COVERING THE VERTICES OF A GRAPH BY VERTEX-DISJOINT PATHS

Shahbaz Noorvash
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Define the path-covering number $\mu(G)$ of a finite graph $G$ to be the minimum number of vertex-disjoint paths required to cover the vertices of $G$. Let $g(n,k)$ be the minimum integer so that every graph, $G$, with $n$ vertices and at least $g(n,k)$ edges has $\mu(G) \leq k$. A relationship between $\mu(G)$ and the degree sequence for a graph $G$ is found; this is used to show that

$$\frac{1}{2}(n-k)(n-k-1) + 1 \leq g(n,k) \leq \frac{1}{2}(n-1)(n-k-1) + 1$$

A further extremal problem is solved.

1. Introduction. A graph $G$ is a finite collection $\mathcal{V}(G)$ of vertices (or points) some pairs of which are joined by a single edge; the collection of edges is denoted by $\mathcal{E}(G)$. $H$ is a subgraph of $G$ if $\mathcal{V}(H) \subseteq \mathcal{V}(G)$ and $\mathcal{E}(H) \subseteq \mathcal{E}(G)$. If $H$ and $K$ are two vertex-disjoint graphs, $H \cup K$ is the graph with $\mathcal{V}(H \cup K) = \mathcal{V}(H) \cup \mathcal{V}(K)$ and $\mathcal{E}(H \cup K) = \mathcal{E}(H) \cup \mathcal{E}(K)$; $H + K$ is $H \cup K$ together with all $|\mathcal{V}(H)| \cdot |\mathcal{V}(K)|$ possible choices of edges joining a vertex of $H$ to a vertex of $K$. $\overline{G}$ denotes the complement of $G$; $\Gamma_n$ denotes the complete graph with $n$ vertices and $\Gamma_{m,n}$ denotes the complete bipartite graph, $\Gamma_m + \Gamma_n$.

Let $G$ be a graph. A path of length $n$ in $G$ is an ordered sequence $P = \langle a_1, a_2, \cdots, a_n \rangle$ of distinct points, where if $n \geq 2$, $a_i$ is adjacent to $a_{i+1}$ for $1 \leq i \leq n - 1$. $\langle a_1, a_2, \cdots, a_n \rangle$ is the same path as $\langle a_n, a_{n-1}, \cdots, a_1 \rangle$. If $P$ and $Q$ are paths, by $P * Q$ we shall mean that one end-point, $a$ of $P$, is adjacent to one end-point, $b$ of $Q$, and that $P * Q$ is formed by joining $a$ to $b$. More specifically we may write $Pa * bQ$ or $P * bQ$ or $Pa * Q$ to specify, in varying degrees, which end-point of $P$ is joined to which end-point of $Q$. Also, $\langle a_1, a_2, \cdots, a_n \rangle * \langle b_1, b_2, \cdots, b_m \rangle = \langle a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_m \rangle$ where $a_n$ must be adjacent to $b_1$. A Hamilton-path is a path of length $|\mathcal{V}(G)|$. A path-cover of $G$ is a collection, $\mathcal{F}$, of vertex-disjoint paths such that every vertex of $G$ lies on some path in $\mathcal{F}$. The path-covering number, denoted by $\mu(G)$, of $G$ is defined by:

$$\mu(G) = \text{Min}\{|\mathcal{F}|: \mathcal{F} \text{ is a path-cover of } G\}.$$
A minimal path-cover (M.P.C.) of $G$ is a path-cover, $\mathcal{P}$ of $G$, with $|\mathcal{P}| = \mu(G)$.

We note that $\mu(G)$ is an invariant of $G$ and remark that a graph, $G$, has a Hamilton-path if and only if $\mu(G) = 1$. It has been shown by Nash-Williams [1] and others that the problem of classifying all Hamiltonian graphs is equivalent to that of classifying all graphs which have a Hamilton-path. Thus a classification of all graphs with $\mu(G) = k$ ($k = 1, 2, 3, \cdots$) would also solve the Hamiltonian problem as a special case.

Historically, O, Ore [3] first introduced the graphical invariant $\mu$. In [2] some elementary properties of $\mu$ are derived. In §2 we generalize a result of O. Ore (Theorem 2.1 in [3]) and in §3 we consider two extremal problems involving $\mu$.

2. Valency considerations. In this section we derive a connection between the path-covering number and the degree sequence of a graph. We begin with some definitions:

**Definition 2.1.** Let $A$ be a finite set and $f$ a real-valued function defined on the collection of subsets of $A$. For $B \subseteq A$ and for any integer $i$ with $1 \leq i \leq |B|$, define the function $S_i$ by:

$$S_i(f, B) = \sum_{C \subseteq B, |C| = i} f(C).$$

**Definition 2.2.** If $G$ is a graph, $B \subseteq V(G)$, and either $H \subseteq V(G)$ or $H$ is a subgraph of $G$, then define the generalized valence function, $\rho$, by

$$\rho_H(B) = \text{the number of vertices of } H \text{ which are adjacent to every member of } B.$$

If $x$ is a vertex of $G$, then we write $\rho(x)$ for $\rho_G(\{x\})$.

**Definition 2.3.** Let $G$ be a graph and $X \subseteq V(G)$ with $|X| = k \geq 2$. Define:

$$D(G, X) = \frac{1}{k} S_i(\rho_G, X) - \sum_{i=1}^{k} (-1)^{i-1} \binom{k-i}{k} S_i(\rho_G, X).$$

The following lemma is easily verified:
Lemma 2.4. If \( X = \{x_1, x_2, \cdots, x_k\} \), and \( 1 \leq m \leq k - 1 \), then
\[
\sum_{i=1}^{k} S_m(f, X - \{x_i\}) = (k - m)S_m(f, X).
\]

We now state the main result of this section:

Theorem 2.5. Let \( G \) be a graph with \( \mu = \mu(G) \geq 2 \), \( |\mathcal{V}(G)| = n \) and \( k \) an integer with \( 2 \leq k \leq \mu \), then there exists a set \( X \) consisting of \( k \) mutually non-adjacent vertices of \( G \), satisfying:

\[
(2.6)
\mu \leq n - D(G, X).
\]

Note that the case \( k = 2 \) reduces to the result of Ore (Theorem 2.1 in [3]):
\[
\mu \leq n - \rho(x_1) - \rho(x_2).
\]

Proof. Let \( \mathcal{S} = \{P_1, P_2, \cdots, P_\mu\} \) be a M.P.C. for \( G \). For each \( 1 \leq i \leq k \), let \( x_i \) be an end-vertex of \( P_i \). Since \( \mathcal{S} \) is a M.P.C., \( x_i \) is not adjacent to \( x_j \) for \( i \neq j \).

Let \( X = \{x_1, x_2, \cdots, x_k\} \). We first show that for \( 1 \leq i \leq k \) and \( 1 \leq j \leq \mu \), the inequality:

\[
(2.7)
\rho_{P_i}(\{x_i\}) \leq |P_i| - \left(1 - \sum_{i=1}^{k-1} (-1)^iS_i(\rho_{P_i}, X - \{x_i\})\right)
\]

holds. Let \( P_j \) be the path \( \langle a_1, a_2, \cdots, a_r \rangle \), let \( 1 \leq m \leq k, m \neq i \), and consider the following cases:

(i) \( i = j \). In this case assume that \( x_i = a_1 \).
(ii) \( m = j \). In this case assume that \( x_m = a_r \).
(iii) \( m \neq j \) and \( i \neq j \).

Let
\[
A = \{r: a_r \text{ is adjacent to } x_i\},
\]
\[
B_m = \{r: a_{r-1} \text{ is adjacent to } x_m\}
\]

and
\[
B = \bigcup_{1 \leq m \leq k, m \neq i} B_m.
\]

We claim that \( A \cap B_m = \phi \), for if \( r \in A \cap B_m \), then in each case we can
construct a path-cover, $\mathcal{F}$ for $G$, as follows (see Figure 2.8):

In case (i), let:

$$\mathcal{F} = \mathcal{F} \cup \{(a_i, a_{i-1}, \ldots, a_n, x_i, a_2, a_3, \ldots, a_{r-1}) \ast x_m P_m \} - \{P_n, P_m\}.$$
In case (ii), let:

\[ \mathcal{F} = \mathcal{F} \cup \{(a_1, a_2, \ldots, a_{r-1}, x_m, a_{r-1}, a_r) * x_1 P_i \} - \{P, P_m\}. \]

In case (iii), let:

\[ \mathcal{F} = \mathcal{F} \cup \{(a_1, \ldots, a_{r-1}) * x_m P_m, (a_r, a_{r-1}, \ldots, a_r) * x_r P_i \} - \{P, P_i, P_m\}. \]

In either case, \(| \mathcal{F} | = | \mathcal{F} | - 1 < | \mathcal{F} |\), contradicting the minimality of \( \mathcal{F} \). Hence \( A \cap B_m = \emptyset \). Also, in each case \( a_1 \notin A \); so \( A \subseteq P - B \cup \{a_i\} \). This gives \(|A| \leq |P_i| - |B \cup \{a_i\}|\), since \( B \cup \{a_i\} \subseteq P_i \). But then, since \( a_i \notin B \), we get:

\[
(2.9) \quad |A| \leq |P_i| - (1 + |B|).
\]

For \( 1 \leq m \leq k \), let:

\[ C_m = \{r: a_r \text{ is adjacent to } x_m\}. \]

Then since \( x_m \) is not adjacent to \( a_1 \), \(|C_m| = |B_m|\) and:

\[
|B| = \bigcup_{1 \leq m \leq k, \ m \neq i} B_m = \bigcup_{1 \leq m \leq k, \ m \neq i} C_m = \sum_{l=1}^{k-1} (-1)^{l+1} \sum_{1 \leq m_1 < m_2 < \cdots < m_i \leq k} |C_{m_1} \cap C_{m_2} \cap \cdots \cap C_{m_i}|.
\]

\[
(2.10) \quad = -\sum_{i=1}^{k-1} (-1)^i S_i(\rho_i, X - \{x_i\}).
\]

So since \(|A| = \rho_i(\{x_i\})\), (2.7) follows from (2.9) and (2.10). Summing (2.7) for \( 1 \leq i \leq k \) and applying Lemma 2.4, we get:

\[
(2.11) \quad S_i(\rho_i, X) \leq k |P_i| - \left( k - \sum_{l=1}^{k-1} (-1)^l (k - l) S_l(\rho_i, X) \right).
\]
Summing (2.11) for $1 \leq j \leq \mu$, we get:

$$S_i(\rho_G, X) \leq kn - \sum_{l=1}^{k-1} (-1)^l(k-l)S_i(\rho_G, X).$$

from which (2.6) follows.

3. Extremal problems.

**Definition 3.1.** Let $k$ and $n$ be integers with $1 \leq k \leq n$. Define:

$$g(n, k) = \min \{m: \text{ every graph, } G, \text{ with } |\mathcal{V}(G)| = n \text{ and } |\mathcal{E}(G)| \geq m \text{ has } \mu(G) \leq k\}. $$

In this section we determine bounds for $g(n, k)$. See [4] for techniques in proving the following:

**Lemma 3.2.**

(3.3) \[ \sum_{i=1}^{k-1} (-1)^i \left(\frac{k-i}{k}\right) \frac{k}{i} = -1 \quad \text{if} \quad k \geq 2, \]

(3.4) \[ \sum_{i=2}^{k} (-1)^i(k-i+1) \frac{k}{i-1} = k \quad \text{if} \quad k \geq 2, \]

(3.5) \[ \sum_{i=2}^{j} (-1)^i(k-i+1) \frac{j-1}{i-1} = k \quad \text{if} \quad 3 \leq j \leq k. \]

**Lemma 3.6.** Let $K$ be a graph with $|\mathcal{V}(K)| = s \geq 1$, and let $k$ be an integer with $k \geq 2$, and suppose $H = \bar{G}_k + K$, then:

$$D(H, \mathcal{V}((\bar{G}_k))) = 2s.$$  

**Proof.** For $1 \leq i \leq k-1$ and $B \subseteq \mathcal{V}((\bar{G}_k))$ with $|B| = i$, each member of $B$ is adjacent to every member of $\mathcal{V}(K)$. There are $\left(\frac{k}{i}\right)$ choices for $B$ and $|\mathcal{V}(K)| = s$; thus:

$$S_i(\rho_H, \mathcal{V}((\bar{G}_k))) = s\left(\frac{k}{i}\right).$$
This gives:

\[
D(H, V(\Gamma_n)) = \frac{s}{k} \binom{k}{1} - \sum_{i=1}^{k-1} (\frac{k-i}{k}) \binom{k}{i} \]

\[
= s \left[ 1 - \sum_{i=1}^{k-1} (\frac{k-i}{k}) \binom{k}{i} \right]
\]

\[
= 2s, \text{ using (3.3).}
\]

**Theorem 3.7.** For \(1 \leq k \leq n\),

(3.8) \[ g(n, k) \leq \frac{1}{2} (n - 1) (n - k - 1) + 1. \]

**Proof.** Let \(G\) be a graph with \(|V(G)| = n\), and \(|E(G)| \geq \frac{1}{2} (n - 1) (n - k - 1) + 1\). Suppose \(\mu(G) > k\) and \(X = \{x_1, x_2, \ldots, x_k\}\) is a set of mutually nonadjacent vertices of \(G\).

\(G\) may be considered to have been obtained from the complete graph \(\Gamma_n\) through the elimination of at most:

\[
\frac{1}{2} n (n - 1) - \frac{1}{2} (n - 1) (n - k - 1) - 1 = \frac{1}{2} (n - k - 1) (k + 1) - 1
\]

edges. \(\frac{1}{2} k(k + 1)\) are removed in obtaining, from \(\Gamma_n\), the graph \(H\) in which only members of \(X\) are nonadjacent. Thus, to obtain \(G\) from \(H\), at most:

(3.9) \[ \frac{1}{2} (n - 1) (k + 1) - 1 - \frac{1}{2} k(k + 1) = \frac{1}{2} (n - k - 1) (k + 1) - 1 \]

edges are removed from \(H\).

We wish to compute \(D(G, X)\). By Lemma 3.6,

(3.10) \[ D(H, X) = 2(n - k - 1). \]

Now suppose that at some stage in the transformation from \(H\) to \(G\), we have obtained a graph \(K\) with \(E(H) \supseteq E(K) \supseteq E(G)\) and \(V(K) = V(H) = V(G)\). Let \(L = K - e\) where \(e \in E(K) - E(G)\). We wish to know the effect, \(f(e) = D(L, X) - D(K, X)\), on \(D\), of removing the edge \(e\). Since \(e\) is an edge of \(H\), it cannot join two points of \(X\). If neither end-point of \(e\) is in \(X\), then \(f(e) = 0\) since \(S_i(\rho_k, X) = S_i(\rho_{k+1}, X)\) for \(1 \leq i \leq k\). Now suppose that one end-point, \(y_1\), of \(e\) is in \(X\) and that the other end-point, \(v\), is not in \(X\). Let \(y_1, y_2, \ldots, y_j\) be the points of \(X\) which are adjacent to \(v\) in the graph \(K\). Note that \(1 \leq j \leq k + 1\).
If $1 \leq i \leq j$ and $B \subseteq \{y_2, y_3, \ldots, y_i\}$ with $|B| = i - 1$, and $C = B \cup \{y_1\}$, then $|C| = i$ and $v$ is adjacent to every member of $C$ in the graph $K$ but not in the graph $L$. There are $\binom{i-1}{i-1}$ choices for such a set $C$. Furthermore, for any other combination of a vertex, $t$, and a set $A \subseteq X$ with $|A| = i$, $t$ is adjacent to every member of $A$ in the graph $L$. Thus:

$$S_i(\rho_t, X) - S_i(\rho_k, X) = \begin{cases} -\frac{(j-1)}{i-1} & \text{for } i \leq i \leq j \\ 0 & \text{for } i > j. \end{cases}$$

This gives:

$$f_i = f(e) = D(L, X) - D(K, X)$$

$$= \begin{cases} \left[rac{1}{k+1} - \sum_{i=1}^{k} (-1)^i \left(\frac{k-i+1}{k+1}\right) \left(\frac{k}{i-1}\right) \right] & \text{if } j = k + 1 \\ \left[rac{1}{k+1} - \sum_{i=1}^{j} (-1)^i \left(\frac{k-i+1}{k+1}\right) \left(\frac{j}{i-1}\right) \right] & \text{if } 1 \leq j \leq k \end{cases}$$

$$= \begin{cases} \left[-\frac{1}{k+1}\left[k+1 - \sum_{i=2}^{k} (-1)^i (k-i+1) \left(\frac{k}{i-1}\right) \right] \right] & \text{if } j = k + 1 \\ \left[-\frac{1}{k+1}\left[k+1 - \sum_{i=2}^{j} (-1)^i (k-i+1) \left(\frac{j}{i-1}\right) \right] \right] & \text{if } 2 \leq j \leq k \end{cases}$$

$$= \begin{cases} -\frac{1}{k+1} & \text{if } 3 \leq j \leq k + 1 \\ -\frac{2}{k+1} & \text{if } j = 2 \\ -1 & \text{if } j = 1 \end{cases}$$

using (3.4) and (3.5).

Notice that $f_1 \leq f_2 \leq \cdots \leq f_k \leq f_{k+1} < 0$ and that in order to realize the effect $f_i$, edges with effects $f_{k+1}, f_k, \ldots, f_{i+1}$ must first be removed. Hence when $(k + 1)$ edges are removed, the combined effect is at least:

$$\sum_{i=1}^{k+1} f_i = -2.$$
So if \( r \) edges are removed in obtaining \( G \) from \( H \),

\[(3.11) \quad D(G, X) - D(H, X) \geq \frac{2r}{k + 1}.
\]

Using (3.9) and (3.10) in (3.11) now gives:

\[(3.12) \quad D(G, X) \geq [2(n - k - 1) - (n - k - 1) + 2/(k + 1)] > n - k - 1.
\]

But Theorem 2.5 guarantees the existence of a set \( X \) as constructed above, and satisfying:

\[D(G, X) \leq n - \mu(G) \leq n - k - 1.
\]

This contradicts (3.12) and completes the proof of the theorem.

**Corollary 3.13.** For \( n \geq 4 \), \( g(n, n - 3) = n \).

*Proof.* The bipartite graph \( \Gamma_{1,n-1} \) is a graph with \( n \) vertices, \( (n - 1) \) edges and path-covering number \( (n - 2) \). Thus \( g(n, n - 3) \geq n \). The reverse inequality is given by Theorem 3.7.

To obtain a lower bound for \( g(n, k) \), consider the graph \( G = \Gamma_{n-1} \cup \Gamma_k \); then \( \mu(G) = k + 1 \), while \( |\mathcal{V}(G)| = n \) and \( |\mathcal{E}(G)| = \frac{1}{2}(n - 1)(n - k - 1) \). This gives:

**Proposition 3.14.** For \( n > k \geq 1 \)

\[(3.15) \quad g(n, k) \geq \frac{1}{2}(n - k)(n - k - 1) + 1.
\]

The following proposition gives some results that are easily verified:

**Proposition 3.15.**

(i) \( g(n, n) = 0, g(n + 1, n) = 1, g(n + 2, n) = 2 \) for \( n \geq 1 \)

(ii) \( g(6, 2) = 7 \)

(iii) \( g(n + 1, k + 1) \geq g(n, k) \) for \( n \geq k \geq 1 \).

Part (iii) can be seen by letting \( G = H \cup \{x\} \) where \( H \) is a graph with \( n \) vertices, \( g(n, k) - 1 \) edges, and \( \mu(H) = k + 1 \), and \( x \) is an isolated vertex with \( x \notin \mathcal{V} \) \( i \)th \( x \notin \mathcal{V}(H) \). Then \( G \) has \( (n + 1) \) vertices, \( g(n, k) - 1 \) edges, and \( (G) = k + 2 \).
In the case $k = 1$, the upper bound in (3.8) is seen to be the same as the lower bound in (3.15) and hence equality holds for $g(n, k)$ in both inequalities. However, Corollary 3.13 shows that the upper bound in (3.8) and not the lower bound in (3.15) is achieved in the case $k = n - 3$. Part (ii) of Proposition 3.15 shows a case where the lower bound and not the upper bound is achieved. It is conjectured that for small values of $k$, $g(n, k)$ is close to the lower bound in (3.15), while for large values of $k$, $g(n, k)$ is closer to the upper bound in (3.8).

We now turn to another extremal problem. Let $v$ and $n$ be integers with $0 \leq v \leq n$. Define:

$$h(n, v) = \operatorname{Min}\{k : \text{every graph, } G, \text{ with } |\mathcal{V}(G)| = n \text{ and } \rho(x) \geq v \text{ for every } x \in \mathcal{V}(G), \text{ has } \mu(G) \leq k\}.$$ 

**Theorem 3.16.**

$$(n, v) = \begin{cases} 1 & \text{if } v \geq \frac{n}{2}, \\ n - 2v & \text{if } v < \frac{n}{2}. \end{cases}$$

**Proof.** The case $v \geq \frac{n}{2}$ and the upper bound $h(n, v) \leq n - 2v$ if $v < \frac{n}{2}$ follows from 0. Ore's result (the note to Theorem 2.5). If $v < \frac{n}{2}$, let $K = \Gamma_{c,v}$. Then clearly $|\mathcal{V}(K)| = n$ and $\rho(x) \geq v$ for every $x \in \mathcal{V}(G)$; and in [2] (Theorem 2.2.10) we show that $\mu(K) = n - 2v$. Hence

$$h(n, v) \geq n - 2v$$

completing the proof of the theorem.

**References**


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