the graph K . Note that $1 \leq j \leq k+1$.

If $1 \leq i \leq j$ and $B \subseteq \{y_2, y_3, \dots, y_j\}$ with $|B| = i - 1$, and $C = B \cup \{y_1\}$, then $|C| = i$ and v is adjacent to every member of C in the graph K but not in the graph L . There are $\binom{j-1}{i-1}$ choices for such a set C . Furthermore, for any other combination of a vertex, t , and a set $A \subseteq X$ with $|A| = i$, t is adjacent to every member of A in the graph L . Thus:

$$S_i(\rho_L, X) - S_i(\rho_K, X) = \begin{cases} -\binom{j-1}{i-1} & \text{for } i \leq j \\ 0 & \text{for } i > j. \end{cases}$$

This gives:

$$\begin{aligned} f_j &= f(e) = D(L, X) - D(K, X) \\ &= \begin{cases} -\left[\frac{1}{k+1} - \sum_{i=1}^k (-1)^i \binom{k-i+1}{k+1} \binom{k}{i-1}\right] & \text{if } j = k+1 \\ -\left[\frac{1}{k+1} - \sum_{i=1}^j (-1)^i \binom{k-i+1}{k+1} \binom{j-1}{i-1}\right] & \text{if } 1 \leq j \leq k \end{cases} \\ &= \begin{cases} -\frac{1}{k+1} \left[k+1 - \sum_{i=2}^k (-1)^i (k-i+1) \binom{k}{i-1}\right] & \text{if } j = k+1 \\ -\frac{1}{k+1} \left[k+1 - \sum_{i=2}^k (-1)^i (k-i+1) \binom{j-1}{i-1}\right] & \text{if } 2 \leq j \leq k \\ -1 & \text{if } j = 1 \end{cases} \\ &= \begin{cases} -\frac{1}{k+1} & \text{if } 3 \leq j \leq k+1 \\ -\frac{2}{k+1} & \text{if } j = 2 \\ -1 & \text{if } j = 1 \end{cases} \end{aligned}$$

using (3.4) and (3.5).

Notice that $f_1 \leq f_2 \leq \dots \leq f_k \leq f_{k+1} < 0$ and that in order to realize the effect f_j , edges with effects $f_{k+1}, f_k, \dots, f_{j+1}$ must first be removed. Hence when $(k+1)$ edges are removed, the combined effect is at least:

$$\sum_{i=1}^{k+1} f_i = -2.$$

So if r edges are removed in obtaining G from H ,

$$(3.11) \quad D(G, X) - D(H, X) \geq -\frac{2r}{k+1}.$$

Using (3.9) and (3.10) in (3.11) now gives:

$$(3.12) \quad D(G, X) \geq [2(n-k-1) - (n-k-1) + 2/(k+1)] > n-k-1.$$

But Theorem 2.5 guarantees the existence of a set X as constructed above, and satisfying:

$$D(G, X) \leq n - \mu(G) \leq n - k - 1.$$

This contradicts (3.12) and completes the proof of the theorem.

COROLLARY 3.13. For $n \geq 4$, $g(n, n-3) = n$.

Proof. The bipartite graph $\Gamma_{1, n-1}$ is a graph with n vertices, $(n-1)$ edges and path-covering number $(n-2)$. Thus $g(n, n-3) \geq n$. The reverse inequality is given by Theorem 3.7.

To obtain a lower bound for $g(n, k)$, consider the graph $G = \Gamma_{n-k} \cup \bar{\Gamma}_k$; then $\mu(G) = k+1$, while $|\mathcal{V}(G)| = n$ and $|\mathcal{E}(G)| = \frac{1}{2}(n-1)(n-k-1)$. This gives:

PROPOSITION 3.14. For $n > k \geq 1$

$$(3.15) \quad g(n, k) \geq \frac{1}{2}(n-k)(n-k-1) + 1.$$

The following proposition gives some results that are easily verified:

PROPOSITION 3.15.

- (i) $g(n, n) = 0$, $g(n+1, n) = 1$, $g(n+2, n) = 2$ for $n \geq 1$
- (ii) $g(6, 2) = 7$
- (iii) $g(n+1, k+1) \geq g(n, k)$ for $n \geq k \geq 1$.

Part (iii) can be seen by letting $G = H \cup \{x\}$ where H is a graph with n vertices, $g(n, k) - 1$ edges, and $\mu(H) = k+1$, and x is an isolated vertex with $x \notin \mathcal{V}(H)$. Then G has $(n+1)$ vertices, $g(n, k) - 1$ edges, and $\mu(G) = k+2$.

In the case $k = 1$, the upper bound in (3.8) is seen to be the same as the lower bound in (3.15) and hence equality holds for $g(n, k)$ in both inequalities. However, Corollary 3.13 shows that the upper bound in (3.8) and not the lower bound in (3.15) is achieved in the case $k = n - 3$. Part (ii) of Proposition 3.15 shows a case where the lower bound and not the upper bound is achieved. It is conjectured that for small values of k , $g(n, k)$ is close to the lower bound in (3.15), while for large values of k , $g(n, k)$ is closer to the upper bound in (3.8).

We now turn to another extremal problem. Let v and n be integers with $0 \leq v \leq n$. Define:

$$h(n, v) = \text{Min} \{k: \text{every graph, } G, \text{ with } |\mathcal{V}(G)| = n \text{ and } \rho(x) \geq v \\ \text{for every } x \in \mathcal{V}(G), \text{ has } \mu(G) \leq k\}.$$

THEOREM 3.16.

$$(n, v) = \begin{cases} 1 & \text{if } v \geq \frac{n}{2} \\ n - 2v & \text{if } v < \frac{n}{2}. \end{cases}$$

Proof. The case $v \geq \frac{n}{2}$ and the upper bound $h(n, v) \leq n - 2v$ if $v < \frac{n}{2}$ follows from Ore's result (the note to Theorem 2.5). If $v < \frac{n}{2}$, let $K = \Gamma_{v, n-v}$. Then clearly $|\mathcal{V}(K)| = n$ and $\rho(x) \geq v$ for every $x \in \mathcal{V}(G)$; and in [2] (Theorem 2.2.10) we show that $\mu(K) = n - 2v$. Hence

$$h(n, v) \geq n - 2v$$

completing the proof of the theorem.

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